

Asset Trading and Valuation with Uncertain Exposure*

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Abstract

This paper considers an asset market where investors have private information not only about asset payoffs, but also about their own exposure to an aggregate risk factor. In equilibrium, rational investors disagree about asset payoffs: those with higher exposure to the risk factor are more optimistic about claims on the risk factor, which leads to less risk sharing than under symmetric information. Moreover, uncertainty about exposure amplifies the effect of aggregate exposure on asset prices, and can thereby help explain the excess volatility of prices and the predictability of excess returns.

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1 Introduction

A lot of modern financial market activity consists of agents trading claims to aggregate risk factors. Examples include secondary markets for government debt, exchange traded index funds, and the burgeoning markets for interest rate, exchange rate and stock index derivatives. These markets allow for gains from trade between agents with different exposure to the aggregate risk factors. As a simple example, suppose one investor has high exposure to the stock market because his wealth consists mostly of stocks, whereas a second investor is not exposed to the stock market. A sale of stock by the first to the second investor can then make both better off.

In models with optimizing agents, asset prices reflect the result of investors' trading of exposure. Indeed, the value of a claim to a risk factor is usually equal to its expected payoff less a risk premium that depends on investors' exposure to the risk factor in equilibrium. Asset prices thus move if there are news about payoffs, but also if there are aggregate shocks that change investors' exposures. For example, consider a shock that carries no news about stock payoffs, but lowers the expected payoff of some other asset like housing or human capital. This shock will increase investors' exposure to the stock market by making stocks a more important component in their portfolios. It thus increases the risk premium on stocks and lowers stock prices.

Most papers that study the role of exposure shocks for asset pricing either consider a representative investor or assume that individual exposures are observable. However, in the typical market, a participant cannot easily observe others' portfolios and risk attitudes; he may not even know their identities. This paper considers asset pricing and trading in economies with private information. Since investors' private information consists not only of their exposure, but also of other signals, market prices do not allow investors to distinguish between news that change expected payoffs and shocks to the distribution of exposures that change risk premia.¹

The main implication is that aggregate shocks to the distribution of exposures become more important drivers of asset prices. Even with symmetric information, a shock that increases

¹For example, a recession will lower the value of human capital of many, but not all stock market participants. Individual market participants see the change in their own exposure to stocks, but cannot discern from aggregate statistics how the entire distribution of exposures has shifted.

equilibrium exposure to a risk factor lowers the price of a claim to that factor. In our model, investors partly mistake such a shock for bad news, which further lowers the price. From the perspective of an econometrician, this effect contributes to predictability of excess returns and excess volatility of prices, a well known puzzle in asset pricing. We show that the extent to which it does so depends on preferences and the covariance of exposure and wealth.

Our model can also help understand the joint arrival of high risk premia and low trading volume that has been observed in recent financial crises. Indeed, suppose an economy where aggregate exposure has been known to be low is hit by a shock that increases the exposure of some, but not all market participants. The increase in average exposure lowers prices and increases risk premia. If the distribution of exposures were known, one would also expect increased trade as high exposure investors sell risky claims. In our model, in contrast, rational inference from prices and private information leads investors to disagree: those with higher exposure to a risk factor become more optimistic about claims on the risk factor, and thus sell fewer of those claims to low exposure agents. The result is less trading that takes place at unusually low prices.

Formally, we study a two period exchange economy with incomplete markets. In the first period, agents trade claims that are contingent on a tradable aggregate risk factor. In the second period, the factor is realized and agents consume. Agents' endowments are tradable. To define exposure, we regress an agent's relative willingness to give up a unit of consumption – minus the ratio of marginal utility to expected marginal utility – on the tradable risk factor. The beta from the regression is the agent's exposure to the risk factor – it measures how his personal valuation of funds moves with changes in the risk factor. Exposure is positive if and only if consumption comoves with the risk factor. It also depends on preferences: it is zero for a risk neutral agent, and its absolute value is increasing in risk aversion for common utility functions.

Before trading takes place in the first period, two other aggregate shocks are realized. A news shock determines the distribution of private signals about the tradable risk factor. There is also an aggregate shock that determines the distribution of endowments and preferences, and hence initial exposures at the endowment point. Agents see their own signal, endowment and preferences, but not the aggregate shocks. We consider rational expectations equilibria of this model.

Most of our results are derived in a setting where the tradable risk factor and the exposure shock take on two values, but the news shock is a continuous random variable. For this environment, we establish properties of nonrevealing equilibria when preferences exhibit linear risk tolerance (LRT). With LRT preferences, there exists a representative agent with LRT preferences whose exposure captures the aggregate exposure of the economy to the risk factor. We then present numerical results for the case where agents have logarithmic utility. We also consider two examples – one with a discrete distribution and one with exponential utility and normal shocks – that allow for more analytical results.

If initial exposures are known, then our model has a rational expectations equilibrium in which (i) the price perfectly reveals the news shock, that is, the relevant pooled information of all agents about the tradable risk factor, (ii) all agents equate their exposures to the tradable risk factor, which is Pareto-efficient and (iii) the risk premium reflects the equal exposure of all agents. However, if the initial exposures are private information, asset prices cannot simultaneously reveal the news shock and the aggregate exposure shock. For example, a low price could be due either to bad news that lowers the expected payoff, or to higher exposure that increases the risk premium.

Investors then disagree in equilibrium about expected payoffs. In particular, investors with higher initial exposure to the risk factor are more optimistic about assets with high betas (that is, high covariance with the factor) than investors with lower initial exposure. This leads to less trading between agents with different initial exposures, and exposures are not equated in equilibrium. The source of disagreement is that investors rely on both private and public information to estimate expected asset payoffs and risk premia. On the one hand, investors take their own initial exposure as a signal of aggregate exposure: agents with high initial exposure believe that risk premia are higher than those with low initial exposure. On the other hand, agents extract from the price a signal about payoffs. Since risk premia drive down prices, high exposure investors rationally view any given price as a better signal about future payoffs than low exposure investors.

The presence of this inference problem makes aggregate exposure shocks more important drivers of prices. Indeed, for any beliefs, there is a direct effect of aggregate exposure on prices: agents with higher initial exposure to a risk factor have a lower net demand for claims on the risk

factor. With asymmetric information, exposure shocks also work through average beliefs: higher aggregate exposure lowers the price of a claim on the risk factor (through the direct effect), but cannot be distinguished from bad news about the risk factor, so that agents become more pessimistic on average, which leads to an even lower price.

The strength of this belief effect depends on preferences as well as the distribution of endowments and exposure. With exponential utility, there are no wealth effects. The relevant average belief reflected in prices is population weighted, and the distribution of exposures matters only because individual exposure is a signal of aggregate exposure. In contrast, with logarithmic utility, the relevant average belief reflected in prices is wealth weighted. This introduces an additional source of average pessimism: a drop in the value of the risk factor lowers the wealth share of agents with high initial exposure to the factor, who are also more optimistic.

In many markets, future excess returns can be predicted by looking at current prices: high prices precede low excess returns and vice versa. Predictability requires shocks that affect prices, but not future payoffs. On average, a price increase driven by such a shock must be followed by a below average return. This is why shocks to aggregate exposure that are orthogonal to future payoffs have been explored in the literature. In our model, the amplification of shocks to aggregate exposure implies that predictability patterns become more pronounced in economies with asymmetric information.

Related literature

Relative to the literature, we make three contributions. First, we provide a general definition of exposure that leads to predictions for the distribution of initial exposures, beliefs, and trading behavior. Second, we establish the importance of exposure shocks for prices when agents have LRT preferences. Finally, we study predictability regressions in an asymmetric information economy. In this context, we provide a numerically tractable setting where asset prices can be studied with log utility, a standard utility function that allows for decreasing absolute risk aversion and wealth effects.

There is a large literature on rational expectations equilibria in economies with asymmetric information, following the seminal work of Radner (1967) and Lucas (1972). The fact that

equilibrium involves inference from prices has made it difficult to provide general proofs of the existence of partially revealing equilibria (see Allen and Jordan (1998) for a survey of early work and Pietra and Siconolfi (2008) for some recent results).² DeMarzo and Skiadas (1998) have characterized existence and asset pricing properties of a class of “quasi-complete” economies with LRT preferences that subsumes many models considered earlier. Our model economies are not quasi-complete because of the presence of aggregate exposure shocks.

For asset pricing applications, a main workhorse has been the framework with exponential utility and normally distributed shocks developed by Grossman (1976), Hellwig (1980), and Admati (1985). In these models, the presence of nonrevealing equilibria is due to a random supply of assets sold by “noise traders”. A net sale by noise traders can be viewed as a shock that increases the equilibrium exposure of the rational agents. It thus generates low prices and high risk premia together with *high* volume. Moreover, volume increases with uncertainty, since there are more speculative trades while the response of volume to noise trades is independent of the information structure. In contrast, a key feature of our model is that volume can be lower when uncertainty about exposure increases.

Diamond and Verrecchia (1981) consider an exponential/normal setup where agents receive stochastic endowments from which they learn about the aggregate endowment. Our exponential-normal example in Section 6 is a version of their model, but with a continuum of traders and news shocks. Ganguli and Yang (2009) consider a special case of this setup in which the pooled information of all agents reveals asset payoffs. In contrast to our exponential-normal example, partially revealing equilibria may fail to exist in their setup. Moreover, they study information acquisition rather than the dependence of trading and risk premia on aggregate exposure shocks.

Recently a few papers have started to examine the quantitative implications of dynamic exponential/normal models with asymmetric information for asset pricing and trading. Biais et al. (2007) show that the interaction of informed and uninformed agents can explain the superior performance of mean-variance efficient portfolio strategies used by uninformed investors relative

²Several authors have studied the role of exogenously given heterogeneous beliefs on asset prices (see for example Calvet et al. (2001), Jouini and Napp (2007), Jouini and Napp (2006), Detemple and Murthy (1994)). In a rational expectations equilibrium, the heterogeneity of beliefs is endogenous.

to the market. Building on the dynamic exponential-normal model proposed by Wang (1994), Albuquerque et al. (2007) show that modelling trades between informed and uninformed traders in international equity markets help understand the joint distribution of cross border equity flows and stock returns. The computational approach in this paper may be useful for further quantitative work in the future.

Many papers consider the role of exposure shocks for the predictability of excess returns and the volatility of prices in the absence of private information. In a representative agent model, any shock to marginal utility that does not affect dividends will move around the exposure of the representative agent to the stock market. Examples include housing and human capital, and also exogenous changes in risk aversion. Heterogeneous agent models with incomplete markets typically allow for changes in the idiosyncratic volatility of labor income which also changes the initial exposure of agents to stocks. In exponential-normal models, Campbell et al. (1993), Campbell and Kyle (1991), and Spiegel (1998) have examined the role of a random supply of assets for volatility.

The rest of the paper is structured as follows. Section 2 introduces the model. Section 3 studies a simple version of the model with a finite number of aggregate states. Here we discuss the relationship of exposure, beliefs and trading, but for rather general preferences. Section 4 considers a setup with a continuum of states and LRT preferences and derives properties of asset prices. Section 5 presents numerical results with log utility. Section 6 contains an example with exponential utility and normal shocks.

2 Model

There are two dates and a continuum of agents of measure 1. At date 1, nature draws a distribution of agent types μ , where an agent's type θ determines his endowment, preferences and information. When agents trade assets at date 1, they know their own type, but not the distribution μ . At date 2, assets pay off a single consumption good, and agents consume. Tradeable risk is captured by a random variable τ that can take the values τ_1 or τ_2 , with $\tau_1 > \tau_2$, drawn by nature at date 2. The assets traded at date 1 are contingent claims on τ .

Types

There is a finite number of agent types, indexed by $\theta \in \Theta$. Types are iid across agents. As a result, $\mu(\theta)$ is both the probability that an individual agent is of type θ and the fraction of agents of type θ in the population. Agents have expected utility preferences over consumption in the two states realized at date 2, with felicity

$$u(c; \theta),$$

where u is continuously differentiable, strictly increasing and strictly concave. An agent's endowment is a vector $\omega(\theta) = (\omega_1(\theta), \omega_2(\theta))' \in \mathfrak{R}_{++}^2$, where $\omega_j(\theta)$ is the endowment that agent θ receives when $\tau = \tau_j$. Endowments are thus tradable, that is, they can be replicated by portfolios of tradable assets.

Agent Problem

Since the tradable risk factor τ takes on only two values, only one relative price must be determined in equilibrium. We normalize prices such that $p \in [0, 1]$ is the price of a contingent claim that pays one unit of consumption when $\tau = \tau_1$, and $1 - p$ is the price of a claim that pays one unit when $\tau = \tau_2$. Individual information sets at date 1 contain an agent's own type θ as well as the price p . A key feature of the model is that the price not only enters the budget constraint, but also serves as a signal. Let $\hat{\delta}$ denote an agent's subjective probability of event $\tau = \tau_1$, given his information. The agent solves

$$\begin{aligned} \max_{(c_1, c_2)} \quad & \hat{\delta} u(c_1, \theta) + (1 - \hat{\delta}) u(c_2, \theta) \\ \text{s.t.} \quad & p c_1 + (1 - p) c_2 = w(\theta, p) := p \omega_1(\theta) + (1 - p) \omega_2(\theta) \end{aligned} \quad (1)$$

We denote the optimal policy for this problem by the vector $c^*(\theta, p, \hat{\delta})$.

Types and exposure

Consider an agent who chooses a consumption plan (or, equivalently, a portfolio) c . Intuitively, the agent is more "exposed" to the risk factor τ if his valuation of consumption varies more with τ . Formally, we measure the agent's relative willingness to give up a unit of consumption by the ratio $-u'(c) / E[u'(c)]$. We then define the exposure to the risk factor τ perceived

by a type θ agent with portfolio c and belief $\hat{\delta}$ by

$$\tilde{e}_\tau(\theta, c, \hat{\delta}) := \frac{\text{cov}\left(\tau, -\frac{u'(c; \theta)}{E[u'(c; \theta)]}\right)}{\text{var}(\tau)}, \quad (2)$$

where the moments are computed using the belief $\hat{\delta}$. Exposure can thus be thought of as the beta of a regression of agent θ 's willingness to give up consumption on the risk factor τ .

An agent with higher exposure to the risk factor τ is relatively more willing to give up consumption when τ is high. Exposure is positive if and only if consumption covaries positively with τ . Exposure is zero when consumption is independent of τ or the agent is risk neutral. If agents have the same beliefs, they equate their exposures to τ in equilibrium. Indeed, for an optimal consumption plan c , the first order condition implies that the price of a claim on state 1 is $p = \hat{\delta}u'(c_1) / E[u'(c)]$, the marginal rate of substitution between wealth in state 1 at date 2 and wealth at date 1. With common beliefs, equating marginal rates of substitution is the same as equating exposures.

Definition (2) describes the exposure perceived by a type θ agent given his own belief $\hat{\delta}$. To compare the behavior of different agents, it is helpful to compare their *initial* exposure under some common reference probability for the event $\tau = \tau_1$, say $\pi \in (0, 1)$. In fact, in our two state setup, the ranking of exposures \tilde{e}_τ across agents does not depend on the common reference probability; it depends only on the ratio of marginal utilities across the 2 states. We thus introduce the following measure of an agent's initial exposure:

$$e_\tau(\theta) := \log\left(\frac{u'(\omega_2(\theta); \theta)}{u'(\omega_1(\theta); \theta)}\right) = \log\left(\frac{1 + \pi\tilde{e}_\tau(\theta, \omega, \pi)}{1 - (1 - \pi)\tilde{e}_\tau(\theta, \omega, \pi)}\right). \quad (3)$$

For any reference probability π , the sign of $e_\tau(\theta)$ is the same as the sign of $\tilde{e}_\tau(\theta, \omega, \pi)$. In addition, for every π, θ and $\tilde{\theta}$ we have that $e_\tau(\theta) > e_\tau(\tilde{\theta})$ if and only if $\tilde{e}_\tau(\theta, \omega, \pi) > \tilde{e}_\tau(\tilde{\theta}, \omega, \pi)$. The measure $e_\tau(\theta)$ is sometimes more convenient than $\tilde{e}_\tau(\theta, \omega, \pi)$, but it can only be used in a 2 state setup.

In our model, the only reason to allow for differences in endowments and felicities across types is to generate differences in initial exposure to the risk factor τ . Accordingly, we assume that two agents cannot have different endowments and felicities if they have the same initial exposure

to τ , that is, for all $\theta, \tilde{\theta} \in \Theta$, $e_\tau(\theta) = e_\tau(\tilde{\theta})$ implies $u(\cdot; \theta) = u(\cdot; \tilde{\theta})$ and $\omega(\theta) = \omega(\tilde{\theta})$. We also assume that the initial exposure parameter takes one of two values: $e_\tau(\theta) \in \{\bar{e}, \underline{e}\}$ with $\bar{e} > \underline{e}$. The economy is thus always populated by high and low exposure agents.

In addition to determining exposure, an agent's type plays a second role in our setup: it serves as a signal about the tradable risk factor τ . For example, suppose the uncertain distribution μ is correlated with τ in a way that, say, $\mu(\theta)$ is higher than $\mu(\tilde{\theta})$ whenever $\tau = \tau_1$ is more likely than $\tau = \tau_2$. In this case, an agent who learns that his type is θ and not $\tilde{\theta}$ thereby receives a noisy signal that $\tau = \tau_1$ is more likely than $\tau = \tau_2$. In other words, the type is informative about τ . Importantly, our setup allows for type to be informative about τ even if the distribution of endowments and felicities is not correlated with τ . For example, we could have that the type θ contains a pure signal component s that is orthogonal to $e_1(\theta)$ but correlated with the factor τ .

Aggregate news and the distribution of exposures

It remains to specify the joint distribution of the aggregates, that is, the type distribution $\mu \in \Delta(\Theta)$ drawn at date 1 and the tradable risk factor $\tau \in (\tau_1, \tau_2)$ drawn at date 2. We define the aggregate state space as (X, Ξ, Pr) , where $X = \Delta(\Theta) \times \{1, 2\}$. The probability Pr governs the joint distribution of the aggregates μ and τ as well as agents' individual types θ . We assume that agents have rational expectations, so that Pr also describes every agent's individual belief. Agents can disagree only if they have different information.

We consider distributions μ that are parameterized by two numbers. First, let $\delta(\mu)$ denote the information carried by μ about the factor τ ,

$$\delta(\mu) = \text{Pr} \{ \tau = \tau_1 | \mu \}$$

Agents care only about τ . If they were to pool their information, and thus knew the distribution μ , only the parameter $\delta(\mu)$ would be relevant to them. In other words, δ represents the “aggregate news” about τ available at date 1.

Second, let $\varepsilon(\mu)$ denote the fraction of high exposure agents in the population (or, equivalently the probability that an individual agent has high exposure):

$$\varepsilon(\mu) = \text{Pr} \{ e_\tau(\theta) = \bar{e} \}$$

Since endowments and felicities differ only if exposures differ, the parameter ε summarizes the effect of the distribution of types on the distribution of endowments and felicities. In particular, it determines the average exposure in the economy.

All our economies have the property that *given the distribution μ , the pair $(\delta(\mu), \varepsilon(\mu))$ is a sufficient statistic for forecasting an individual's type θ* . This parametrization rules out distributions where agents have unequal information quality. For example, suppose that endowments are constant ($\varepsilon(\mu) = 1$, say), and that some random fraction $\alpha(\mu)$ of the agents is told what the event τ is, whereas the other half receives no signal. It follows that $\delta(\mu) = 1$ if $\tau = \tau_1$ and $\delta(\mu) = 0$ otherwise. Given μ , $(\delta(\mu), \varepsilon(\mu))$ is then not a sufficient statistic for forecasting θ . This is because the type θ can take the value "no signal". Forecasting θ given μ thus involves $\alpha(\mu)$; this information is lost when attention is restricted to the statistic $(\delta(\mu), \varepsilon(\mu))$.

Equilibrium

Since the types of individual agents are iid, the aggregate demand for assets, and hence the equilibrium price, depend only on the distribution of types μ . We can write the price as $p = P(\mu)$. For any price function P , let

$$\hat{\delta}(\theta, p; P)$$

denote the posterior probability that an agent of type θ assigns to the event $\tau = \tau_1$ if he observes the price function P take the value p . This probability can be derived by Bayes' rule from the joint distribution of μ , τ and θ , given knowledge of the price function P .

Definition. A *rational expectations equilibrium (REE)* consists of a price function $P : \Delta(\Theta) \rightarrow [0, 1]$ and a consumption allocation $c : \Theta \times \Delta(\Theta) \rightarrow \mathbb{R}^2$ such that:

1. *The individual consumption plan $c(\theta, \mu)$ solves the problem (1) for the price $p = P(\mu)$ and the belief $\delta = \hat{\delta}(\theta, p; P)$, that is, for every μ and θ ,*

$$c(\theta, \mu) = c^*(\hat{\delta}(\theta, P(\mu); P), \theta; P(\mu)).$$

2. *Markets clear: for every μ*

$$\sum_{\theta \in \Theta} \mu(\theta) c(\theta, \mu) = \sum_{\theta \in \Theta} \mu(\theta) \omega(\theta).$$

The Revelation of Information by Prices

Since the pair $(\delta(\mu), \varepsilon(\mu))$ is a sufficient statistic for θ , the distribution of agents' individual demands depends only on these parameters. The same is true for the aggregate excess demand at some price p , which can be written as

$$\sum_{\theta \in \Theta} \mu(\theta) \left(c^*(\hat{\delta}(\theta, p; P), \theta, p) - \omega(\theta) \right) =: Z(p; \delta(\mu), \varepsilon(\mu)).$$

It follows that the equilibrium price can also be represented as a function of the parameters, that is, $P(\mu) = \tilde{P}(\delta(\mu), \varepsilon(\mu))$ where \tilde{P} is defined by

$$Z(\tilde{P}(\delta, \varepsilon); \delta, \varepsilon) = 0. \tag{4}$$

Equation (4) illustrates why a fully revealing equilibrium – where agents' beliefs all agree at δ – need not exist in our model. The existence of such an equilibrium requires that agents can infer the relevant aggregate news δ from observing the price and their own type. However, the unobservable distribution μ has two unknown parameters δ and ε , whereas there is only one price signal \tilde{P} . In general, inferring δ from \tilde{P} need not be possible. At the same time, there may be a partially revealing equilibrium in which different parameter pairs (δ, ε) lead to the same solution for the price.

Section 3 considers the simplest possible example of a partially revealing equilibrium. We assume that there are only two possible distributions μ , and choose two pairs of parameters $(\delta(\mu), \varepsilon(\mu))$ such that one price solves (4) for both distributions. The example shows when the separate influence of beliefs and exposure on demand allows such a choice, and what beliefs and trades emerge in equilibrium. While this example is useful, it is not sufficient for our purposes. On the one hand, it implies a constant price and we cannot talk about volatility of asset prices. On the other hand, it cannot be robust: it must rely on a knife edge choice of parameters which small perturbations to endowments or preferences would destroy (as in Radner (1979)).

In Section 4, we consider a version of the model where the parameter δ varies continuously. This implies that there cannot exist a fully revealing equilibrium. The intuition is apparent from (4). Suppose that Z is continuous in δ and that variations in beliefs δ lead to sufficient variation in Z for given average exposure ε and price p . We can then find, for every ε and p , a δ such that

$p = \tilde{P}(\delta, \varepsilon)$. But then the price can never reveal δ . In contrast, a partially revealing equilibrium is a robust feature of the economy. We can then explore the relative importance of news and exposure for asset prices.

Asset Pricing, Beliefs and Aggregate Exposure

Suppose that agents know the distribution μ , so that individual beliefs agree at $\hat{\delta} = \delta(\mu)$ for all types θ . Since there is a complete set of contingent claims for τ -risk and agents maximize expected utility with common beliefs, standard arguments imply the existence of a representative agent with expected utility preferences and the same belief whose marginal utility serves as a stochastic discount factor.³ Let $v_{FI}(c; \mu)$ denote the felicity of this representative agent. It depends on μ , which governs both beliefs and the distribution of endowments and preferences. Since the parameter $\varepsilon(\mu)$ summarizes the distribution of endowments, the aggregate endowment depends on μ only through $\varepsilon(\mu)$: we write

$$\sum_{\theta \in \Theta} \mu(\theta) \omega_{\tau}(\theta) =: \Omega_{\tau}(\varepsilon(\mu))$$

for the aggregate endowment if the distribution μ is drawn and the event τ occurs.

The equilibrium price with pooled information $P_{FI}(\mu)$ can be read off the representative agent's marginal rate of substitution at the aggregate endowment:

$$\frac{P_{FI}(\mu)}{1 - P_{FI}(\mu)} = \frac{\delta(\mu)}{1 - \delta(\mu)} \frac{v'_{FI}(\Omega_1(\varepsilon(\mu)); \mu)}{v'_{FI}(\Omega_2(\varepsilon(\mu)); \mu)} = \frac{\delta(\mu)}{1 - \delta(\mu)} \exp(-E_{\tau}(\mu)), \quad (5)$$

where $E_1(\mu)$ is a measure of the initial exposure of the representative agent to the risk factor τ , analogously to (3). In general, asset prices depend on both beliefs and aggregate exposure. Other things equal, the relative price of a claim on the event $\tau = \tau_1$ is higher if state 1 is more likely or if the representative agent is less exposed to τ .

Aggregate exposure drives risk premia. Let $\tilde{E}_{\tau}(C, \delta)$ denote the exposure of the representative agent to τ for the consumption plan C and belief δ , defined as in (2). Consider an asset with payoff X contingent on the factor τ . We use X to denote both the payoff vector (that is, X_j is the payoff if $\tau = \tau_j$) and for the corresponding random variable. We can write the value of X in

³See, for example, Duffie (1997), Section 1.4.

state μ as

$$\begin{aligned} V_{FI}(X; \mu) &= P_{FI}(\mu) X_1 + (1 - P_{FI}(\mu)) X_2 \\ &= E[X|\mu] - \text{var}(X|\delta) \tilde{E}_\tau(\Omega(\varepsilon(\mu)), \delta(\mu)) \end{aligned} \quad (6)$$

In the second line, the expectation of X is the expected present value of payoffs because we have normalized the riskless interest rate to zero. The second term is the risk premium: written as the variance of X multiplied by aggregate exposure to X . Risk matters only if the representative agent is exposed to it. Equation (6) offers another way to think about the information revealed by prices. Agents in a standard full information environment agree on the probability of $\tau = \tau_1$, and therefore also on the risk premium contained in the price. In contrast, agents in a world with uncertain exposure are not sure about either.

3 Sharing exposure: a discrete example

In this section, we explore a discrete version of the model to show how uncertain exposure inhibits the sharing of exposure in asset markets. There are two types of agents that differ only on initial exposure: $\Theta = \{\bar{\theta}, \underline{\theta}\}$, and $e_\tau(\bar{\theta}) > e_\tau(\underline{\theta})$. A type distribution is therefore summarized by the fraction of agents with high initial exposure. There are only two possible type distributions. With probability η , nature draws the distribution μ^h with a high number of high exposure agents $\varepsilon^h = \varepsilon(\mu^h)$. With probability $1 - \eta$, nature draws the type distribution is μ^l , with a low fraction of high exposure agents $\varepsilon^l = \varepsilon(\mu^l) < \varepsilon^h$.

Since exposure is iid across agents, an individual agent's initial exposure is a signal about μ . Pooling all agents' information about their own exposure reveals the distribution μ , or, equivalently, the aggregate shock ε . Moreover, since there are only two states, the news δ carried by the type distribution must be perfectly correlated with ε . We use $\delta^j = \delta(\mu^j)$ for $j = h, l$ to denote $\Pr(\tau = \tau_1 | \mu^j)$, the aggregate news about the risk factor τ that is carried by the distribution μ^j . To summarize, an economy is described by

$$\mathcal{E} = (\delta^h, \delta^l, \varepsilon^h, \varepsilon^l, \omega(\bar{\theta}), \omega(\underline{\theta}), u(\cdot; \bar{\theta}), u(\cdot; \underline{\theta}))$$

Since the number of states in $\Delta(\Theta) \times \{1, 2\}$ is finite, REE prices are fully revealing for a generic economy \mathcal{E} . To illustrate the effect of uncertain exposure on trading, we thus construct nongeneric economies that have nonrevealing equilibria. The economic mechanisms that emerge are also relevant in the model of Section 4, where $\Delta(\Theta)$ is uncountable and fully revealing equilibria do not exist. What is special about the example is that, in a nonrevealing equilibrium, the price is constant across distributions μ^j and carries no information at all. Individual beliefs are thus independent of μ^j and depend only on agents' individual types; we write $\hat{\delta}(\theta)$ for the probability that type θ assigns to the event $\tau = \tau_1$. Individual consumption is also independent of μ ; we write $c_\tau(\theta)$, suppressing the dependence on μ .

The following proposition constructs economies that have nonrevealing equilibria and characterizes their properties.

Proposition 3.1.

1. *An economy \mathcal{E} can have at most one nonrevealing REE. In a nonrevealing REE,*
 - (a) *$c(\theta) = \omega(\theta)$ for all θ there is no trade)*
 - (b) *$\hat{\delta}(\bar{\theta}) > \hat{\delta}(\underline{\theta})$ (agents with higher exposure to the risk factor τ are more optimistic about τ – they find it more likely that τ takes on the high value τ_1)*
 - (c) *$\delta^h > \delta^l$ (the aggregate news about the risk factor τ is better when there are more agents with higher exposure to the risk factor τ)*

2. *The following conditions are equivalent:*

- (a) *there exist $\delta^h, \delta^l \in (0, 1)$ such that the economy*

$$\mathcal{E} = (\delta^h, \delta^l, \varepsilon^h, \varepsilon^l, \omega(\bar{\theta}), \omega(\underline{\theta}), u(\cdot; \bar{\theta}), u(\cdot; \underline{\theta}))$$

has a nonrevealing REE.

- (b) *the endowments, felicities and distribution parameters ε satisfy*

$$e_\tau(\bar{\theta}) - e_\tau(\underline{\theta}) \leq \log \left(\frac{\varepsilon^h}{1 - \varepsilon^h} \frac{1 - \varepsilon^l}{\varepsilon^l} \right). \tag{7}$$

The proof of part 1 is straightforward. In a nonrevealing equilibrium, an agent of type θ must have the same beliefs in the two aggregate states h and l . His net demand $c(\theta) - \omega(\bar{\theta})$ must therefore be the same in both aggregate states. Market clearing requires that, for $j = h, l$ and $\tau = 1, 2$,

$$\varepsilon^j (c_\tau(\bar{\theta}) - \omega_\tau(\bar{\theta})) + (1 - \varepsilon^j) (c_\tau(\underline{\theta}) - \omega_\tau(\underline{\theta})) = 0.$$

For fixed τ , this equation can only hold for both $j = h$ and $j = l$ if the net demands $c_\tau(\theta) - \omega_\tau(\theta)$ are zero for both types, which shows part 1.a.

Under our assumptions on felicities, agents' first order conditions must hold in equilibrium. If agents consume their endowments, this means

$$\frac{p}{1-p} = \frac{\hat{\delta}(\theta)}{1 - \hat{\delta}(\theta)} \frac{u'(\omega_1(\theta), \theta)}{u'(\omega_2(\theta), \theta)} = \frac{\hat{\delta}(\theta)}{1 - \hat{\delta}(\theta)} \exp(-e(\theta)). \quad (8)$$

In an autarkic equilibrium, the type with higher initial exposure to the factor τ must be more optimistic that τ takes on the high value τ_1 , than the type with lower exposure, that is $\hat{\delta}(\bar{\theta}) > \hat{\delta}(\underline{\theta})$ (part 1.b). Otherwise, agents with high and low initial exposure cannot both rationalize the observed price.

The existence of an equilibrium requires further that the beliefs in (8) can be derived by Bayesian updating from agents' individual types, which serve as noisy signals of the type distribution. In particular, an agent's subjective probability that $\tau = \tau_1$ must be his conditional expectation of the aggregate news δ , conditional on his type:

$$\begin{aligned} \hat{\delta}(\bar{\theta}) &= \eta \frac{\varepsilon^h}{\bar{\varepsilon}} \delta^h + (1 - \eta) \frac{\varepsilon^l}{\bar{\varepsilon}} \delta^l, \\ \hat{\delta}(\underline{\theta}) &= \eta \frac{1 - \varepsilon^h}{1 - \bar{\varepsilon}} \delta^h + (1 - \eta) \frac{1 - \varepsilon^l}{1 - \bar{\varepsilon}} \delta^l, \end{aligned} \quad (9)$$

where we have defined $\bar{\varepsilon} := \eta \varepsilon^h + (1 - \eta) \varepsilon^l$, the unconditional probability of type $\bar{\theta}$. Since $\varepsilon^h > \varepsilon^l$ and $\hat{\delta}(\bar{\theta}) > \hat{\delta}(\underline{\theta})$, we must have $\delta^h > \delta^l$ (part 1.c). Moreover, given a pair of δ^j s in $(0, 1)$ and hence an economy, the formulas (9) deliver a unique pair of posteriors, and (8) the unique nonrevealing equilibrium price.

Intuitively, existence of a nonrevealing equilibrium requires that the probability δ is higher when more agents have high exposure to the factor τ . High exposure agents then interpret their

type as a signal that $\tau = \tau_1$. Since they are more optimistic about τ , they are happy to consume their endowment at the price p , even though they have higher exposure than other agents (so that gains from trade would exist with symmetric information).

The proof of part 2 is provided in the appendix. It starts from the fact that for given $\hat{\delta}(\theta)$ s, (9) can be viewed as a pair of linear equations in (δ^h, δ^l) with a unique solution. The existence problem then amounts to finding a price such that, if the posteriors satisfy (8), then the solutions (δ^h, δ^l) to (9) are indeed between zero and one. Such a price exists if and only if condition (7) is satisfied. The condition requires that there should not be “too much” heterogeneity in individual exposure, relative to the differences in type distributions across states.

4 Uncertain Exposure and Asset Pricing

In this section, we explore the role of uncertain exposure under a more general distribution of aggregate news and aggregate exposure that gives rise to asset price volatility. This is in contrast to the previous section where news and exposure were perfectly correlated, and the price was constant. We retain the assumption that the number of agents with high initial exposure is either ε^h or ε^l , where the former occurs with probability η . However, the parameter $\delta(\mu) = \Pr(\tau = 1|\mu)$, the news contained in agents’ pooled information, can now vary over the whole unit interval. The distribution of δ conditional on ε is described by a pair of continuous strictly positive densities $f(\delta|\varepsilon^j)$ on $[0, 1]$.

We assume that the pooled information is reflected in private signals observed by agents. In particular, every agent receives a private signal $s(\theta) \in \{s_1, s_2\}$ about the event $\tau = \tau_1$. The signals are iid across agents, and independent of initial exposure. The probability of receiving a “good” signal s_1 is $\delta(\mu)$. By the law of large numbers, a fraction δ of agents thus receive a good signal, while a fraction $1 - \delta$ receive a bad signal. As a result, the value of $\delta(\mu)$ can be recovered if all signals are pooled. In contrast to the previous section, an agent’s type θ is now not only identified with an initial exposure $e_\tau(\theta)$, but also with a signal $s(\theta)$. To simplify the notation below, we will simply write $\theta = (s, e)$.

Preferences are restricted to the Linear Risk Tolerance (LRT) class, with marginal risk tol-

erance equal across agents. Felicities are

$$u(c; \theta) = \begin{cases} \frac{\sigma}{\sigma-1} (\alpha(\theta) + \sigma c)^{1-\frac{1}{\sigma}} & \text{if } \sigma \notin \{0, 1\} \\ \log(\alpha(\theta) + c) & \text{if } \sigma = 1 \\ -\alpha(\theta) \exp(-c/\alpha(\theta)) & \text{if } \sigma = 0 \text{ and } \alpha(\theta) > 0 \end{cases}$$

The common denominator of these preferences is that risk tolerance $-u'/u''$ (the inverse of the coefficient of absolute risk aversion) is given by the linear function $\alpha(\theta) + \sigma c$. Important special cases of LRT preferences are CRRA utility ($\alpha(\theta) = 0$, with $1/\sigma > 0$ the coefficient of relative risk aversion), CARA utility ($\sigma = 0$, with $\alpha(\theta) > 0$ the coefficient of absolute risk aversion), and quadratic utility ($\sigma = -1$). We require that the coefficient of marginal risk tolerance σ be equal across agents. However, there can be differences in risk attitude independent of income that are captured by differences in $\alpha(\theta)$. For the case $\sigma > 0$, an intuitive way to think about the coefficient $\alpha(\theta)$ in our context is as a riskless endowment that cannot be traded away.

Full Information Benchmark and Non-Revelation

A convenient feature of the LRT family of preferences is that the price function $P_{FI}(\mu) = \tilde{P}_{FI}(\delta(\mu), \varepsilon(\mu))$ in the full information case is available in closed form. Indeed, there exists a representative agent who has an LRT felicity function with the same coefficient of marginal risk tolerance σ as the individual agents and the average coefficient α :

$$\sum_{\theta \in \Theta} \mu(\theta) \alpha(\theta) =: \bar{\alpha}(\varepsilon(\mu)).$$

Here $\bar{\alpha}$ is well defined as a function of ε only, because we have assumed that types with the same exposure have the same felicity function.

For $\sigma \neq 0$, the full information price (5) can now be written as

$$\frac{\tilde{P}_{FI}(\delta, \varepsilon)}{1 - \tilde{P}_{FI}(\delta, \varepsilon)} = \frac{\delta}{1 - \delta} \left(\frac{\bar{\alpha}(\varepsilon) + \sigma \Omega_1(\varepsilon)}{\bar{\alpha}(\varepsilon) + \sigma \Omega_2(\varepsilon)} \right)^{-\frac{1}{\sigma}} = \frac{\delta}{1 - \delta} \exp(-E_1(\varepsilon)). \quad (10)$$

With LRT utilities, aggregate exposure E_1 to the risk factor τ thus depends on the type distribution μ only via the fraction of agents who have high initial exposure to the risk factor τ ; the beliefs δ do not matter. Moreover, it follows from (10) that aggregate exposure is strictly increasing in ε . In other words, the parameter ε can be taken as a measure of aggregate exposure.

The formula (10) also implies that there cannot be a fully revealing REE in which agents learn the relevant pooled information δ . Indeed, suppose there is such a fully revealing equilibrium. Since agents' beliefs agree at δ , the REE price function must be \tilde{P}_{FI} . Moreover, agents must have been able to arrive at the belief δ based on their information, that is, their own types and the price. Since an individual's type θ alone does not reveal δ , agents must have been able to invert the price function \tilde{P}_{FI} to infer δ . However, (10) implies that for every price $p \in (0, 1)$, there exist $\delta^h, \delta^l \in (0, 1)$ with $\delta^h \neq \delta^l$ such that

$$\tilde{P}_{FI}(\delta^h, \varepsilon^h) = \tilde{P}_{FI}(\delta^l, \varepsilon^l) = p$$

In other words, agents who see the price p (and their own type) cannot know whether the relevant information carried by the distribution of types is δ^h or δ^l . Since the densities f^j are continuous and strictly positive on $[0, 1]$, Bayes' Rule says that agents place positive probability on both distributions, contradicting the fact that their beliefs agree at a single number δ . It follows that P_{FI} cannot be an equilibrium price function when there is asymmetric information.

Beliefs in Partially Revealing Equilibrium

In the simple example of the previous section, the price is constant and conveys no information, so that individual beliefs depend only on individual types. In this section, the price will generally convey some information. To describe inference from prices, it is helpful to define, for $j = h, l$, functions $\tilde{P}^j : [0, 1] \rightarrow [0, 1]$ by $\tilde{P}^j(\delta) := \tilde{P}(\delta, \varepsilon^j)$. We focus on partially revealing equilibria in which the \tilde{P}_j are continuous and strictly increasing in δ . In this case, there are well-defined inverse functions defined by $\delta_j(p) = \tilde{P}_j^{-1}(p)$.

Consider the inference problem of a type θ agent. His posterior probability $\hat{\delta}(\theta, p)$ of the event $\tau = \tau_1$ is his posterior mean of the aggregate news δ . If he observes a price p , knowledge of the price function tells him that the distribution μ is parametrized either by $(\delta_h(p), \varepsilon^h)$ or by $(\delta_l(p), \varepsilon^l)$. We denote the two distributions by $\mu_h(\cdot; p)$ and $\mu_l(\cdot; p)$, respectively. The probability that $\mu = \mu_h(\cdot; p)$ based on the price alone is

$$\eta_p = \frac{\eta \delta'_h(p) f(\delta_h(p); \varepsilon^h)}{\eta \delta'_h(p) f(\delta_h(p); \varepsilon^h) + (1 - \eta) \delta'_l(p) f(\delta_l(p); \varepsilon^l)}. \quad (11)$$

In addition to the price, the agent observes his type, which is also a signal about the distribution μ . Let $\hat{\eta}(\theta, p)$ denote the posterior probability that a type θ agent observing a price p assigns to $\mu = \mu_h(\cdot|p)$. It is given by

$$\hat{\eta}(\theta, p) = \frac{\eta_p \mu_h(\theta; p)}{\eta_p \mu_h(\theta; p) + (1 - \eta_p) \mu_l(\theta; p)}.$$

An agent believes that the distribution μ_h is more likely if his own type θ is more likely to be drawn from μ_h relative to μ_l . For example, a high exposure agent believes that it is more likely that there are many high exposure agents. His posterior probability of the event $\tau = \tau_1$ is then

$$\hat{\delta}(\theta, p) = \hat{\eta}(\theta, p) \delta^h(p) + (1 - \hat{\eta}(\theta, p)) \delta^l(p).$$

The following proposition establishes basic properties of beliefs and the news contains in prices that are similar to those for the discrete example in Proposition 3.1.

Proposition 4.1. Consider a nonrevealing equilibrium with a price function \tilde{P} that is continuous and increasing in δ . Then

1. Individual beliefs are ranked by
 - (a) $\hat{\delta}(s, \bar{e}, p) > \hat{\delta}(s, \underline{e}, p)$ (for a given signal, agents with higher exposure to the risk factor τ believe that τ is more likely to be high)
 - (b) $\hat{\delta}(s_1, e, p) > \hat{\delta}(s_2, e, p)$ (for given exposure, agents with a better signal about the risk factor τ believe that τ is more likely to be high).
2. $\delta^h(p) > \delta^l(p)$ (holding fixed the price, aggregate news about the risk factor τ is better when more agents have high exposure to τ).

Part 1.a of the proposition says that agents with higher exposure to the risk factor τ more optimistic about the factor, that is, they believe that τ is more likely to take on the high value $\tau_1 > \tau_2$. Intuitively, high exposure agents find it more likely that other agents also have high exposure, which would lower the price. As a result, they extract from any given price a more

favorable signal about $\tau = \tau_1$ than do low exposure agents. Part 1.b simply says that, in a nonrevealing equilibrium, agents' beliefs also respond to their signals. This is because the signals add information over and above that contained in the price.

The ranking of beliefs translates directly into differences in trading behavior across agents. Indeed, compared to the full information case, equilibrium disagreement has two effects on trading volume that work in opposite directions. On the one hand, Part 1.b.says that agents with the same endowments and preferences will choose different portfolios: agents speculate based on their private signals. This effect tends to increase trading volume.

On the other hand, agents with different initial exposures to τ will not equate their exposures as they would in a full information equilibrium. Instead, precisely those agents who start with more claims that covary with τ , or who are more risk averse, end up more optimistic and hold on to their exposure. They do not receive as much insurance from agents with a lot of riskless claims or are less risk averse, because the latter are pessimistic about claims on τ . This effect tends to lower trading volume relative to the full information case.

Part 2 of the proposition says that the news about $\tau = \tau_1$ given the price and the number of agents with high initial exposure ε must be better if ε is higher. In the example of the previous section, this property held unconditionally and was assumed exogenously – it was a result of our “reverse engineering” a discrete economy with a nonrevealing equilibrium. Here, in contrast, it is a property of the endogenous price function and holds only conditional on the price. Unconditionally, δ and ε may or may not be negatively correlated, depending on the properties of the densities f . Part 2 also means that the price function \tilde{P} is decreasing in ε . As in the full information case, higher aggregate exposure thus lowers prices.

Exposure Shocks and Asset Prices

Asset price behavior in a partially revealing equilibrium is best explained in the case of log utility. The proof of Proposition 4.1 shows that, in the log case, the price can be represented by

$$\frac{\tilde{P}(\delta, \varepsilon)}{1 - \tilde{P}(\delta, \varepsilon)} = \frac{\bar{\delta}_2(\delta, \varepsilon)}{1 - \bar{\delta}_1(\delta, \varepsilon)} \exp(-E_1(\varepsilon)), \quad (12)$$

where $\bar{\delta}_\tau$ is an average of individual agents' beliefs $\hat{\delta}(\theta, p)$ weighted by agents' endowments in the event τ , namely

$$\bar{\delta}_\tau(\delta, \varepsilon) = \sum_{\theta} \frac{\omega_\tau(\theta)}{\Omega_\tau(\varepsilon)} \hat{\delta}(\theta, p(\delta, \varepsilon)) \mu(\theta; \delta, \varepsilon).$$

The equation above shows that the averages $\bar{\delta}_\tau$ depend on ε and δ as these two variables determine the distribution of endowments and signals.

The functional form is thus very similar to the full information case; in fact, if all agents agree on δ , we are back to (10). The key difference is in the impact of the “true” unobservable news δ on the price. In the full information case, a change in δ directly changes the price. With asymmetric information, δ affects the price only to the extent that it shifts the mean of the (now nondegenerate) distribution of individual beliefs. Since agents’ signals of δ are imperfect, the sensitivity of price to δ is typically smaller than in the full information case. As the opposite extreme from full information, consider the case where δ and ε are independent and individual signals are uninformative, i.e. $s_1 = s_2$. In this case, all agents would agree at the prior η . The news δ then has no effect, but the effect of ε on the price is the same as in the full information case.

The bottom line is that, with asymmetric information, aggregate news tends to matter less, whereas aggregate exposure tends to matter more for prices. The following proposition shows that this general intuition holds for the whole LRT class.

Proposition 4.2 Consider a nonrevealing equilibrium with price function \tilde{P} that is continuous and strictly increasing in δ . The equilibrium price depends more strongly on aggregate exposure than in the full information case: for every $\delta \in (0, 1)$,

$$\tilde{P}(\delta, \varepsilon^l) > \tilde{P}_{FI}(\delta, \varepsilon^l) > \tilde{P}_{FI}(\delta, \varepsilon^h) > \tilde{P}(\delta, \varepsilon^h).$$

Interpretation: risk premia and trading in financial crises

While our model is stylized, the basic mechanism – changes in aggregate exposure matter more with asymmetric information – may help understand the behavior of asset prices and asset trading in the recent US financial crisis. It has been widely reported that the market for mortgage-backed securities “seized up”, and that the trades that did take place were done at “fire

sale prices”. One explanation discussed in the press runs as follows. Securities are heterogeneous and there is a lemons problem: market participants cannot tell the good securities from the bad. As a result only bad securities are traded, at low prices. This story assumes that securities are valued at their expected payoffs. The price of traded securities does not appear low to an agent who knows their expected payoff. In other words, there is not an unusually high risk premium.

Our model points to a different mechanism that also generates low trading activity at low sale prices. However, it works via low risk premia driven by uncertain exposure to aggregate risk. It does not rely on idiosyncratic risk in mortgage backed securities. Instead, we think of two standardized securities. ”Top-rated” mortgage backed securities are riskless claims. ”Junk” securities are bets on a risk factor τ which affects repayment on mortgages, such as house prices or economic activity. The agents in the model can be thought of as financial institutions, with risk aversion taken to be a stand-in for some imperfection (for example, risk aversion of undiversified managers, or an upward sloping cost of external finance).

Banks’ exposure to mortgage risk depends on how many top-rated versus junk securities they have in their portfolios. Consider first an initial scenario, which may capture the situation before summer 2007. There are relatively few junk securities, and the exposure of the financial system to the risk factor is perceived to be small. Formally the number of banks with high initial exposure is at the low value ε^l and everybody knows this. Banks are therefore able to efficiently spread around exposure among themselves; for example banks that have originated subprime mortgages and have a high initial exposure to τ are able to package them into junk securities and sell them to other banks at relatively low risk premia reflected in the price $\tilde{P}_{FI}(\delta, \varepsilon^l)$.

Consider next a second scenario where the risk assessment of some top-rated securities has changed. Suddenly, many securities that were previously considered top-rated are no longer considered riskless. In terms of the model, assume that we move to a situation where some top-rated securities held by some banks are converted to junk. We thus consider the comparative static whereby the number of banks with high initial exposure increases to the high value ε^h . If the new distribution of exposure were known then the shock would lower the prices to $\tilde{P}_{FI}(\delta, \varepsilon^h)$ as risk premia increase, but should also lead to efficient sharing of exposure as banks who had a lot of top-rated securities turn to junk sell some of their junk to other, less exposed banks.

Assume now, however, that exposures are uncertain: nobody knows precisely which banks and how many banks altogether have become more exposed. Formally, we consider the asymmetric information economy, where agents do not know whether the aggregate exposure is ε^l or ε^h (but the true exposure is ε^h). In addition, they do not know the true distribution δ of the risk factor τ that governs mortgage losses. Banks only see their own exposure, but need to estimate the aggregate exposure of the whole financial system, as well as the expected losses. Since banks know that everyone used similar risk assessment tools in the past, they take their own exposure as a signal of aggregate exposure. In addition, they observe the low price $\tilde{P}(\delta, \varepsilon^h)$.

Banks with high exposure believe that many other banks are similarly exposed. They therefore perceive the low price as largely due to an increase in aggregate exposure, rather than an increase in default probabilities (lower δ). In contrast, banks with low exposure believe that the overall exposure of the financial system is low. They conclude that the low price must be reflecting higher default probabilities. As a result, they hesitate in purchasing securities from the high exposure banks, who hold on to their exposures.

At the same time, from the perspective of an observer, the price $\tilde{P}(\delta, \varepsilon^h)$ – and therefore the price on any security that loads on the factor τ – looks “too low”, like a “fire sale price”. Indeed, consider an observer who has a good estimate of the actual expected payoff on the junk securities, or equivalently the true δ , and who suspects that exposure might have increased. This observer knows that higher exposure will imply a higher risk premium and hence a lower price. From experience, he knows the size of risk premia in times when banks know aggregate exposure. He can thus compute the price that would obtain, in his experience, in the “worst case” state for aggregate exposure, namely $\tilde{P}_{FI}(\delta, \varepsilon^h)$. Comparing this price to the observed price $\tilde{P}(\delta, \varepsilon^h)$, the observer will then be puzzled to find that the market price on assets that depend on τ is even lower, and their risk premium even higher.

5 Asset prices with log utility

In the section we use the setup of the previous section, but we restrict attention to logarithmic utility. Subsection 5.1 introduces an example where a partially revealing equilibrium can be found

$\omega_1(\bar{e})$	$\omega_2(\bar{e})$	$\omega_1(\underline{e})$	$\omega_2(\underline{e})$	$E_1(\epsilon_l)$	$E_1(\epsilon_h)$
2	1	1	1	0.1	0.65

Table 1: Baseline parameterization

using numerical techniques. Subsection 5.2 explores what would happen if an econometrician ran standard predictability regressions from data generated by the example economy. Subsection 5.3. considers an econometrician who knows that agents have log utility, but tests the Euler equation of the representative agent while ignoring the presence of asymmetric information.

5.1 A numerical example

Consider an economy where all agents have log utility. Table 1 describes the parameters used in the baseline of endowments and information. The endowment of state 2 goods is the same for the two agents and is normalized to 1. Agents with high exposure have twice as much state 1 goods as agents with low exposure. Individual preferences are represented by a logarithmic felicity functions. As a result, the values for aggregate exposure imply that the fraction of highly exposed agents is 10.5% (91.6%) when aggregate exposure is low (high). It is assumed that δ is drawn from a uniform distribution with support $[0, 1]$ and that the random variables δ and ϵ are independent.⁴

Figure 1 compares the equilibrium price of a contingent claim that pays one unit of the consumption good in state $\tau = 1$ and zero otherwise. The graph presents prices for the full and asymmetric information cases. The horizontal axis measures the continuous news shock δ . For every information structure, there are two lines, the price functions given low and high aggregate exposure, $\tilde{P}(\cdot, \epsilon^l)$ and $\tilde{P}(\cdot, \epsilon^h)$, respectively. The price functions found by our algorithm satisfy the strict monotonicity properties in terms of δ . As predicted by Proposition 4.1, the price is

⁴The model is solved using Chebychev collocation. The code approximate the functions $P(\delta; \epsilon_l)$ and $P(\delta; \epsilon_h)$ as the weighted sum of fifteen Chebychev polynomials. The equilibrium is defined by the solution of a fixed point problem.

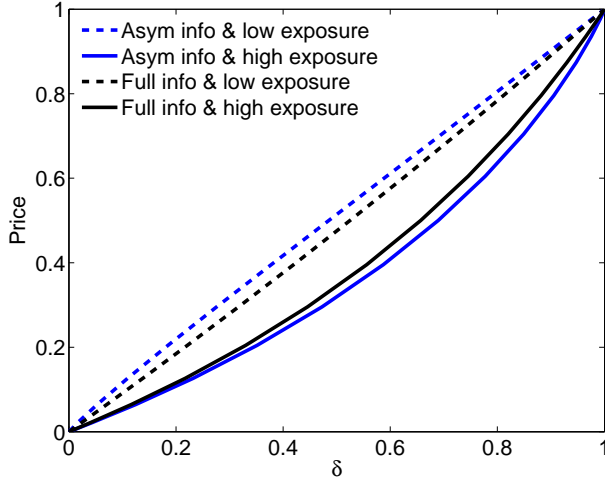


Figure 1: Price functions $P(\delta, \epsilon)$ in the economies with full and asymmetric information.

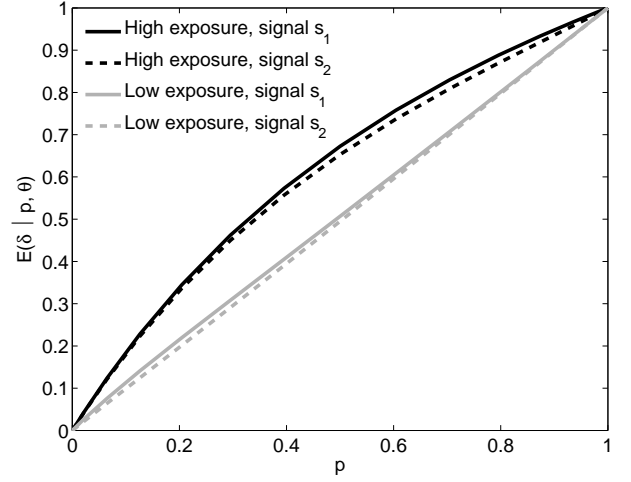


Figure 2: Distribution of beliefs in the economy with asymmetric information.

decreasing in ϵ for fixed δ . Moreover, as predicted by Proposition 4.2, a shift in exposure has a larger effect in the economy with asymmetric information.

Figure 2 describes the distribution of beliefs across types. As shown in proposition 4.1, the beliefs of high exposure types lie above the beliefs of low exposure types. Similarly, the beliefs of agents with signals favorable to state $\tau = \tau_1$ lie above the beliefs of agents with signals favorable to $\tau = \tau_2$.

In the economy with full information, agents with high exposure to state 1 reduce their exposure by trading away some of their state 1 goods in exchange for state 2 goods. There is no dispersion of beliefs, which means that all agents with the same exposure display the same trading strategies. The economy with asymmetric information introduces a “speculative” motive for trading: agents have different assessments about the probability of state 1. Conditional on other individual characteristics, more optimistic agents exhibit a higher net demand for state 1 contingent claims.

Figure 3 shows the effect of information on trading volume. In the asymmetric information economy, the correlation between individual exposure and beliefs implies that there are less sharing of exposure. Agents with high exposure are optimistic about state 1, which moderates

their desire to reduce their exposure that state.

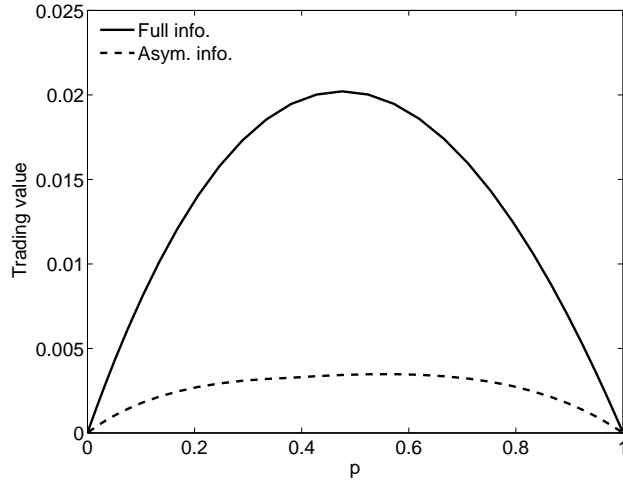


Figure 3: Aggregate trading value of contingent claims paying when $\tau = 1$.

5.2 Predictability of excess returns

Several authors have documented that there appears to be a sizable forecastable component of stock returns, especially at longer horizons (see for example Campbell and Shiller (1988a), Campbell and Shiller (1988b), and Fama and French (1988)). After controlling for dividends, earnings or book value, higher current stock prices tend to be followed by low stock returns. Different studies have analyzed how this finding can be reconciled with cross-time variations in the level of stock market risk or the price of risk demanded by market participants (see Campbell (2003)). The idea is that either an increase in the volatility of future returns or an increase in the price of risk would induce an increase in expected stock returns and require a downward adjustment in current stock prices.

We consider how the information structure affects the predictability of the excess return on a contingent claim that pays one unit when $\tau = \tau_1$ and zero otherwise, that is,

$$r^e = 1_{\{\tau=\tau_1\}} - p.$$

Suppose an econometrician sees many realizations of excess returns and prices generated from

the model. On this data, he runs a regression of x on the price of the contingent claim:

$$r^e = \alpha + \beta p + \nu,$$

where ν is an error term. In our two state setting, the value of β is independent of the asset used to run the regression. If the econometrician used the contingent claim paying when $\tau = 2$ the only coefficient that would change is α . Similarly, the econometrician could use an asset with some other payoff contingent on τ .

	Asym. info.	Full info.
α	0.073	0.069
β	-0.031	-0.017
R^2	0.105	0.033

Table 2: Regressions of excess returns on prices

Table 2 presents summary statistics of the regression coefficients that the econometrician would obtain if the data were generated in an economy with asymmetric information or with full information. It shows that there is a —linear—negative relationship in both cases, between excess returns and prices becomes more pronounced in the economy with asymmetric information. In addition, the R-squared is higher in the economy with asymmetric information.

The intuition for the result is as follows. The predictability of excess returns requires shocks that affect the price without affecting the conditional expectation if the payoff conditional on the price. In our setup, shocks to aggregate exposure (ϵ) serve this purpose. Exposure shocks are present in both economies. However, Proposition 4.2. says that they have a bigger effect on prices in the economy with asymmetric information. Figure 1 shows that the larger sensitivity of prices to aggregate exposure shocks is more pronounced at “intermediate” price values. At these prices there is more uncertainty about the actual values of δ and thus, the discrepancy between agents’ beliefs and δ become larger.

Figure 4 further illustrates this point. It plots the derivative of the expected excess return

$E[\delta|p] - p$ with respect to the price in the economies with asymmetric and full information. The figure shows that there is an intermediate range of price values where the sensitivity of expected excess returns is lower in the asymmetric information case.

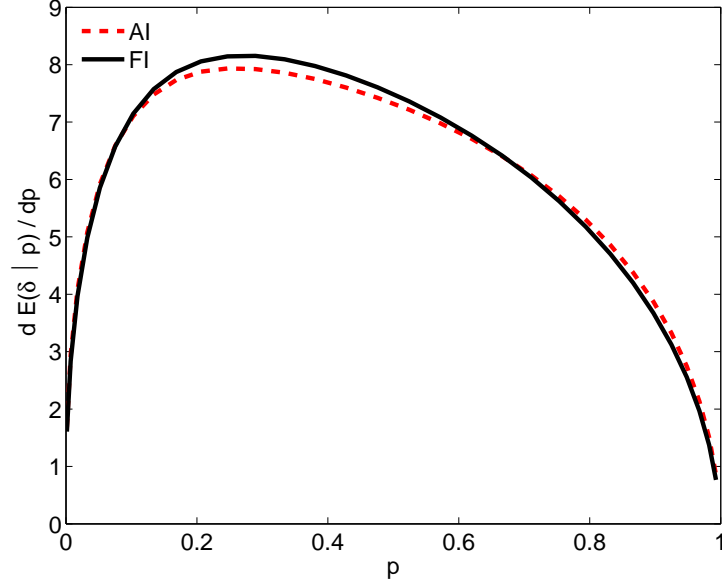


Figure 4: Sensitivity of expected returns to price changes in the economies with full and asymmetric information. Measured by the derivative $\frac{\partial E(\delta|p)}{\partial p}$.

The presence of return predictability suggests an exploitable portfolio strategy based on public information: at low prices, borrow at the riskless rate and buy claims on τ , at high prices, sell short claims on τ and invest in the riskless asset. It is interesting to ask why the rational agents in the model do not exploit this strategy. The first order condition of type θ implies that the expected excess return perceived by type θ must be equal to the risk premium perceived by type θ . We can therefore write the expected excess return conditional on public information – namely, the price – as

$$E(x|p) - p_x = E[x|p] - E[x|\theta] + var(x|\theta) \tilde{e}_\tau(\theta, c(\theta), \hat{\delta}(\theta, p)) \quad (13)$$

Here the difference in expectations – the first term on the right hand side – is a measure of pessimism of a type θ agent. The second term is the subjective risk premium of a type θ agent

– his subjective variance of x multiplied by his exposure. Consider now a low price, where the expected excess return is positive. This can be consistent with optimal behavior for two reasons: the agent can be more pessimistic than what one would be on the basis of public information, or he could demand a risk premium because of his exposure.

In the equilibrium of our model, *both* reasons are at work. Consider a contingent claim that pays in the event $\tau = \tau_1$. High exposure types demand a high risk premium to hold these securities as their consumption level is more volatile and correlated to the payoff of these assets. Yet they are also optimistic about event $\tau = \tau_1$, which deters them from selling too many of these contingent claims. For low exposure types, the consumption allocation is less sensitive to the realization of τ or it is negatively correlated with the payoff of this contingent claim. This implies that they demand a lower risk premium. At the same time, they they are more pessimistic. This deters them from buying too many of these contingent claims.

5.3 Representative agent asset pricing

In this section we consider an econometrician who studies the Euler equation of a representative agent. The motivation comes from results in the asset pricing literature that suggest high and time varying risk aversion (that is, higher risk aversion when asset prices are low) can reconcile representative agent models with the data. We ask whether the presence of asymmetric information can help understand these results.

Suppose that the risk factor τ represents the aggregate stock market. An econometrician assumes that the data generating process comes from a representative agent model with log utility. He observes the joint distribution of (τ, Ω, p) : stock payoffs, aggregate consumption and the price. The econometrician does not know a priori the information structure of the agent. He is aware of this, and therefore estimates the model by maximum likelihood, allowing for prices to depend on signals about future aggregate consumption and stock payoffs that agents receive at date 1.

An unrestricted estimation will recover true joint distribution, summarized by the number η , the distribution of δ and the price function. In particular, the econometrician will find that

movements in ε (i.e. changes in aggregate consumption that are not in stock payoffs) are reflected in the price. He infers from this that the representative agent receives a signal that reveals ε .

However, when the econometrician imposes the cross equation restrictions implied by log preferences, he will reject the model. Satisfying the cross equation restrictions would require that, for all p ,

$$\frac{p}{1-p} = \frac{\delta^j(p)}{1-\delta^j(p)} \frac{\Omega_2(\varepsilon^j)}{\Omega_1(\varepsilon^j)} \quad \text{for } j = h, l.$$

Proposition 4.2 implies that this condition is violated.

To fix this problem, the econometrician can introduce preference shocks to fit the data exactly using his representative agent model. We capture the preference shock by specifying subjective beliefs $\tilde{\delta}^j(p)$ which depend on the price as well as on the state j . The econometrician thus determines $\tilde{\delta}^j(p)$ such that

$$\frac{p}{1-p} = \frac{\tilde{\delta}^j(p)}{1-\tilde{\delta}^j(p)} \frac{\Omega_2(\varepsilon^j)}{\Omega_1(\varepsilon^j)} \quad \text{for } j = h, l.$$

Proposition 4.2 now implies that $\tilde{\delta}^h(p) > \delta^h(p)$ and $\tilde{\delta}^l(p) < \delta^l(p)$. In other words, the econometrician's model will make agents more optimistic about τ in times of high aggregate exposure to τ , and more pessimistic in times of low aggregate exposure. Since the price is decreasing in exposure, the econometrician has thus introduced a force that induces additional pessimism at low prices and optimism at high prices.

Of course, the price also depends on δ , so we do not yet know whether the econometrician will conclude that the agent is pessimistic on average. The following proposition considers the econometrician's belief conditional on the price. It shows that the econometrician concludes agents are pessimistic at a price p if aggregate wealth $W(\varepsilon, p) = p\Omega_1(\varepsilon) + (1-p)\Omega_2(\varepsilon)$ is positively correlated with aggregate exposure conditional on the price.

Proposition 5.1 With log utility, the econometrician's belief is more pessimistic conditional on the price if and only if

$$W(\varepsilon^l, p) > W(\varepsilon^h, p),$$

that is, there is more wealth in states with less aggregate exposure.

The condition in Proposition 5.1 depends on the endogenous price p . We have $W(\varepsilon^l, p) > W(\varepsilon^h, p)$ if and only if

$$p(\Omega_1(\varepsilon^l) - \Omega_1(\varepsilon^h)) + (1-p)(\Omega_2(\varepsilon^l) - \Omega_2(\varepsilon^h)) > 0 \quad (14)$$

If moreover the aggregate endowment vectors are clearly ranked

$$\Omega_j(\varepsilon^l) > \Omega_j(\varepsilon^h), \quad j = 1, 2 \quad (15)$$

then (14) holds for all values of $p \in (0, 1)$. Therefore, the econometrician will conclude that the agent is more pessimistic than what the data warrants given any price.

To sum up, suppose data are generated by an economy with log investors with rational expectations, where (15) holds. An econometrician who observes the data and studies the Euler equation of a log representative agent will reject the model. In particular, he will conclude that the agent is “too pessimistic”. In other words, he will discover an “equity premium puzzle” – exogenously assumed pessimism has the same effect on unconditional moments as higher risk aversion. Moreover, he will discover a force that increases risk aversion at low prices and vice versa. These findings do not reflect preferences with high or time and risk aversion, but instead the econometrician’s mistaken assumption that agents have symmetric information, so that standard representative agent analysis applies.

6 Exponential utility and normal distributions

In this section we consider a version of the model where agents have exponential utility and face normally distributed shocks. We show that several properties of asset prices that we have found in our two state setup extend to this environment. The tradable aggregate risk factor τ realized at date 2 is now

$$\tau = \delta + w,$$

where δ and w are independent and normally distributed with mean zero and variances $1/\pi_\delta$ and $1/\pi_w$, respectively.

Consumption set, types and exposure

Agents choose normally distributed consumption plans from the set

$$C = \{c : \text{there are } a_c, b_c \in \mathbb{R} \text{ s.t. } c = a_c + b_c\tau\}$$

Every feasible consumption plan is identified with a pair of coefficients, namely mean consumption a_c and the “loading” b_c on the tradable risk factor τ . Agents trade contingent claims on τ . Two assets are sufficient to span the consumption set C . For example, a riskless asset and an asset with payoff τ would work. The value of a consumption bundle in C is $P(c) = a_c + b_cp$, where p is a parameter of the price function. As in our two-state example in earlier sections, finding the equilibrium price in for a given aggregate state boils down to finding one number p . Here it can be interpreted as the relative price of a the claim on the factor relative to the price of a riskless asset.

The endowment of a type θ agent is the random variable $\omega(\theta) = a_\omega + b_\omega(\theta)\tau$. The agent’s type determines both the loading $b_\omega(\theta)$ of his endowment on the risk factor τ , and a signal $s(\theta)$, which satisfy

$$\begin{aligned}\bar{b}(\theta) &= \varepsilon + v(\theta) \\ s(\theta) &= \delta + u(\theta)\end{aligned}$$

where $v(\theta)$ and $u(\theta)$ are normally distributed with mean zero and variances $1/\pi_v$ and $1/\pi_u$, respectively, and are both independent of all other random variables. The aggregate news shock and aggregate exposure are denote δ and ε , as in the previous sections.

The preferences of all types are represented by exponential utility with coefficient of absolute risk aversion coefficient ρ , that is, $u(c) = -\exp(-\rho c)$. Assume that the belief of a type θ agent about τ can be represented by a normal density \hat{f} . The exposure to the risk factor τ of a type θ agent with consumption plan $c = a_c + b_c\tau$ and belief \hat{f} is defined as in (2), with moments evaluated using the density \hat{f} . It simplifies to

$$\tilde{e}_\tau(\theta, c, \hat{f}) = -\frac{\text{cov}(\tau, u'(c))}{\text{var}(\tau|I(\theta)) E[u'(c)]} = b_c\rho,$$

and thus does not depend on \hat{f} .

As in the two-state example, exposure is positive if and only if the consumption plan is positively correlated with τ , and exposure is zero if the consumption plan is independent of τ or the agent is risk neutral. The initial exposure of type θ , defined using some arbitrary normal reference density f , is $\tilde{e}_\tau(\theta, \omega, f) = \bar{b}(\theta) \rho$.⁵ Since preferences are identical LRT preferences, there is a representative agent with the same utility function. The aggregate endowment is $\Omega(\varepsilon) = \bar{a}_\omega + \varepsilon\tau$, so aggregate exposure is given by $\tilde{e}_\tau(\theta, \Omega(\varepsilon), f) = \varepsilon\rho$.

Agent's problem

The agent solves

$$\begin{aligned} & \max_{c \in \mathcal{C}} -E[\exp(-\rho c) | I(\theta)] \\ \text{s.t. } & P(c) = P(\omega). \end{aligned}$$

Using the fact that a consumption plan can be represented as $c = a + b\tau$, and using the properties of normal distributions, the agent's problem simplifies to a linear quadratic problem in the coefficients:

$$\begin{aligned} & \max_{a,b} \{ \rho a_c + \rho b_c E[\tau | I(\theta)] - \frac{1}{2} \rho^2 b_c^2 \text{var}(\tau | \theta) \} \\ \text{s.t. } & a_c + b_c p = a_\omega(\theta) + b_\omega(\theta) p \end{aligned}$$

The coefficients of the optimal consumption bundle are then

$$\begin{aligned} b_c(\theta) &= \frac{E[\tau | I(\theta)] - p}{\rho \text{var}(\tau | I(\theta))} \\ a_c(\theta) &= a_\omega(\theta) + b_\omega(\theta) p - p \frac{E[\tau | I(\theta)] - p}{\rho \text{var}(\tau | I(\theta))} \end{aligned}$$

The agent will load more on the factor if the expected excess return on the factor is higher, and when risk and risk aversion is lower. The endowment does not matter for the loading on the factor, or the agent's choice of risky assets. However, riskless claims are chosen so as to satisfy the budget constraint.

⁵Since the exposure measure does not depend on beliefs, there is no need for a second measure such as e_τ that we used in the 2 state example above.

Equilibrium

Market clearing requires $\int c(\theta) d\theta = \int \omega(\theta) d\theta$, or, in terms of coefficients,

$$\begin{aligned}\int a_c(\theta) d\theta &= \int a_\omega(\theta) d\theta \\ \int b_c(\theta) d\theta &= \int b_\omega(\theta) d\theta\end{aligned}$$

The first equation can be thought of as market clearing for riskless claims, and the second equation as market clearing for claims with payoff τ . Walras' law holds for these two assets: if one equation holds, and the budget constraints, too, then the other market clears as well.

An equilibrium again consists of a price function $\tilde{P}(\delta, \varepsilon)$ together with a consumption allocation, such that consumers update their beliefs using Bayes' rule and choose consumption optimally given beliefs, and market clear. The following proposition shows that the main features of price movement and belief formation emphasized earlier carry over to the exponential/normal case.

Proposition 6.1. In the exponential/normal economy,

1. there exists an equilibrium with a linear price function

$$\tilde{P}(\delta, \varepsilon) = \alpha + \beta\delta + \gamma\varepsilon.$$

2. the price function is increasing in the news δ (that is, $\beta > 0$) and decreasing in aggregate exposure ε (that is, $\gamma < 0$)
3. in equilibrium, agents with higher initial exposure are more optimistic τ , that is, the conditional expectations $E[\tau|\theta, p]$ is higher if the endowment loads more on τ ($b_\omega(\theta)$ is higher).

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7 Appendix

Proof of Proposition 3.1, part 2.

We need to establish the existence of (δ^h, δ^l) and a price p such that a price function that is constant at p together with the autarkic allocation constitute an equilibrium in the economy parametrized by the δ s.

Given our assumptions on utility, it is optimal for agents to consume their endowment at the price p and for belief $\hat{\delta}(\theta)$ if and only if the first order conditions

$$\frac{\hat{\delta}(\theta)}{1 - \hat{\delta}(\theta)} \exp(-e_1(\theta)) = \frac{p}{1 - p}.$$

hold for every θ . In other words, equilibrium posteriors must be

$$\hat{\delta}(\theta) = \left(1 + \frac{1-p}{p} \exp(-e_1(\theta)) \right)^{-1}. \quad (16)$$

We are done if we can show that there exist (δ^h, δ^l) and p such that the posteriors $\hat{\delta}(\theta)$ not only satisfy (16), but are also derived from agents' individual types by Bayes' Rule. If this is true, then the $\hat{\delta}(\theta)$ are also posteriors given a constant, and hence uninformative, price function. (16) thus says that the autarkic allocation is optimal in every state given the price p . Finally, markets clear in all states if each consumer chooses his endowment.

Consider agents' updating given their individual type. Bayes' Rule says

$$\begin{aligned} \hat{\delta}(\bar{\theta}) &= \frac{\eta \varepsilon^h \delta^h + (1 - \eta) \varepsilon^l \delta^l}{\eta \varepsilon^h + (1 - \eta) \varepsilon^l} = \eta \frac{\varepsilon^h}{\bar{\varepsilon}} \delta^h + (1 - \eta) \frac{\varepsilon^l}{\bar{\varepsilon}} \delta^l, \\ \hat{\delta}(\underline{\theta}) &= \frac{\eta (1 - \varepsilon^h) \delta^h + (1 - \eta) (1 - \varepsilon^l) \delta^l}{\eta (1 - \varepsilon^h) + (1 - \eta) (1 - \varepsilon^l)} = \eta \frac{1 - \varepsilon^h}{1 - \bar{\varepsilon}} \delta^h + (1 - \eta) \frac{1 - \varepsilon^l}{1 - \bar{\varepsilon}} \delta^l, \end{aligned}$$

where we have defined $\bar{\varepsilon} = \eta \varepsilon^h + (1 - \eta) \varepsilon^l$.

For fixed p , this can be viewed as a linear equation in (δ^h, δ^l) with unique solution

$$\begin{aligned} \delta^h &= \frac{(1 - \varepsilon^l) \bar{\varepsilon} \hat{\delta}(\bar{\theta}) - \varepsilon^l (1 - \bar{\varepsilon}) \hat{\delta}(\underline{\theta})}{\eta (\varepsilon^h - \varepsilon^l)}, \\ \delta^l &= \frac{\varepsilon^h (1 - \bar{\varepsilon}) \hat{\delta}(\underline{\theta}) - (1 - \varepsilon^h) \bar{\varepsilon} \hat{\delta}(\bar{\theta})}{(1 - \eta) (\varepsilon^h - \varepsilon^l)}. \end{aligned} \quad (17)$$

We must ensure that δ^h and δ^l are between zero and one. The inequalities $\delta^h > 0$ and $\delta^l > 0$ are equivalent to the two inequalities in

$$\frac{\varepsilon^l}{1 - \varepsilon^l} < \frac{\bar{\varepsilon}}{1 - \bar{\varepsilon}} \frac{\hat{\delta}(\bar{\theta})}{\hat{\delta}(\underline{\theta})} < \frac{\varepsilon^h}{1 - \varepsilon^h}, \quad (18)$$

respectively. Moreover, the inequalities $\delta^h < 1$ and $\delta^l < 1$ are equivalent to the inequalities in

$$\frac{\varepsilon^l}{1 - \varepsilon^l} < \frac{\bar{\varepsilon}}{1 - \bar{\varepsilon}} \frac{1 - \hat{\delta}(\bar{\theta})}{1 - \hat{\delta}(\underline{\theta})} < \frac{\varepsilon^h}{1 - \varepsilon^h}, \quad (19)$$

respectively.

To simplify notation, we write $\bar{\rho} = \exp(e_1(\bar{\theta}))$ and $\underline{\rho} = \exp(e_1(\underline{\theta}))$. From agents' first order conditions, we know

$$\begin{aligned} \frac{\hat{\delta}(\bar{\theta})}{\hat{\delta}(\underline{\theta})} &= \frac{p + (1 - p)/\underline{\rho}}{p + (1 - p)/\bar{\rho}} =: f(p) \\ \frac{1 - \hat{\delta}(\bar{\theta})}{1 - \hat{\delta}(\underline{\theta})} &= \frac{1 - p + p\underline{\rho}}{1 - p + p\bar{\rho}} =: g(p) \end{aligned}$$

We want to show that there exists a price $p \in (0, 1)$ such that

$$f(p), g(p) \in \left[\frac{\varepsilon^l}{1 - \varepsilon^l} \frac{1 - \bar{\varepsilon}}{\bar{\varepsilon}}, \frac{\varepsilon^h}{1 - \varepsilon^h} \frac{1 - \bar{\varepsilon}}{\bar{\varepsilon}} \right] =: [\underline{b}, \bar{b}]$$

If such a price exists, then the δ s in (17) are between zero and one, and therefore p is a non-revealing equilibrium price for the economy parameterized by those δ s. By construction, we have $\bar{b} > 1 > \underline{b}$. This already shows that there exists an equilibrium price if the differences in exposure are not “too large”: if $\bar{\rho} = \underline{\rho}$, then $f(p) = g(p) = 1$ for any price. By continuity, an equilibrium also exists for “small enough” heterogeneity. We now establish that condition (7) provides tight bounds for this heterogeneity.

Using the fact that $\bar{\rho} > 1$ and $\bar{\rho} > \underline{\rho}$, it can be verified that the function f is continuous and strictly decreasing for all $p > p_f$, where

$$p_f = -\frac{\bar{\rho}}{\bar{\rho} - 1}$$

Furthermore $f(0) = \bar{\rho}/\underline{\rho} > 1$ and $f(1) = 1$ and f tends to $+\infty$ as p tends to p_f from above. It follows that $f(p) \geq \underline{b}$ for all $p \in (0, 1)$. and that there exists a unique price $p^u > p_f$ such that $f(p^u) \leq \bar{b}$ for all $p \geq p^u$.

It can also be verified that the function g is continuous and strictly decreasing for all $p > p_g$, where

$$p_g = -\frac{1}{\bar{\rho} - 1} > p_f.$$

Furthermore $g(0) = 1$ and $g(1) = \underline{\rho}/\bar{\rho} < 1$ and g tends to $+\infty$ as p tends to p_g from above. It follows that $g(p) \leq \bar{b}$ for all $p \in (0, 1)$. We also know that $f(p) > g(p)$ for all $p \in (0, 1)$.

It follows that there exists a price in $p \in (0, 1)$ such that $f(p), g(p) \in [\underline{b}, \bar{b}]$ if and only if $g(p^u) \geq \underline{b}$. Indeed, suppose that $g(p^u) \geq \underline{b}$. Since $f(1) = 1$, we know that $p^u < 1$. If $p^u \in (0, 1)$, then $f(p^u), g(p^u) \in [\underline{b}, \bar{b}]$. If $p^u < 0$, then $f(0) < \bar{b}$. But we also have $f(0) > g(0) = 1 > \underline{b}$. Using continuity of f and g , we can therefore pick a small positive price p such that $f(p), g(p) \in [\underline{b}, \bar{b}]$. To show the converse, suppose that $g(p^u) < \underline{b}$. Since $g(0) = 1$, it must be that $p^u > 0$. Since g is decreasing, we have $g(p) < \underline{b}$ for all $p \in [p^u, 1)$. But at the same time, $f(p) > \bar{b}$ for all $p \in (0, p^u)$. As a result there exists no price in the unit interval such that $f(p), g(p) \in [\underline{b}, \bar{b}]$.

We now show that the condition $g(p^u) \geq \underline{b}$ is equivalent to condition (7). We first solve for p^u from the equation $f(p^u) = \bar{b}$ to find

$$\frac{p^u}{1 - p^u} = \frac{\underline{\rho}^{-1} - \bar{b}\bar{\rho}^{-1}}{\bar{b} - 1}$$

The condition $g(p^u) \geq \underline{b}$ is

$$\frac{\frac{1-p^u}{p^u} + \underline{\rho}}{\frac{1-p^u}{p^u} + \bar{\rho}} \geq \underline{b}.$$

Substituting in for $\frac{p^u}{1-p^u}$ and multiplying the numerator and denominator by $\underline{\rho}^{-1} - \bar{b}\bar{\rho}^{-1}$, we obtain equivalently

$$\frac{(\bar{b} - 1) + \underline{\rho}(\underline{\rho}^{-1} - \bar{b}\bar{\rho}^{-1})}{(\bar{b} - 1) + \bar{\rho}(\underline{\rho}^{-1} - \bar{b}\bar{\rho}^{-1})} \geq \underline{b},$$

which simplifies to

$$\frac{\bar{b}(1 - \underline{\rho}/\bar{\rho})}{\bar{\rho}/\underline{\rho} - 1} \geq \underline{b}.$$

and further to

$$\bar{\rho}/\underline{\rho} \leq \bar{b}/\underline{b}$$

Using the definitions of $\bar{\rho}$, $\underline{\rho}$, \bar{b} and \underline{b} we arrive at the condition (7). ■

Proof of Proposition 4.1.

As a preliminary step, we establish

Lemma 1. (a) $\hat{\delta}(s, \bar{e}, p) > \hat{\delta}(s, \underline{e}, p)$ for all s if and only if $\delta^h(p) > \delta^l(p)$.
(b) $\hat{\delta}(s_1, e, p) > \hat{\delta}(s_2, e, p)$ for all e .

Proof. The individual belief $\hat{\delta}$ can be viewed as an average of δ^h and δ^l ,

$$\hat{\delta}(s, e, p) = \hat{\eta}(s, e, p) \delta^h(p) + (1 - \hat{\eta}(s, e, p)) \delta^l(p) \quad (20)$$

where the individual weights are

$$\hat{\eta}(s, e, p) = \frac{\eta_p \mu_h(s, e)}{\eta_p \mu_h(s, e) + (1 - \eta_p) \mu_l(s, e)}.$$

By independence of s and e , the population weights are,

$$\begin{aligned} \mu_j(s_1, \bar{e}; p) &= \delta^j(p) \varepsilon^j \\ \mu_j(s_1, \underline{e}; p) &= \delta^j(p) (1 - \varepsilon^j) \\ \mu_j(s_2, \bar{e}; p) &= (1 - \delta^j(p)) \varepsilon^j \\ \mu_j(s_2, \underline{e}; p) &= (1 - \delta^j(p)) (1 - \varepsilon^j) \end{aligned}$$

The implication (a) follows from the fact that $\hat{\eta}(s, \bar{e}, p) > \hat{\eta}(s, \underline{e}, p)$. Indeed, that statement is equivalent to

$$\frac{\mu_h(s, \bar{e}; p)}{\mu_l(s, \bar{e}; p)} > \frac{\mu_h(s, \underline{e}; p)}{\mu_l(s, \underline{e}; p)} \quad (21)$$

which is in turn equivalent to

$$\frac{\mu_h(s, \bar{e}; p)}{\mu_h(s, \underline{e}; p)} = \frac{\varepsilon^h}{1 - \varepsilon^h} > \frac{\varepsilon^l}{1 - \varepsilon^l} = \frac{\mu_l(s, \bar{e}; p)}{\mu_l(s, \underline{e}; p)}.$$

To show implication (b), consider first the case $\delta^h > \delta^l$. We want to show that $\hat{\eta}(s_1, e, p) > \hat{\eta}(s_2, e, p)$, which is equivalent to

$$\frac{\mu_h(s_1, e; p)}{\mu_l(s_1, e; p)} > \frac{\mu_h(s_2, e; p)}{\mu_l(s_2, e; p)}. \quad (22)$$

For any e , this is equivalent to

$$\frac{\delta^h}{\delta^l} > \frac{1 - \delta^h}{1 - \delta^l},$$

and thus holds if and only if $\delta^h > \delta^l$.

In the case $\delta^h < \delta^l$, we want to show that $\hat{\eta}(s_1, e, p) < \hat{\eta}(s_2, e, p)$, that is, the reverse of (22), which holds iff $\delta^h < \delta^l$. ■

We now establish **Part 2** of the proposition. **Part 1** then follows immediately from Lemma 1

We begin with the case $\sigma \neq 0$. Start from the market clearing condition for the claim on $\tau = \tau_1$:

$$p \sum_{\theta} \mu_j(\theta; p) c_1(\theta; \mu_j) = p \Omega_1(\varepsilon^j)$$

Multiplying the equation by σ , adding $\bar{\alpha}(\varepsilon^j)$ and rearranging, we obtain

$$p \sum_{\theta} \mu_j(\theta; p) (\alpha(\theta) + \sigma c_1(\theta; \mu_j)) = p (\bar{\alpha}(\varepsilon^j) + \sigma \Omega_1(\varepsilon^j))$$

The first order conditions and budget constraint for agent θ can be written as

$$\begin{aligned} \frac{\hat{\delta}(\theta, p)}{1 - \hat{\delta}(\theta, p)} \left(\frac{\alpha(\theta) + \sigma c_1}{\alpha(\theta) + \sigma c_2} \right)^{-\frac{1}{\sigma}} &= \frac{p}{1 - p} \\ p(\alpha(\theta) + c_1) + (1 - p)(\alpha(\theta) + c_2) &= w(\theta, p) + \alpha(\theta) \end{aligned}$$

Define the expenditure share in the case of power utility with belief δ by

$$\psi(\delta, p) := \frac{1}{1 + \left(\frac{1-p}{p}\right)^{1-\sigma} \left(\frac{1-\delta}{\delta}\right)^{\sigma}}.$$

Combining the first order conditions and the definition of ψ , we can then rewrite the market clearing conditions as

$$\sum_{\theta} \mu_j(\theta; p) \psi(\hat{\delta}(\theta, p), p) (\alpha(\theta) + \sigma w(\theta, p)) = p (\bar{\alpha}(\varepsilon^j) + \sigma \Omega_g(\varepsilon^j))$$

Intuitively, the function ψ acts as an “adjusted” expenditure share in a world where endowments and consumption have been linearly translated using the parameters σ and α . Using the definition of wealth, we now have

$$\frac{p}{1-p} = \left(\frac{\bar{\psi}_2(\varepsilon^j, p)}{1 - \bar{\psi}_1(\varepsilon^j, p)} \right) \left(\frac{\bar{\alpha}(\varepsilon^j) + \sigma\Omega_2(\varepsilon^j)}{\bar{\alpha}(\varepsilon^j) + \sigma\Omega_1(\varepsilon^j)} \right) \quad \text{for } j = h, l. \quad (23)$$

where

$$\bar{\psi}_i(\varepsilon^j, p) = \sum \mu_j(\theta) \frac{\alpha(\theta) + \sigma\omega_i(\theta)}{\bar{\alpha}(\varepsilon^j) + \sigma\Omega_i(\varepsilon^j)} \psi(\hat{\delta}(\theta, p), p)$$

is an average of the adjusted expenditure shares ψ formed by weighting individual adjusted expenditure shares by adjusted endowments in states i and j .

Now suppose towards a contradiction that $\delta^h(p) \leq \delta^l(p)$. By Lemma 1, the individual beliefs are ordered as $\hat{\delta}(s, \bar{e}, p) \leq \hat{\delta}(s, \underline{e}, p)$ and $\hat{\delta}(s_1, e, p) \geq \hat{\delta}(s_2, e, p)$.

The effect of beliefs on the adjusted expenditure shares ψ depends on the sign of σ . We begin with the case $\sigma > 0$. To simplify notation, write $\hat{\psi}(e, s, p) := \psi(\hat{\delta}(s, \bar{e}, p), p)$. If $\sigma > 0$, the function ψ is strictly increasing in δ , which implies $\hat{\psi}(s, \bar{e}, p) \leq \hat{\psi}(s, \underline{e}, p)$ and $\hat{\psi}(s_1, e, p) \geq \hat{\psi}(s_2, e, p)$.

The averages $\bar{\psi}_i(\varepsilon^j)$ can now be ranked, for $i = g, b$:

$$\begin{aligned} \bar{\psi}_i(\varepsilon^h) &= \frac{\varepsilon^h \omega_i(\bar{e})}{\Omega_i(\varepsilon^h)} \left[\delta^h \hat{\psi}(s_1, \bar{e}, p) + (1 - \delta^h) \hat{\psi}(s_2, \bar{e}, p) \right] \\ &\quad + \frac{(1 - \varepsilon^h) \omega_i(\underline{e})}{\Omega_i(\varepsilon^h)} \left[\delta^h \hat{\psi}(s_1, \underline{e}, p) + (1 - \delta^h) \hat{\psi}(s_2, \underline{e}, p) \right] \\ &\leq \frac{\varepsilon^l \omega_i(\bar{e})}{\Omega_i(\varepsilon^l)} \left[\delta^h \hat{\psi}(\bar{s}, \bar{e}, p) + (1 - \delta^h) \hat{\psi}(\underline{s}, \bar{e}, p) \right] \\ &\quad + \frac{(1 - \varepsilon^l) \omega_i(\underline{e})}{\Omega_i(\varepsilon^l)} \left[\delta^h \hat{\psi}(s_1, \underline{e}, p) + (1 - \delta^h) \hat{\psi}(s_2, \underline{e}, p) \right] \\ &\leq \frac{\varepsilon^l \omega_i(\bar{e})}{\Omega_i(\varepsilon^l)} \left[\delta^l \hat{\psi}(\bar{s}, \bar{e}, p) + (1 - \delta^l) \hat{\psi}(\underline{s}, \bar{e}, p) \right] \\ &\quad + \frac{(1 - \varepsilon^l) \omega_i(\underline{e})}{\Omega_i(\varepsilon^l)} \left[\delta^l \hat{\psi}(s_1, \underline{e}, p) + (1 - \delta^l) \hat{\psi}(s_2, \underline{e}, p) \right] \\ &= \bar{\psi}_i(\varepsilon^l), \end{aligned} \quad (24)$$

where the first inequality holds because $\hat{\psi}(s, \bar{e}, p) \leq \hat{\psi}(s, \underline{e}, p)$ and $\varepsilon^h > \varepsilon^l$, and where the second inequality holds because $\hat{\psi}(s_1, e, p) \geq \hat{\psi}(s_2, e, p)$ and $\delta^h \leq \delta^l$.

We also know that aggregate exposure is strictly increasing in ε . If $\sigma > 0$, it thus follows that

$$\frac{\bar{\alpha}(\varepsilon^l) + \sigma\Omega_2(\varepsilon^l)}{\bar{\alpha}(\varepsilon^l) + \sigma\Omega_1(\varepsilon^l)} > \frac{\bar{\alpha}(\varepsilon^h) + \sigma\Omega_2(\varepsilon^h)}{\bar{\alpha}(\varepsilon^h) + \sigma\Omega_1(\varepsilon^h)} \quad (25)$$

Putting together inequalities (24) and (25), we have

$$\begin{aligned} \frac{p}{1-p} &= \frac{\bar{\psi}_2(\varepsilon^h)}{1-\bar{\psi}_1(\varepsilon^h)} \frac{\bar{\alpha}(\varepsilon^h) + \sigma\Omega_2(\varepsilon^h)}{\bar{\alpha}(\varepsilon^h) + \sigma\Omega_1(\varepsilon^h)} \\ &< \frac{\bar{\psi}_2(\varepsilon^l)}{1-\bar{\psi}_1(\varepsilon^l)} \frac{\bar{\alpha}(\varepsilon^l) + \sigma\Omega_2(\varepsilon^l)}{\bar{\alpha}(\varepsilon^l) + \sigma\Omega_1(\varepsilon^l)}, \end{aligned}$$

which contradicts the equilibrium condition (23).

Now suppose instead that $\sigma < 0$. In this case, the function ψ is decreasing in δ . As a result, we also have $\hat{\psi}(s, \bar{e}, p) \geq \hat{\psi}(s, \underline{e}, p)$ and $\hat{\psi}(s_1, e, p) \leq \hat{\psi}(s_2, e, p)$. The inequalities in (24) are thus reversed, and we have $\bar{\psi}_i(\varepsilon^h) \geq \bar{\psi}_i(\varepsilon^l)$. At the same time, the fact that aggregate exposure is increasing in ε implies that (25) is reversed as well. But then

$$\begin{aligned} \frac{p}{1-p} &= \frac{\bar{\psi}_2(\varepsilon^h)}{1-\bar{\psi}_1(\varepsilon^h)} \frac{\bar{\alpha}(\varepsilon^h) + \sigma\Omega_2(\varepsilon^h)}{\bar{\alpha}(\varepsilon^h) + \sigma\Omega_1(\varepsilon^h)} \\ &> \frac{\bar{\psi}_2(\varepsilon^l)}{1-\bar{\psi}_1(\varepsilon^l)} \frac{\bar{\alpha}(\varepsilon^l) + \sigma\Omega_2(\varepsilon^l)}{\bar{\alpha}(\varepsilon^l) + \sigma\Omega_1(\varepsilon^l)}, \end{aligned}$$

which again contradicts the equilibrium condition (23).

Finally, consider now the case of exponential utility ($\sigma = 0$). The first order conditions for agent θ imply that

$$c_2(\theta) = c_1(\theta) + \alpha(\theta) \ln \left(\frac{p(1 - \hat{\delta}(\theta, p))}{\hat{\delta}(\theta, p)(1-p)} \right).$$

This equation, the individual budget constraints, and the market clearing condition for state 1 imply that

$$\sum_{\theta} \mu_j(\theta; p) (\omega_1(\theta) - \omega_2(\theta)) = \sum_{\theta} \mu_j(\theta; p) \alpha(\theta) \left[\ln \left(\frac{\hat{\delta}(\theta, p)}{1 - \hat{\delta}(\theta, p)} \right) - \ln \left(\frac{p}{1-p} \right) \right] \quad \text{for } j = l, h.$$

If $\sigma = 0$, the exposure of a type θ agent is determined by the ratio

$$e_1(\theta) = \frac{\omega_1(\theta) - \omega_2(\theta)}{\alpha(\theta)}.$$

The previous two equations imply that

$$\ln\left(\frac{p}{1-p}\right) = \sum_{\theta} \mu_j(\theta; p) \frac{\alpha(\theta)}{\bar{\alpha}(\varepsilon^j)} \varphi\left(\hat{\delta}(\theta, p), p\right) \quad \text{for } j = l, h,$$

where

$$\varphi(\delta, p) = \ln\left(\frac{\hat{\delta}(\theta, p)}{1 - \hat{\delta}(\theta, p)}\right) - e_1(\theta).$$

Assume towards a contradiction that $\delta^h(p) \leq \delta^l(p)$. The function φ is strictly increasing in δ and strictly decreasing in e_1 , which implies $\varphi(s, \bar{e}, p) < \varphi(s, \underline{e}, p)$ and $\varphi(s_1, e, p) \geq \varphi(s_2, e, p)$.

Now define $\tilde{\varepsilon}^j$ as

$$\tilde{\varepsilon}^j := \frac{\varepsilon^j \alpha(\bar{e})}{\varepsilon^j \alpha(\bar{e}) + (1 - \varepsilon^j) \alpha(\underline{e})} = \frac{1}{1 + \frac{(1 - \varepsilon^j) \alpha(\underline{e})}{\varepsilon^j \alpha(\bar{e})}}.$$

For $\alpha(\underline{e}), \alpha(\bar{e}) > 0$, it is easy to verify that $\tilde{\varepsilon}^j$ is strictly increasing in ε^j .

Therefore,

$$\begin{aligned} \ln\left(\frac{p}{1-p}\right) &= \tilde{\varepsilon}^h [\delta^h \varphi(s_1, \bar{e}, p) + (1 - \delta^h) \varphi(s_2, \bar{e}, p)] + (1 - \tilde{\varepsilon}^h) [\delta^h \varphi(s_1, \underline{e}, p) + (1 - \delta^h) \varphi(s_2, \underline{e}, p)] \\ &< \tilde{\varepsilon}^l [\delta^h \varphi(s_1, \bar{e}, p) + (1 - \delta^h) \varphi(s_2, \bar{e}, p)] + (1 - \tilde{\varepsilon}^l) [\delta^h \varphi(s_1, \underline{e}, p) + (1 - \delta^h) \varphi(s_2, \underline{e}, p)] \\ &\leq \tilde{\varepsilon}^l [\delta^l \varphi(s_1, \bar{e}, p) + (1 - \delta^l) \varphi(s_2, \bar{e}, p)] + (1 - \tilde{\varepsilon}^l) [\delta^l \varphi(s_1, \underline{e}, p) + (1 - \delta^l) \varphi(s_2, \underline{e}, p)] \\ &= \ln\left(\frac{p}{1-p}\right), \end{aligned} \tag{26}$$

where the first inequality holds because $\varphi(s, \bar{e}, p) < \varphi(s, \underline{e}, p)$ and $\varepsilon^h > \varepsilon^l$, and where the second inequality holds because $\varphi(s_1, e, p) \geq \varphi(s_2, e, p)$ and $\delta^h \leq \delta^l$.

■

Proof of Proposition 4.2.

Using analogous notation as for the REE price function, we define $P_{FI}^j(\delta) := P_{FI}(\delta, \varepsilon^j)$. By (10) in the proof of Proposition 4.1, the functions P_{FI}^j are strictly increasing and thus have well-defined inverse functions $\delta_{FI}^j(p) = (P_{FI}^j)^{-1}(p)$.

We now establish that for all $p \in (0, 1)$,

$$\delta^h(p) > \delta_{FI}^h(p) > \delta_{FI}^l(p) > \delta_l(p).$$

Since both \tilde{P} and \tilde{P}_{FI} are strictly increasing and continuous in δ , this proves part 3.

In the full information case, we can follow the same algebra as in the proof of Proposition 4.1. above to arrive at equation (23). If all beliefs are equal at $\hat{\delta}(\theta, p) = \delta^j(p)$, that equation simplifies to

$$\frac{p}{1-p} = \frac{\psi(\delta_{FI}^j(p), p)}{1 - \psi(\delta_{FI}^j(p), p)} \frac{\bar{\alpha}(\varepsilon^j) + \sigma\Omega_2(\varepsilon^j)}{\bar{\alpha}(\varepsilon^j) + \sigma\Omega_1(\varepsilon^j)} \quad \text{for } j = h, l. \quad (27)$$

To establish $\delta^h(p) > \delta_{FI}^h(p)$, start again with the case $\sigma > 0$. Since all the $\bar{\psi}_i(\varepsilon^j)$ are averages of the $\psi(\hat{\delta}(\theta, p), p)$, and moreover the $\hat{\delta}(\theta, p)$ are averages of δ^h and δ^l , we have $\bar{\psi}_i(\varepsilon^h) < \max_{\theta} \psi(\hat{\delta}(\theta, p), p) < \psi(\delta^h, p)$ for $i = 1, 2$. Therefore

$$\frac{p}{1-p} < \frac{\psi(\delta^h, p)}{1 - \psi(\delta^h, p)} \frac{\bar{\alpha}(\varepsilon^h) + \sigma\Omega_2(\varepsilon^h)}{\bar{\alpha}(\varepsilon^h) + \sigma\Omega_1(\varepsilon^h)}. \quad (28)$$

The definition of $\hat{\delta}^h$ in (27) together with the fact that ψ is strictly increasing thus implies $\hat{\delta}^h < \delta^h$. The argument for $\hat{\delta}^l > \delta^l$ follows analogously from the fact that $\bar{\delta}_i(\delta, \varepsilon^l) > \delta^l$ for $i = 1, 2$.

Now if $\sigma < 0$, the ψ s are decreasing in δ , so that

$$\bar{\psi}_i(\varepsilon^h) > \min_{\theta} \psi(\hat{\delta}(\theta, p), p) > \psi(\delta^h, p)$$

and the inequality (28) is reversed. However, The definition of $\hat{\delta}^h$ in (27) together with the fact that ψ is strictly decreasing once more implies $\hat{\delta}^h < \delta^h$. Again, the argument for $\hat{\delta}^l > \delta^l$ follows analogously from the fact that $\bar{\delta}_i(\delta, \varepsilon^l) > \delta^l$ for $i = 1, 2$.

case $\sigma = 0$: to be written

■

Proof of Proposition 5.1. We want to show

$$\eta_p^h \hat{\delta} + (1 - \eta_p) \hat{\delta}^l < \eta_p \delta^h + (1 - \eta_p) \delta^l.$$

or equivalently

$$\eta_p \left(\hat{\delta}^h - \delta^h \right) + (1 - \eta_p) \left(\hat{\delta}^l - \delta^l \right) < 0$$

Market clearing at the price p is

$$\sum_{\theta} \mu^j(\theta) \psi \left(\hat{\delta}(\theta, p), p \right) w(\theta, p) = p \sum_{\theta} \mu^j(\theta) \omega_g(\theta), \quad (29)$$

where

$$\begin{aligned} \hat{\delta}(\theta, p) &= \hat{\eta}(\theta, p) \delta^h(p) + (1 - \hat{\eta}(\theta, p)) \delta^l(p) \\ \hat{\eta}(\theta, p) &= \frac{\eta_p \mu_h(\theta)}{\eta_p \mu_h(\theta) + (1 - \eta_p) \mu_l(\theta)} \\ \eta_p &= \frac{\eta \delta^{h'} f(\delta_h)}{\eta \delta^{h'} f(\delta_h) + (1 - \eta) \delta^{l'} f(\delta_l)} \end{aligned}$$

Multiply the market clearing equation for state h by η_p , multiply that for state l by $(1 - \eta_p)$ and add the two equations to get

$$\begin{aligned} &\eta_p \sum_{\theta} \mu^h(\theta) \frac{\eta_p \mu_h(\theta) \delta^h + (1 - \eta_p) \mu_l(\theta) \delta^l}{\eta_p \mu_h(\theta) + (1 - \eta_p) \mu_l(\theta)} w(\theta, p) \\ &+ (1 - \eta_p) \sum_{\theta} \mu^l(\theta) \frac{\eta_p \mu_h(\theta) \delta^h + (1 - \eta_p) \mu_l(\theta) \delta^l}{\eta_p \mu_h(\theta) + (1 - \eta_p) \mu_l(\theta)} w(\theta, p) \\ &= p \left(\eta_p \sum_{\theta} \mu^l(\theta) \omega_g(\theta) + (1 - \eta_p) \sum_{\theta} \mu^l(\theta) \omega_g(\theta) \right) \end{aligned}$$

Rearranging terms we get

$$\eta_p \delta^h W(\varepsilon^h, p) + (1 - \eta_p) \delta^l W(\varepsilon^l, p) = p (\eta_p \Omega_g(\varepsilon^h) + (1 - \eta_p) \Omega_g(\varepsilon^l)), \quad (30)$$

where $W(\varepsilon^j, p)$ is aggregate wealth in state j .

Now the subjective beliefs fit by the econometrician satisfy

$$\hat{\delta}^j W(\varepsilon^j, p) = p \Omega_g(\varepsilon^j), \quad j = h, l.$$

We can again multiply the equations for h and l by η_p and $1 - \eta_p$, respectively. We get that

$$\eta_p^h \hat{\delta}^h W(\varepsilon^h, p) + (1 - \eta_p) \hat{\delta}^l W(\varepsilon^l, p) = p (\eta_p \Omega_g(\varepsilon^h) + (1 - \eta_p) \Omega_g(\varepsilon^l)), \quad (31)$$

Combining (30) and (31), we have

$$\eta_p \left(\hat{\delta}^h - \delta^h \right) W \left(\varepsilon^h, p \right) + (1 - \eta_p) \left(\hat{\delta}^l - \delta^l \right) W \left(\varepsilon^l, p \right) = 0$$

But then

$$\begin{aligned} \eta_p \left(\hat{\delta}^h - \delta^h \right) + (1 - \eta_p) \left(\hat{\delta}^l - \delta^l \right) &= \eta_p \left(\hat{\delta}^h - \delta^h \right) - (1 - \eta_p) \frac{\eta_p \left(\hat{\delta}^h - \delta^h \right) W \left(\varepsilon^h, p \right)}{(1 - \eta_p) W \left(\varepsilon^l, p \right)} \\ &= \frac{\eta_p}{W \left(\varepsilon^l, p \right)} \left(\hat{\delta}^h - \delta^h \right) \left(W \left(\varepsilon^l, p \right) - W \left(\varepsilon^h, p \right) \right) \end{aligned}$$

Since $\hat{\delta}^h < \delta^h$ from Proposition 4.2., the condition follows. ■