Incorporating Diagnostic Expectations into the New Keynesian Framework*

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Abstract

Diagnostic expectations constitute a realistic behavioral model of inference. This paper shows that this approach for expectation formation can be productively integrated into the New Keynesian framework. To this end, we start by offering a first technical treatment of diagnostic expectations in linear macroeconomic models. Diagnostic expectations generate *endogenous* extrapolation in general equilibrium. We show that diagnostic expectations generate extra amplification in the presence of nominal frictions; a fall in aggregate supply generates a Keynesian recession; fiscal policy is more effective at stimulating the economy; with imperfect information, diagnostic expectations generate *delayed* overreaction of aggregate variables. Bayesian estimation of a rich medium-scale model delivers estimates of the diagnosticity parameter that is in line with previous studies. Moreover, we find strong empirical evidence in favor of the diagnostic model.

Keywords: Heuristics, representativeness, general equilibrium, shocks, volatility.

JEL codes: E12, E32, E71.

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1 Introduction

Diagnostic expectations (DE) have emerged as an important departure from rational expectations in macroeconomics and finance. Among the host of possible deviations from rational expectations, there are three broad reasons that make diagnostic expectations a leading alternative to consider for macroeconomic modeling. First, diagnostic expectations constitute a microfounded deviation immune to the Lucas critique. Second, this approach lends itself to a great deal of tractability, as a number of recent efforts in macroeconomics and finance have demonstrated (see Bordalo, Gennaioli, and Shleifer 2018; Bordalo, Gennaioli, Ma, and Shleifer 2020; Bordalo, Gennaioli, Shleifer, and Terry 2021, among others). Third, based on the pathbreaking and influential work on the "representativeness heuristic" by Kahneman and Tversky (1972), one ought to consider this behavioral model as fundamentally realistic, and thereby portable across fields of economics.[1]

In this paper, we argue that diagnostic expectations can be productively incorporated into the New Keynesian (NK) framework. To this end, we start off with a substantial technical contribution: We develop a solution method for a general class of linear DSGE models with diagnostic expectations. The key to our method is to formally establish the existence and uniqueness of a rational expectations representation of the diagnostic expectations model, a challenging task in the presence of endogenous states. This result allows us to compute the equilibrium diagnostic expectation of endogenous variables.

Armed with this method, we demonstrate the usefulness of diagnostic expectations in two parts, analytical and empirical. Analytically, using a three-equation NK model, we show how diagnostic expectations bring rich insights on four issues raised by the literature. The first issue we tackle is that of amplification and propagation in general equilibrium. As shown in previous work (Bordalo, Gennaioli, and Shleifer 2018, henceforth BGS), diagnostic expectations (DE) imply an extrapolation of current shocks into the future. Intuitively, this could generate extra volatility for endogenous variables. We show that this intuition is in fact not guaranteed. In the presence of nominal frictions (as in the NK model) DE generate extra volatility; in a frictionless representative agent real business cycle (RBC) model, general equilibrium channels shut down the effect of DE, and output is *less* volatile under DE than under rational expectations (RE).[2]

[1]Simply put, the representativeness heuristic is the general human tendency to over-estimate how representative a small sample is, a pattern documented in a large body of literature in psychology and behavioral economics. For a survey and more detailed discussion, see Kahneman, Slovic, and Tversky (1982).

[2]Bordalo, Gennaioli, Shleifer, and Terry (2021) consider financial frictions and how DE generate realistic credit

The second issue considered is whether a fall in aggregate supply can cause a demand shortage. Since the onset of the COVID-19 pandemic, there is a renewed interest on whether supply-side disruptions can ultimately generate shortfalls in aggregate demand (see Guerrieri, Lorenzoni, Straub, and Werning 2020; Fornaro and Wolf 2020; Caballero and Simsek 2020; Bilbiie and Melitz 2020, among others.) Whereas the rational expectations NK (RE-NK) model generates the opposite prediction, we show that adding DE into the NK framework (DE-NK) allows for the possibility of "Keynesian supply shocks": Following a negative supply shock, diagnostic agents extrapolate the shock into the future, and hence become excessively pessimistic. This pushes them to reduce consumption drastically, generating a Keynesian recession.

The third issue we tackle concerns government policy. We show how endogenous extrapolation arising from the evaluation of the inflation process by diagnostic agents can significantly raise the government spending multiplier. Current surprise inflation causes the diagnostic agent to expect future inflation thereby reducing the subjective real interest rate. When the diagnosticity parameter is higher than the coefficient governing the reaction of the monetary authority to inflation, the DE-NK model is able to generate a multiplier greater than 1 even with i.i.d. government spending shocks. We show how this analytical conclusion can be challenged by the degree of exogenous extrapolation, which depends on the persistence of the shock. If the shock is persistent enough, the DE of future spending can completely crowd out current consumption and lead to a multiplier that is equal to 0, or even negative. Hence, the degree of diagnosticity allows the model to span a wide range of multipliers, highlighting the importance of the behavioral friction in this context.

With an eye to the large macroeconomics literature on information frictions, the fourth question we consider concerns under- and overreaction of expectations (Coibion and Gorodnichenko 2015a; Bordalo, Gennaioli, Ma, and Shleifer 2020). Based on previous work (Lorenzoni 2009; Blanchard, L'Huillier, and Lorenzoni 2013), we extend DE-NK model to a setting where the consumers receive noisy signals about the future path of their income. Beliefs about their long-run income determine aggregate consumption and output due to nominal rigidities. We show that a plausible calibration of the imperfect information DE-NK model can generate both short-run underreaction, and an overreaction over the medium-term. Combining diagnostic expectations with information frictions can deliver rich implications for the path of agents' beliefs in general equilibrium models.

On the empirical front, we let DE and RE compete within a standard medium-

cycles in a real economy. See Section 3 for a broader discussion.

scale DSGE model. Using Bayesian methods, we evaluate the relative fitness of both approaches when applied to post-war U.S. business cycles data. In order to submit the behavioral expectational friction to a stringent empirical test, the model we consider contains a large number of benchmark frictions and shocks drawn from the seminal works by Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2007). For the same reason, we include information frictions following an influential literature as recently emphasized by Coibion and Gorodnichenko (2015a). We find very strong empirical evidence in favor of DE versus RE.

A recurrent theme in our paper is that when agents have diagnostic beliefs about endogenous variables, instead of exogenous processes, new behavioral insights emerge. Endogenous extrapolation, as highlighted in our fiscal policy exercise, has remarkable economic implications. We provide two examples of models with endogenous extrapolation at the end of Section 2.

We briefly make a few technical remarks about our solution method. First, the computation of the DE of linear combinations involving endogenous state variables in the form of predetermined variables is challenging, and is, thus far, an open question in the literature. We make use of the mathematics of the Dirac delta distribution to overcome this technical challenge. Our results clarify that predetermined variables cannot be treated as constants in the diagnostic model. The reason is that DE introduce a form of 'behavioral inattention', whereby aggregate predetermined variables are perfectly observed by the atomistic diagnostic agent only with a lag.[3] This introduces path dependence in beliefs, while maintaining tractability of the recursive, diagnostic, model. Section 2 provides further motivation and intuition for these results; the formal derivations are presented in Appendix A. Second, we show that incorporating DE requires researchers to loglinearize the model from scratch rather than simply replacing the rational expectations operator with the corresponding diagnostic expectations operator in linear economies. For a given set of equilibrium conditions obtained from first principles, the presence of DE actually changes the loglinear equilibrium conditions that constitute a correct approximation.[4] We explain, in detail, how to obtain the correct approximation and provide a few examples. Log-linearization under DE brings forward novel economic insights in forward looking models, discussed in depth below. Third, we provide sharp results on the stability and the existence of a bounded solution with DE. While the stability conditions are same as in the corresponding RE

[3]Relatedly, Gennaioli and Shleifer (2010) emphasize how memory accessibility generates judgment errors that depend on representative scenarios.

[4]This is different from many other departures from the full-information rational expectations case, as for example the introduction of imperfect information (Woodford 2002) or other behavioral models (Garcia-Schmidt and Woodford 2019), where the structure of equilibrium conditions of the loglinear model does not change.

model, we note that the solution under DE can be explosive for certain limiting values of the diagnosticity parameter. Researchers may need to exercise caution when applying DE to endogenous variables.

Related Literature. The paper is primarily related to the emerging literature on DE. See Gennaioli and Shleifer (2018) for a review. Most closely related are papers by Maxted (2020) and Bordalo, Gennaioli, Shleifer, and Terry (2021), who incorporate DE in macro-finance frameworks.[5] Maxted (2020) shows that incorporating DE into a macro-finance framework can reproduce several facts surrounding financial crises (see also Krishnamurthy and Li 2020). Bordalo, Gennaioli, Shleifer, and Terry (2021) show that DE can quantitatively generate countercyclical credit spreads in a heterogeneous firms business-cycle model. We complement these efforts by providing a general treatment of DE in linear macroeconomic models. In particular, we show how incorporating DE into NK models (Woodford 2003; Galí 2015) delivers rich new insights and significantly improves the fit to the data.

In parallel and complementary work, Bianchi, Ilut, and Saijo (2021) also investigate applications of DE in linear models. The main focus of their paper is distant memory, the notion that agents' reference distribution looks back more than 1 period. In such settings, the law of iterated expectations fails, and therefore the model with distant memory is time inconsistent. In their paper, Bianchi, Ilut, and Saijo (2021) investigate the rich implications of this type of time inconsistency. Our paper focuses exclusively on linear settings with time consistency, and shows that this baseline setup offers a number of insights useful for the NK literature. We also provide an empirical evaluation in a benchmark medium-scale DSGE model. Moreover, ours is the first general technical treatment of linear models. We formally prove how to evaluate the DE of linear combinations involving endogenous state variables using the mathematics of the Dirac delta distribution, and outline, in detail, the steps from the exact equilibrium conditions to the loglinear approximation of medium-scale models.

Our paper also speaks to the literature proposing deviations from the full-information rational expectations (FIRE) hypothesis. See, for example, Mankiw and Reis (2002), Coibion and Gorodnichenko (2015a), Angeletos, Huo, and Sastry (2020), Bordalo, Gennaioli, Ma, and Shleifer (2020), Kohlhas and Walther (2020), among others.

Angeletos, Huo, and Sastry (2020) document delayed overreaction of beliefs in response to business cycle shocks. Bordalo, Gennaioli, Ma, and Shleifer (2020) propose

[5] D'Arienzo (2020) investigates the ability of DE to reconcile the overreaction of expectations of long rates relative to the expectations of short rates to news in bond markets. Ma, Ropele, Sraer, and Thesmar (2020) quantify the costs of managerial biases.

a model of DE with dispersed information to study underreaction and overreaction in survey forecasts. See also Ma, Ropele, Sraer, and Thesmar (2020) and Afrouzi, Kwon, Landier, Ma, and Thesmar (2020). We complement these analyses by showing that one can obtain delayed overreaction in an imperfect information DE-NK model. With respect to earlier work, there are two innovations in our procedure. First, we use a microfounded behavioral friction. Second, we generate these patterns with expectations in general equilibrium models. In a related vein, our estimated DSGE model builds on work exploring business cycle models where agents receive advance information about future productivity that is subject to an information friction (Blanchard, L'Huillier, and Lorenzoni 2013; Chahrour and Jurado 2018).

Our paper fits into the macroeconomics literature that models departures from rational expectations with various behavioral assumptions. Some of the recent applications have focused on resolving puzzles in New Keynesian models by introducing behavioral assumptions. Angeletos and Lian (2018), Farhi and Werning (2019), Gabaix (2020), and Garcia-Schmidt and Woodford (2019) are some of the papers that propose departures from rational expectations to attenuate the strength of forward guidance. Iovino and Sergeyev (2020) study the effectiveness of central bank balance sheet policies with level-k thinking. Farhi and Werning (2020) study the role of monetary policy as a macro-prudential tool when agents form extrapolative expectations.

Paper Organization. The paper is organized as follows. Section 2 presents our solution method, discusses stability, and provides examples illustrating endogenous propagation of diagnostic beliefs. Section 3 presents the analytical results from a 3-equation NK model. Section 4 presents the empirical evaluation of diagnostic expectations in a medium scale DSGE model. Section 5 concludes. The Appendix provides supplementary materials and collects all the proofs.

2 Solution Method

In this section we present a solution method for a general class of linear models. Agents use diagnostic expectations to form beliefs about the evolution of all variables, exogenous and endogenous. Our strategy consists in obtaining a rational expectations (RE) representation of the diagnostic expectations (DE) model. Based on this step, the model can be solved using standard techniques.

2.1 Preliminary Considerations: Handling Predetermined Variables

The main goal here is to establish a strong additivity result for the DE operator. This result will be useful to get linearity in the context of the general linear model below. For purposes of the arguments in this subsection, define the following two AR(1) processes for random variables x_t and y_t:

$$x_t = \rho_x x_{t-1} + \varepsilon_t \tag{1}$$

$$y_t = \rho_y y_{t-1} + \eta_t \tag{2}$$

where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ and $\eta_t \sim N(0, \sigma_\eta^2)$ are Gaussian and orthogonal exogenous shocks, ρ_x and ρ_y are persistence parameters satisfying $\rho_x, \rho_y \in [0, 1)$, and σ_ε^2 and σ_η^2 are the shocks' variances.

We first focus on x_t. The true (or non-distorted) pdf of x_{t+1} is $f(x_{t+1}|x_t) \propto \varphi\left(\frac{x_{t+1}-\rho_x x_t}{\sigma_\varepsilon}\right)$, where $\varphi(x)$ is the density of a standard normal distribution $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. Following Bordalo, Gennaioli, and Shleifer (2018) (henceforth BGS), the diagnostic distribution is defined as

$$f_t^\theta(x_{t+1}) = f(x_{t+1}|G_t) \cdot \left[\frac{f(x_{t+1}|G_t)}{f(x_{t+1}|-G_t)}\right]^\theta \cdot C$$

where G_t and $-G_t$ are conditioning events. G_t encodes current conditions: $G_t \equiv \{x_t = \check{x}_t\}$, where $\check{x}_t$ denotes the realization of x_t.[6] $-G_t$ encodes a reference group (i.e. a reference event), that is used to compute the reference distribution $f(x_{t+1}|-G_t)$. Due to the representativeness heuristic, agents overweight the last realization of x_t (relative to the reference group) when forming beliefs about the future realization of x_{t+1}. The likelihood ratio $f(x_{t+1}|G_t)/f(x_{t+1}|-G_t)$ distorts beliefs to a degree governed by the diagnosticity parameter $\theta \geq 0$. C is a constant ensuring that $f_t^\theta(x_{t+1})$ integrates to 1.

Following BGS, we impose that the event $-G_t$ carries "no news" at time t (henceforth no-news assumption or NNA).

Assumption 1 (Univariate No-News Assumption)

$$f(x_{t+1}|-G_t) = f(x_{t+1}|x_t = \rho_x \check{x}_{t-1})$$

[6] We do not use the same notation $\hat{x}_t$ for realizations as BGS, since we have reserved hats over variables for loglinear deviations below.

Beliefs about future x_{t+1} are formed conditional on the event that the random variable x_t, conditional on the past realization $\check{x}_{t-1}$, is what it was expected to be, so $\varepsilon_t = E[\varepsilon_t] = 0$, which is equivalent to $x_t = \rho_x \check{x}_{t-1}$. Under the NNA, the diagnostic distribution is then written as

$$f_t^{\theta}(x_{t+1}) = f(x_{t+1}|x_t = \check{x}_t) \cdot \left[\frac{f(x_{t+1}|x_t = \check{x}_t)}{f(x_{t+1}|x_t = \rho_x \check{x}_{t-1})}\right]^{\theta} \cdot C \tag{3}$$

Notice that the distribution (3) is conditional on two elements: first, it is conditional on the current realization of x_t, written $\check{x}_t$, because this enters the true distribution of x_{t+1}; second, it is conditional on the reference event $-G_t \equiv \{x_t = \rho_x \check{x}_{t-1}\}$, which depends on the realization at $t-1$, $\check{x}_{t-1}$. This earlier realization defines the reference distribution. In this dynamic setting, the reference event happens "between $t-1$ and t".

Following previous literature, we denote the diagnostic expectation operator at time t by $\mathbb{E}_t^{\theta}[\,\cdot\,]$. The diagnostic expectation is formally defined as

$$\mathbb{E}_t^{\theta}[x_{t+1}] = \int_{-\infty}^{\infty} x f_t^{\theta}(x) dx$$

Thanks to the NNA (Assumption 1), one can obtain the following tractable expression for the diagnostic expectation of a future variable in terms of current and lagged true (or 'rational') expectations:[7]

$$\mathbb{E}_t^{\theta}[x_{t+1}] = \mathbb{E}_t[x_{t+1}] + \theta(\mathbb{E}_t[x_{t+1}] - \mathbb{E}_{t-1}[x_{t+1}]) \tag{4}$$

Equation (4) reveals that diagnostic beliefs are path dependent. The shock ε_{t-1} affects the information set at $t-1$ and thereby the expectation $\mathbb{E}_{t-1}[x_{t+1}]$. Previous beliefs constitute a state variable.[8] This path dependence generates extrapolation.

The challenging aspect is computing the DE of linear combinations involving predetermined variables. Examples include the capital stock, or past consumption in models with habit formation. Consider the object $\mathbb{E}_t^{\theta}[x_{t+1} + y_t]$, where the degenerate random variable y_t plays the role of the predetermined variable. The properties of this object depends crucially on the assumptions imposed on the reference distribution of each of

[7] See Bordalo et al. (2018, Proposition 1). For completeness, a proof that closely follows Bordalo et al. (2018) is presented in the appendix.

[8] Alternatively, careful inspection of the pdf (3) shows the dependence on the shock ε_{t-1} through the realization $\check{x}_{t-1}$. The shock ε_{t-1} determines $\check{x}_{t-1}$, and hence the expected value of x_t, together with the reference event $-G_t$. Different from the rational case, the current state x_t does not fully determine the subjective density of the process going forward.

the variables x_{t+1} and y_t.

The solution we adopt is motivated by a consistency requirement and by a tractability concern. For consistency, in the context of the general linear model, we impose a multivariate NNA on all shocks. By way of implication, this imposes the NNA on predetermined variables present in linear combinations. In terms of tractability, just as the NNA was useful to obtain the formula for the DE of future variables in isolation (equation 4), we show that the same assumption, imposed on predetermined variables, leads to a tractable solution of the recursive model.

A technical aspect worth highlighting is that the NNA implies that predetermined variables cannot be treated as constants. In fact, going back to the linear combination above,

$$\mathbb{E}_t^\theta[x_{t+1} + y_t] \neq \mathbb{E}_t^\theta[x_{t+1}] + y_t$$

The intuition for this property is that diagnosticity introduces behavioral inattention, whereby aggregate predetermined variables are perfectly observed by the atomistic diagnostic agent only with a lag. To understand how this arises, consider the information sets $G_t = \{x_t = \check{x}_t; y_t = \check{y}_t\}$ and $-G_t = \{x_t = \rho_x \check{x}_{t-1}; y_t = \rho_y \check{y}_{t-1}\}$ that define the diagnostic distribution $f_t^\theta\,(x_{t+1} + y_t)$. With respect to G_t, y_t is known and equal to $\check{y}_t$. However, with respect to $-G_t$, y_t is not known (instead, the agent's memory leads her/him to believe that $y_t = \rho_y \check{y}_{t-1}$.) Hence, the agent is forming inference based on two minds. Because one of these minds is inattentive, overall inference also features inattention. This is closely linked to the path dependence mentioned above.[9] This type of inattention lasts only one period, since

$$\mathbb{E}_t^\theta[x_{t+1} + y_{t-1}] = \mathbb{E}_t^\theta[x_{t+1}] + y_{t-1}$$

This is because $-G_t$ now includes y_{t-1}. In conclusion, note that the predetermined variable is actually *not* predetermined with respect to the reference distribution, showing how the computation of the DE above involves uncertainty *also* about y_t.

With the consistent use of the NNA, we obtain the following strong additivity result, extending BGS's result to settings with predetermined variables.[10]

Proposition 1 (Strong Additivity of the Diagnostic Expectation)

$$\mathbb{E}_t^\theta[x_{t+r} + y_{t+s}] = \mathbb{E}_t^\theta[x_{t+r}] + \mathbb{E}_t^\theta[y_{t+s}], \quad r, s \geq 0$$

[9] In order to make this feature more apparent, one is tempted to denote the diagnostic expectation operator as $\mathbb{E}_{t,t-1}^\theta[\,\cdot\,]$. However, we have decide to follow BGS closely and avoid this type of cumbersome notation.

[10] See BGS, proof of Corollary 1, Online Appendix, for the case $r, s \geq 1$.

In order to prove this result, we make use of the mathematics of the Dirac delta distribution, useful to deal with the joint density of x_{t+1} and y_t. Those developments are presented in Appendix A. There, we also offer two counterexamples showing that one cannot get a consistent recursive representation without the consistent use of the NNA. This also implies that, in that case, the RE representation below does not exist.

2.2 General Formulation of Linear Diagnostic Expectations Model, and Solution

2.2.1 Exogenous Processes.

We start by specifying the exogenous drivers of the economy. Exogenous variables are stacked in a $(n \times 1)$ vector $\mathbf{x}_t$ that is assumed to follow the AR(1) stochastic process

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{v}_t \tag{5}$$

where $\mathbf{v}_t$ is a $(k \times 1)$ vector of Gaussian and orthogonal exogenous shocks $\mathbf{v}_t \sim N(0, \Sigma_{\mathbf{v}})$, and A is a diagonal matrix of persistence parameters.

Following Bordalo et al. (2018), we make a no-news assumption for this multivariate setup.

Assumption 2 (Multivariate No-News Assumption)

$$f(\mathbf{x}_{t+1}| - G_t) = f(\mathbf{x}_{t+1}|\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1})$$

2.2.2 Stochastic Difference Equation

The class of forward-looking models we analyze is written as a stochastic difference equation. To this end, let $\mathbf{y}_t$ denote a $(m \times 1)$ vector of endogenous variables (including jump variables and states) and $\mathbf{x}_t$, as above, denote the $(n \times 1)$ vector of exogenous states. The model is:

$$\mathbb{E}_t^{\theta}[\mathbf{F}\mathbf{y}_{t+1} + \mathbf{G_1}\mathbf{y}_t + \mathbf{M}\mathbf{x}_{t+1} + \mathbf{N_1}\mathbf{x}_t] + \mathbf{G_2}\mathbf{y}_t + \mathbf{H}\mathbf{y}_{t-1} + \mathbf{N_2}\mathbf{x}_t = 0 \tag{6}$$

where $\mathbf{F}$, $\mathbf{G_1}$, $\mathbf{G_2}$, $\mathbf{M}$, $\mathbf{N_1}$, $\mathbf{N_2}$, and $\mathbf{H}$, are matrices of parameters. $\mathbf{F}$, $\mathbf{G_1}$, $\mathbf{G_2}$, and $\mathbf{H}$ are $(m \times m)$ matrices, $\mathbf{N_1}$ and $\mathbf{N_2}$ are $(m \times n)$ matrices. $\mathbb{E}_t^{\theta}[\ \cdot\ ]$, as above, denotes the diagnostic expectation operator, which is now taken over *endogenous and exogenous* variables. Notice that in writing model (6), we were careful in allowing both the expectation of time t variables (e.g. $\mathbb{E}_t^{\theta}[\mathbf{N_1}\mathbf{x}_t]$), and the variables themselves (e.g.

$\mathbf{N_2 x}_t$). This is because at the stage of loglinearization we may encounter expressions of this form, and at this stage, we have not invoked linearity of the DE operator yet.[11]

2.2.3 Solution Procedure

The remaining steps are the following. First, postulate a form for the solution. Second, obtain a rational expectations representation of the model. Third, solve for the model expressed in terms of rational expectations using standard tools (as the method of undetermined coefficients, for instance).

Form of the Solution. We look for a solution of the form

$$\mathbf{y}_t = \mathbf{P}\mathbf{y}_{t-1} + \mathbf{Q}\mathbf{x}_t + \mathbf{R}\mathbf{v}_t \tag{7}$$

We make this guess based on the extrapolative nature of DE. Under (7), $\mathbf{y}_t$ follows a multivariate Gaussian distribution. Thus, using strong additivity (Proposition 1), we can write equation (6) in the more convenient form[12]

$$\mathbf{F}\mathbb{E}_t^\theta[\mathbf{y}_{t+1}] + \mathbf{G_1}\mathbb{E}_t^\theta[\mathbf{y}_t] + \mathbf{G_2}\mathbf{y}_t + \mathbf{H}\mathbf{y}_{t-1} + \mathbf{M}\mathbb{E}_t^\theta[\mathbf{x}_{t+1}] + \mathbf{N_1}\mathbb{E}_t^\theta[\mathbf{x}_t] + \mathbf{N_2}\mathbf{x}_t = 0$$

We now can obtain the representation of the model in terms of rational expectations.

Proposition 2 (Multivariate Rational Expectations Representation) *Assume the multivariate NNA (Assumption 2). Model (6) admits the following rational expectations representation:*

$$\begin{aligned} \mathbf{F}\mathbb{E}_t[\mathbf{y}_{t+1}] + \mathbf{G}\mathbf{y}_t + \mathbf{H}\mathbf{y}_{t-1} + \mathbf{M}\mathbb{E}_t[\mathbf{x}_{t+1}] + \mathbf{N}\mathbf{x}_t \\ +\mathbf{F}\theta\big(\mathbb{E}_t[\mathbf{y}_{t+1}] - \mathbb{E}_{t-1}[\mathbf{y}_{t+1}]\big) \\ +\mathbf{M}\theta\big(\mathbb{E}_t[\mathbf{x}_{t+1}] - \mathbb{E}_{t-1}[\mathbf{x}_{t+1}]\big) \\ +\mathbf{G}_1\theta\big(\mathbf{y}_t - \mathbb{E}_{t-1}[\mathbf{y}_t]\big) \\ +\mathbf{N}_1\theta\big(\mathbf{x}_t - \mathbb{E}_{t-1}[\mathbf{x}_t]\big) = 0 \end{aligned} \tag{8}$$

where $\mathbf{G} = \mathbf{G}_1 + \mathbf{G}_2$ *and* $\mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2$. *Moreover, this representation is unique.*

[11] It is possible that the linear model, in its original form, is written with this expectation broken up into different terms $\mathbb{E}_t^\theta[\mathbf{F}\mathbf{y}_{t+1} + \mathbf{G_1}\mathbf{y}_t + \mathbf{M}\mathbf{x}_{t+1}] + \mathbb{E}_t^\theta[\mathbf{N_1}\mathbf{x}_t]$, say, or with sums of expectations that involve the same variables, i.e. $\mathbb{E}_t^\theta[\mathbf{F_1}\mathbf{y}_{t+1}] + \mathbb{E}_t^\theta[\mathbf{F_2}\mathbf{y}_{t+1}]$, for example. As we will explain below, due to the structure of the solution, the additivity property established by Proposition 1 will render these issues moot.

[12] Together with the property of the diagnostic expectation that for any constant c and random variable Z_{t+1}, $\mathbb{E}_t^\theta[cZ_{t+1}] = c\mathbb{E}_t^\theta[Z_{t+1}]$, which follows from the theorem of the expectation of a monotonic transformation of a random variable.

Armed with this representation, we verify that equation (7) is a solution. Appendix B presents the detailed steps to arrive at the solution matrices following the presentation by Uhlig (1995).

2.3 Stability

It turns out that the model under DE is subject to the same stability conditions as the model under RE. More precisely, consider the same model above, but under rational expectations:

$$\mathbf{F}\mathbb{E}_t[\mathbf{y}_{t+1}] + \mathbf{G}\mathbf{y}_t + \mathbf{H}\mathbf{y}_{t-1} + \mathbf{M}\mathbb{E}_t[\mathbf{x}_{t+1}] + \mathbf{N}\mathbf{x}_t = 0 \tag{9}$$

where the matrices $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$, $\mathbf{M}$ and $\mathbf{N}$ are defined above. The following result holds.

Proposition 3 (Stability) *Assume a bounded solution exists for the DE model given by equations* (5) *and* (6)*. The stability conditions for this DE model are identical to the stability conditions for the RE model given by* (5) *and* (9)*.*

While the stability conditions are exactly same as under the RE model, we note that the existence of a bounded solution under DE requires an additional assumption. We formalize this requirement in the following proposition.

Proposition 4 (Existence of a Bounded Solution) *Assume a bounded solution exists for the RE model given by equations* (5) *and* (6) *with* $\theta = 0$*. Then a bounded solution for the DE model exists if* $(1+\theta)\mathbf{FP} + \mathbf{G} + \theta\mathbf{G}_1$ *is full-rank.*

Example 1 below will illustrate how DE may affect the existence of a bounded solution, even when RE models have a bounded and stable solution.

2.4 Examples: Endogenous Extrapolation

These two examples illustrate how DE generate endogenous extrapolation in dynamic models. (Example 1 also discusses unbounded solutions; example 2 also discusses the loglinearization of equations with non-stationary variables.)

2.4.1 Example 1: Univariate Endogenous State Variable Model

Consider the following model:

$$y_t = a\mathbb{E}_t^\theta[y_{t+1}] + cy_{t-1} + \varepsilon_t$$

where $|a + c| < 1$ and ε_t is white noise.

The solution of the RE model ($\theta = 0$) can be derived analytically using the minimum state variable solution method:

$$y_t = \phi_1 \; y_{t-1} + \frac{1}{1 - a\phi_1}\varepsilon_t$$

where $\phi_1 \equiv \frac{1-\sqrt{1-4ac}}{2a}$.[13] Under DE, the minimum state variable solution is given by

$$y_t = \phi_1 \; y_{t-1} + \frac{1}{1 - (1+\theta)a\phi_1}\varepsilon_t \tag{10}$$

We get two conclusions from these calculations.

First, notice from equation (10) that computing the DE over the endogenous variable y_{t+1} delivers extrapolation, even though the exogenous process is i.i.d. When diagnostic expectations are imposed on endogenous variables, then the solution of the model is altered even with i.i.d. shocks. If instead the diagnostic expectations were imposed on exogenous shock processes only, then the RE and the DE solution would coincide with i.i.d. shocks. This example illustrates qualitatively how modeling diagnostic expectations on endogenous variables provides an internal propagation mechanism for DSGE models.

Second, with this example, we can also illustrate the result obtained in Proposition 4: When $\theta \to \frac{1}{a\phi_1} - 1$ or $\theta \to \infty$, then the DE solution explodes even though there exists a unique bounded RE solution. The lesson of this example is therefore that in practice the researcher may need to be mindful of bifurcation points. In particular, bifurcation values may affect search over the parameter space in the context of structural estimation. In our application to NK models, we compute the conditions such that the DE solution explodes, and verify that the associated limit values for θ are very large. Therefore, this does not materially affect our results.

2.4.2 Example 2: Nominal Euler Equation

Consider the following Euler equation of a nominal economy:[14]

$$\frac{u'(C_t)}{P_t} = \beta(1 + i_t)\mathbb{E}_t^\theta \left[\frac{u'(C_{t+1})}{P_{t+1}}\right] \tag{11}$$

[13] Specifically, using the method of undetermined coefficients, we get the following requirement: $\phi_1 = a \; \phi_1^2 + c$. Imposing that $\phi_1 \to 0$ as $c \to 0$, we arrive at the solution. $|a + c| < 1$ ensures that the model is stable in the sense of Proposition 3 and that the RE solution is bounded.

[14] Section 3 derives this equation from first principles.

where C_t is consumption, P_t is the price level, i_t is the nominal rate, $u(\cdot) = \log(\cdot)$ is period utility, and β is the discount factor.

Loglinearizing:

$$\hat{c}_t = \mathbb{E}_t^\theta \left[\hat{c}_{t+1}\right] - (\hat{i}_t - (\mathbb{E}_t^\theta[\hat{p}_{t+1}] - \hat{p}_t))$$

where $\{\hat{c}_t, \hat{i}_t, \hat{p}_t\}$ denote loglinear deviations of consumption and the interest rate from their respective steady states, and of the price level from an initial price level, respectively. Using the BGS formula (4) and algebraic manipulation delivers the loglinear diagnostic Euler equation[15]

$$\hat{c}_t = \mathbb{E}_t^\theta[\hat{c}_{t+1}] - (\hat{i}_t - \mathbb{E}_t^\theta[\hat{\pi}_{t+1}]) + \theta(\hat{\pi}_t - \mathbb{E}_{t-1}[\hat{\pi}_t]) \tag{12}$$

Notice that current inflation induces an *expansionary* channel by reducing the subjective real rate computed by diagnostic agents. The reason is as follows. Due to path dependence, computation of a real rate of interest involves the price level at $t-1$.[16] Since the agent is extrapolating from yesterday $(t-1)$ into tomorrow $(t+1)$, today's inflation innovation $\hat{\pi}_t - \mathbb{E}_{t-1}[\hat{\pi}_t]$ is extrapolated into tomorrow: Current surprise inflation causes the diagnostic agent to expect future inflation, to a degree θ, thereby reducing the subjective real interest rate. Furthermore, this effect is present even in the case of *i.i.d. shocks*, once again highlighting the endogenous extrapolation generated by *equilibrium* diagnosticity. Presence of this endogenous extrapolation term underscores the importance of computing DE on endogenous variables. We will exploit this channel in Section 3 by emphasizing its implications for fiscal policy.

2.5 A Practical Guide to the Implementation of Diagnostic Expectations in DSGE Models

We conclude this section with the following summary. A researcher interested in using diagnostic expectations within a (loglinear) DSGE model can take the following simple steps.

1. Obtain the exact equilibrium conditions of the model. (Section 3 provides an example in the context of a 3-equation NK model, and Section 4 in the context

[15] See Appendix C.1.1.

[16] To see this, multiply on both sides of (11) by P_{t-1} and use P_t inside the DE to obtain:

$$u'(C_t)\frac{P_{t-1}}{P_t} = \beta(1+i_t)\mathbb{E}_t^\theta\left[u'(C_{t+1})\frac{P_{t-1}}{P_t}\frac{P_t}{P_{t+1}}\right]$$

which can then be loglinearized to arrive at (12), after using strong additivity.

of a medium-scale DSGE model.)

2. Loglinearize the model, being careful not to introduce contemporaneous variables in-and-out of the DE operator. (See the appendix for examples.)
3. Obtain the RE representation of the model (Proposition 2) using the additivity result.
4. Use any solver for RE models that can handle expectations conditional on previous period's information set ($\mathbb{E}_{t-1}[\,\cdot\,]$). (For instance, in Dynare, this is accomplished using the EXPECTATION(-1) command.)
5. Check that the parameter space considered does not cover bifurcation values (Proposition 4 and Example 1).

3 Analysis Using a New Keynesian Model

In this section, we derive a three-equation New Keynesian model augmented by diagnostic expectations. Our goal is to revisit a number of prominent themes in this context.

3.1 Diagnostic New Keynesian Model

We set up the model from first principles. We discuss a number of novel aspects that arise due to DE, such as the possibility of time-inconsistency due to the behavioral distortion, and the need to take belief path dependence into consideration to obtain the loglinear approximation.

There are three sets of agents in the economy: households, firms and the government.

3.1.1 Households

Households maximize the following lifetime utility

$$\log C_t - \frac{\omega}{1+\nu} L_t^{1+\nu} + \mathbb{E}_t^{\theta}\left[\Sigma_{s=t+1}^{\infty}\beta^{s-t}\left(\log(C_s) - \frac{\omega}{1+\nu}L_s^{1+\nu}\right)\right]$$

where L_t is labor supply, $\nu > 0$ is the inverse of the Frisch elasticity of labor supply, β is the discount factor β, satisfying $0 < \beta < 1$, $\omega > 0$ is a parameter that pins down the steady-state level of hours. Maximization is subject to a budget constraint:

$$P_t C_t + \frac{B_{t+1}}{(1+i_t)} = B_t + W_t L_t + D_t + T_t$$

where P_t is the price level, B_{t+1} is the demand of nominal bonds that pay off $1 + i_t$ interest rate in the following period, W_t is the wage, D_t and T_t are dividends from firm-ownership and lump-sum government transfers, respectively.

Notice that we write dynamic maximization problems, as this one, by explicitly separating time t choice variables from the expectation of future choice variables. This separation is crucial for solving the model with diagnostic expectations, and is a consequence of the DE path dependence discussed in Section 2.[17]

3.1.2 Firms

Monopolistically competitive firms, indexed by $j \in [0, 1]$, produce a differentiated good, $Y_t(j)$. We assume a Dixit-Stiglitz aggregator that aggregates intermediate goods into a final good, Y_t. Intermediate goods' demand is given by $Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t$, where $\epsilon_p > 1$ is the elasticity of substitution, $P_t(j)$ is the price of intermediate good j, and P_t is the price of final good Y_t. Each intermediate good is produced using the technology $Y_t(j) = A_t L_t(j)$, where $\hat{a}_t \equiv \log(A_t)$ is an aggregate TFP process that follows an AR(1) process with persistence coefficient ρ_a:

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{a,t}$$

and $\varepsilon_{a,t} \sim iid\, N(0, \sigma_a^2)$. The firm pays a quadratic adjustment cost $\frac{\psi_p}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)^2 P_t Y_t$, in units of the final good (Rotemberg 1982) to adjust prices. Firms' per period profits are given by $D_t \equiv P_t(j)Y_t(j) - W_t L_t(j) - \frac{\psi_p}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)^2 P_t Y_t$. The firm's profit maximization problem is

$$\max_{P_t(j)} \left\{ P_t(j)Y_t(j) - W_t L_t(j) - \frac{\psi_p}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)^2 P_t Y_t + \mathbb{E}_t^\theta \left[\sum_{s=1}^{\infty} \beta^s Q_{t,t+s} D_{t+s}\right]\right\}$$

where $Q_{t,t+s}$ is the household's nominal stochastic discount factor.

3.1.3 Government

The government sets nominal interest rate with the following rule $1 + i_t = (1 + i_{ss})\Pi_t^{\phi_\pi}\left(\frac{Y_t}{Y_t^*}\right)^{\phi_x}$, where $Y_t^* = A_t$ is the natural rate allocation, $i_{ss} = \frac{1}{\beta} - 1$ is the

[17] The reader may wonder whether DE introduces time inconsistency in agents' choices. It turns out that this is not the case in the loglinear approximation when the reference distribution is based on $t-1$. By the law of iterated expectations (which then holds for the diagnostic expectation), time $t+1$ policy functions are in fact consistent with agents' expectations (about their time $t+1$ policy functions). The paper by Bianchi, Ilut, and Saijo (2021) analyzes the interesting implications of time-inconsistency in linear models when the agent's reference distribution is more than one-period backward looking.

steady state nominal interest rate, $\phi_\pi \geq 0$, $\phi_x \geq 0$, and steady state inflation $\Pi = 1$. Total output produced is equal to household consumption expenditure and adjustment costs spent when adjusting prices. We first consider a model where is no government spending, and nominal bonds are in zero net supply.[18]

3.1.4 Equilibrium

Appendix C presents the equilibrium conditions. (The DE operator is the expectation over a continuous density, hence one gets these first-order conditions by taking derivatives, as usual.) In particular, it shows that the household intertemporal first order condition is equation (11). This appendix also goes over the log-linear approximation in detail. The resulting equilibrium is given by following three equations:

$$\hat{y}_t = \mathbb{E}_t^\theta\left[\hat{y}_{t+1}\right] - (\hat{i}_t - (\mathbb{E}_t^\theta[\hat{p}_{t+1}] - \hat{p}_t)) \tag{13}$$

$$\hat{\pi}_t = \beta \mathbb{E}_t^\theta[\hat{\pi}_{t+1}] + \kappa(\hat{y}_t - \hat{a}_t) \tag{14}$$

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_x(\hat{y}_t - \hat{a}_t) \tag{15}$$

where $\kappa \equiv \frac{\epsilon_p - 1}{\psi_p}(1+\nu)$, $\hat{y}_t$ is the log deviation of output, $\hat{p}_t$ is the log deviation of the price level, $\hat{i}_t$ is the log deviation of the interest rate, and $\hat{\pi}_t$ is the log deviation of inflation from the zero-inflation steady state. The shock process is given by:

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{a,t} \tag{16}$$

where $\varepsilon_{a,t} \sim i.i.d.\ N(0, \sigma_a^2)$.

As explain in the context of Example 2 in Section 2, equation (13) can be written as (12), showing that DE change the expression for the approximated Euler equation by adding an extra term.[19] Mathematically, this is a consequence of path dependence.[20] The economic implication is endogenous extrapolation of inflation from $t-1$ to $t+1$, as explained in Example 2, Section 2.

Notwithstanding path dependence, we obtain a similar Phillips curve to the RE case using Rotemberg (1982) pricing. The key to this result is that, different than with Calvo pricing, Rotemberg pricing with DE allows one to obtain a recursion that only involves one expectation forward. This turns out to be key for tractability. The appendix presents the detailed derivation.

[18]In the fiscal multiplier analysis below we will introduce government spending shocks.

[19]Since there is no government spending, $\hat{y}_t = \hat{c}_t$.

[20]Indeed, as shown in the appendix, the loglinearization of the exact Euler equation involves P_{t-1}, since, for any set of random variables X_{t+1} and Y_{t-1}, $\mathbb{E}_t^\theta[X_{t+1}Y_{t-1}] = \mathbb{E}_t^\theta[X_{t+1}] \cdot Y_{t-1}$. A similar operation is not allowed with Y_t.

We make the following assumption in order to guarantee the existence of a bounded solution (Proposition 4).[21]

Assumption 3 (Boundedness) $\theta < \phi_\pi + \kappa^{-1}(1 + \phi_x)$

We provide an explicit solution for the model in Appendix C.

3.2 Diagnostic Expectations and the Possibility of Extra Amplification

A classic challenge in macroeconomic modeling is finding ways to generate realistic business cycles with shocks of moderate size. The literature has relied on multiple types of frictions (e.g. nominal, as in Christiano, Eichenbaum, and Evans 2005, or financial, as in Bernanke and Gertler 1989; Kiyotaki and Moore 1997), interactions in the form of strong complementarities (Benhabib and Farmer 1994), or multiple shocks (Smets and Wouters 2007) to fit this pattern of the data.

We demonstrate that diagnosticity provides a viable behavioral alternative to understand the large size of observed fluctuations within the NK model. Because diagnosticity leads agents to extrapolate the impact of exogenous shocks, expectations are more volatile. Intuitively, one would expect the DE-NK model to predict a higher volatility of output than under RE. Indeed, the following proposition establishes that when diagnosticity is strong enough, it does generate extra endogenous volatility in the NK model. We are able to analytically prove this result when prices are completely rigid ($\psi_p \to \infty$).[22]

Proposition 5 (Extra Volatility: NK Model) *Consider the model given by* (13)-(16)*. Assume that $\psi_p \to \infty$ and that the diagnosticity parameter is high enough, that is, $\theta > 2(1 - \rho_a)(1 + \phi_x)/(\phi_x \rho_a)$. Then, output is more volatile under diagnostic expectations than that under rational expectations: $Var(\hat{y}_t)_{DE} > Var(\hat{y}_t)_{RE}$.*

For standard parameter values, this condition is satisfied. For example, when $\rho_a = 0.9$ and $\phi_x = 0.5$, this condition requires that θ be greater than 0.67 for diagnostic expectations to generate extra volatility. Most estimated values for this parameter provided in Bordalo, Gennaioli, Ma, and Shleifer (2020), and used in Bordalo, Gennaioli, Shleifer, and Terry (2021) are well above 0.67.

[21] We also assume that $\kappa(\phi_\pi - 1) + (1 - \beta)\phi_x > 0$ to ensure a stable solution in the sense of Proposition 3.

[22] Away from this limit, we can use the solution of the model presented in the appendix and obtain a condition for extra volatility, but this condition is messy and does not lend itself to any clear interpretation.

The analytical results noted in the case of completely rigid prices also hold when prices are sticky (but not-completely rigid), i.e. when $0 < \psi_p < \infty$. To numerically demonstrate this, we use a standard calibration of the NK model.[23] θ is set to 1 following Bordalo, Gennaioli, Shleifer, and Terry (2021). We obtain a standard deviation of output of 2.96%, relative to 1.82% under RE. Thus, output volatility increases by 63% due to DE.

DE interact with the nominal frictions embedded in the NK model in order to generate extra output volatility. In order to demonstrate this, we consider the polar opposite case of a frictionless real business cycle (RBC) model.[24] In this case, the standard deviation of output is actually lower under DE (1.88%) than under RE (2.04%). In order to clarify that this does not depend on the particular calibration used for the simulation but it is a general property of the model, we also present the following analytical result in the tractable case of full depreciation.

Proposition 6 (Extra Volatility: RBC Model) *Consider the model given by* (16), (47)-(53)*. Assume that the depreciation rate* $\delta = 1$ *and that* $\rho_a = 0$*. Output is less volatile under DE than under RE:* $Var(\hat{y}_t)_{DE} < Var(\hat{y}_t)_{RE}$.

In order to understand these results, it is useful to draw a parallel to the news shocks literature originating in the seminal work by Beaudry and Portier (2004) and Beaudry and Portier (2006). The addition of DE to the NK model can be seen as a way of generating errors in expectations that resemble news about the future. For instance, in the case of a positive TFP shock, agents extrapolate this shock, expecting a further positive TFP shock in the next period. Therefore, the TFP shock generates a contemporaneous raise in TFP, and an excessive increase in expectations about TFP in the next period. As discussed in this literature, shocks to expectations generally have difficulties in generating amplification and comovement in a baseline, frictionless, RBC model (Beaudry and Portier 2006; Jaimovich and Rebelo 2009). Indeed, in the case of a positive news shock, the implied income effect produces a fall of labor supply and hence output (Barro and King 1984). However, as shown in Blanchard, L'Huillier, and Lorenzoni (2013), nominal rigidities are a solution to this counterfactual prediction of the RBC model. When prices are sticky, output is demand determined: The positive income effect raises consumption and in general equilibrium this effect dominates. Output ultimately increases. This explains the extra volatility afforded by the DE-NK model.

[23]Following Galí (2015), we set $\beta = 0.99$, $\epsilon_p = 9$, $\phi_\pi = 1.50$, and $\phi_x = 0.5$. We set $\nu = 2$, and ψ_p such that $\kappa = 0.050$. The TFP process is calibrated with persistence 0.90 and standard deviation of 2%.

[24]The model is standard. See Appendix D for a detailed exposition. The calibration is presented in Appendix E.

We note that the recent important paper by Bordalo, Gennaioli, Shleifer, and Terry (2021) presents another case in which DE interact with frictions to generate extra volatility. The paper looks at an RBC model with financial frictions on the firm side. Firms are heterogeneous. The paper shows that the interaction of firms' expectations with financial frictions successfully generate amplification of investment and output dynamics, and fits a number of facts relating to credit cycles.

3.3 Keynesian Supply Shocks

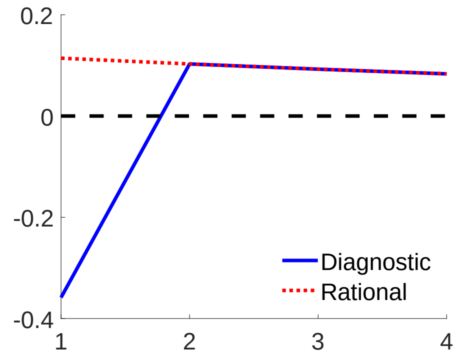
Motivated by economic crisis caused by the COVID-19 pandemic, a rapidly growing literature focuses on constructing models that have the ability to generate a demand shortfall that is fundamentally caused by a disruption on the supply side of the economy, that is, a 'Keynesian' supply shock. Thus far, some of the candidate explanations for this phenomenon include multiple consumption goods (Guerrieri, Lorenzoni, Straub, and Werning 2020), endogenous firm-entry (Bilbiie and Melitz 2020), heterogenous risk-tolerance (Caballero and Simsek 2020), and endogenous TFP growth (Fornaro and Wolf 2020). As the following proposition shows, DE present a behavioral mechanism capable of producing Keynesian supply shocks.

Proposition 7 (Keynesian Supply Shocks) *Consider the model given by* (13)-(16). *Assume that* $\psi_p \to \infty$ *and that the diagnosticity parameter is high enough, that is,* $\theta > 2(1-\rho_a)(1+\phi_x)/(\phi_x\rho_a)$. *Then, the output gap* $\hat{x}_t$ *positively co-moves with the unanticipated component of TFP:* $\frac{\partial \hat{x}_t}{\partial \varepsilon_{a,t}} > 0$.

Similar to Proposition 5, the proposition imposes completely rigid prices for tractability. The result extends to the case of moderately rigid prices, as Figure 1 shows. We use the same calibration as for the volatility result above. The figure plots the evolution of the output gap. Following a negative TFP shock, the economy enters a recession: the output gap and employment falls under DE. In the RE case, the output gap moves in the opposite direction.

The key to this striking result is extrapolation: following the shock, agents extrapolate and become excessively pessimistic about future output. This leads to a large drop in consumption, which due to nominal rigidities, leads to contemporaneous fall in output. Due to diagnosticity, expectations become sufficiently pessimistic to induce a fall in output larger than the initial drop in TFP, generating a Keynesian recession. This is in contrast to the result under RE where the fall in TFP, being only transitory, does not lead to a fall in aggregate demand. Hence, there is a boom: lower TFP for

Figure 1: Output Gap Response to a Negative TFP Shock, Baseline NK Model



Notes: The figure depicts the impulse response of the output gap to a unit negative shock to TFP. The productivity shock process is given by equation (16). The blue solid line denote impulses responses with diagnostic expectations, whereas the red dotted line denote responses with rational expectations. The dynamics of employment are exactly the same as the output gap.

the same level of aggregate demand increases the demand for labor; this generates a boom in the labor market, together with a *rise* in the output gap.

3.4 Fiscal Policy Multiplier

Here we address the implications of DE for the size of the fiscal policy multiplier. There are two reasons to do this.

First, given the recent unprecedented fiscal response to the COVID-19 crisis in the U.S. and other countries, understanding the effects of fiscal policy is central. Also, substantial empirical evidence indicates that marginal propensities to consume are large (see Fagereng, Holm, and Natvik 2021, among others), or similarly, that fiscal multipliers are large in the cross section (Nakamura and Steinsson 2014).[25] We show that DE constitute a useful addition to the NK framework, because it generates novel, rich implications for the fiscal multiplier.

Second, this exercise is a natural path for understanding the endogenous extrapolation generated by the diagnostic Fisher equation embedded in equation (11). This endogenous extrapolation channel highlights the implication of the extra term arising due to belief path-dependence, as explained in Example 2, Section 2.

We add government spending shocks to the NK model. To keep the exposition brief, the model is presented in Appendix C. For convenience, we write the diagnostic Fisher equation here:

$$\hat{r}_t = \hat{i}_t - \mathbb{E}_t[\pi_{t+1}] - \theta(\mathbb{E}_t[\pi_{t+1}] - \mathbb{E}_{t-1}[\pi_{t+1}]) - \theta(\pi_t - \mathbb{E}_{t-1}[\pi_t])$$

[25]See Steinsson (2021) for a similar discussion.

Extrapolation implied by DE reduces the real interest rate, and hence leads to higher multipliers.

To make this point in a transparent way, we start by looking at i.i.d. government spending shocks. The reason is that with i.i.d. shocks, there is no exogenous extrapolation.[26] We obtain the following proposition.

Proposition 8 (Fiscal Policy Multiplier) *Consider the model given by* (28)-(30) *and* (32). *Assume that* $\phi_x = 0$ *and that the persistence of the shock* $\rho_g = 0$. *Then:*

1. *Under rational expectations, the fiscal policy multiplier is always strictly less than 1. Under diagnostic expectations, the fiscal policy multiplier is greater than 1 if* $\theta > \phi_\pi$, *and less than 1 if* $\theta < \phi_\pi$.
2. *The fiscal policy multiplier is greater under diagnostic expectations than under rational expectations.*
3. *The fiscal policy multiplier is increasing in* θ, *and tends to infinity as* $\theta \longrightarrow \phi_\pi + \kappa^{-1}$.

Hence, when the degree of diagnosticity is above the reaction parameter of the monetary authority, the multiplier is greater than one. The intuition for this result is as follows. The diagnostic real rate moves, in response to current inflation, due to the endogenous extrapolation (governed by θ), and by the response of the central bank. In the RE benchmark, the multiplier is always smaller than 1 because the central bank moves the nominal rate to dampen the effect of fiscal policy. The condition $\theta > \phi_\pi$ ensures that endogenous extrapolation offsets this dampening.

The degree of diagnosticity parametrizes the multiplier, increasing it above the RE multiplier, and spanning the full range of values to infinity. (We assume that $\phi_x = 0$ in order to get a clean and easy to interpret condition such that the multiplier is greater than 1 in the DE model.[27])

This analytical case highlights that the higher multiplier under DE is only working through the term $\theta(\pi_t - E_{t-1}[\pi_t])$ in the diagnostic Fisher equation. Extrapolation is endogenous, generating the expansionary effect discussed in Example 2 above. Given that the government spending shock is i.i.d., there is no *exogenous* extrapolation of the shock due to diagnosticity.

In order to illustrate a case where the multiplier is greater than 1, we consider a dovish interest rate rule ($\phi_\pi = 1.1$) and a moderately higher diagnosticity parameter of $\theta = 1.5$. Using a persistence of the government shock equal to 0.5 generates a DE

[26] To see this, notice that equation (4) implies, for an AR(1) process, $\mathbb{E}_t^\theta[x_{t+1}] = \rho_x \check{x}_t + \theta \rho_x \check{\varepsilon}_t$.

[27] The general condition is $\theta \geq \phi_\pi + \frac{\phi_x}{(1-\psi)\kappa}$.

multiplier of 1.06, and an RE multiplier of 0.91. Raising the diagnosticity parameter slightly generates much larger multipliers. Furthermore, using a steeper Phillips curve (say, $\kappa = 0.20$) strengthens the endogenous inflation extrapolation channel: the DE multiplier is now 1.13, for an RE multiplier of 0.73.

We conclude this section by noting that DE do not always lead to higher multipliers. When government shocks are persistent, the expectation of future spending crowds out current consumption, reducing output. With DE, expectations of future spending are exaggerated, and can considerably reduce multipliers when persistence is high. To illustrate this, we go back to our baseline calibration. In addition, we set the persistence of the shock to 0.9. In this case, the RE multiplier is 0.17, for a DE multiplier of -0.32. In this simulation, the exogenous extrapolation channel is so strong that it dominates the endogenous extrapolation channel, leading to a negative multiplier.

3.5 Overreaction and Delayed Overreaction

Whether beliefs as measured by surveys feature under- or overreaction is the subject of an important debate in recent literature. Indeed, Coibion and Gorodnichenko (2012) provide evidence of underreaction of consensus forecasts, whereas Bordalo, Gennaioli, Ma, and Shleifer (2020) provide evidence of overreaction at the level of the individual forecaster. Kohlhas and Walther (2020) find that there is overreaction, in some cases, even at the aggregate level. In a complementary way, Angeletos, Huo, and Sastry (2020) stress that at the aggregate level one can observe both under- and overreation. According to them, what matters is the horizon: there is underreaction in the short run, whereas overreaction dominates in the medium run.

The importance of the horizon at which one observes the dynamics of forecasts has also been stressed in an application to stock returns by Bordalo, Gennaioli, La Porta, and Shleifer (2019). The authors stress that the key is to look at the medium-term forecast errors to find evidence of overreaction to news. The explanation is the following. A gradual arrival of news can happen some time after an anticipated event, and a buildup of the overreaction can move forecasts away from the underreaction generated by imperfect information on impact.

Based on the premise by Bordalo et al. (2019), our broad aim in this section is to contribute to this debate by presenting an extension of the NK model in which long-term beliefs are guided by the diagnostic Kalman filter. The key innovation of our setup compared to previous exercises in the literature is that agents form beliefs about a hidden component that features both sizeable *persistence*, and is also *permanent* (in the sense that the underlying process has a unit root.) To model the long-term nature of this hidden object, we calibrate this persistence to a high value, which conceptually connects our exercise to the long-run risks approach (Bansal and Yaron 2004). However, ours is a general equilibrium representative-agent macroeconomic model where consumers are concerned with the long run path of income.

Assume prices are completely rigid. Consumption is pinned down solely by beliefs about long-run income.[28] The information structure is as follows. TFP, in logs, now has a permanent component ζ_t and a temporary component ξ_t. Agents do not observe these components separately. Instead, they observe realized TFP and a noisy signal about the permanent component $s_t = \zeta_t + \varepsilon_{s,t}$ where $\varepsilon_{s,t} \sim i.i.d.\ N(0, \sigma_s^2)$, and form beliefs using the diagnostic Kalman filter introduced by Bordalo et al. (2020).[29]

The following analytical result offers a simple comparison of beliefs about the long-run under a) the diagnostic Kalman filter (DKF), b) the rational Kalman filter (RKF), and c) the full information RE benchmark (FIRE).

Proposition 9 (Overreaction) *Assume that $\psi_p \to \infty$, $\phi_x = 0$, and the persistence of the permanent component $\rho_\zeta = 0$. Consider a positive shock to ζ_t. Then,*

1. *Beliefs about the long-run are greater under the DKF than under the RKF.*
2. *If θ is high enough, beliefs about the long-run under the DKF are greater than under FIRE.*

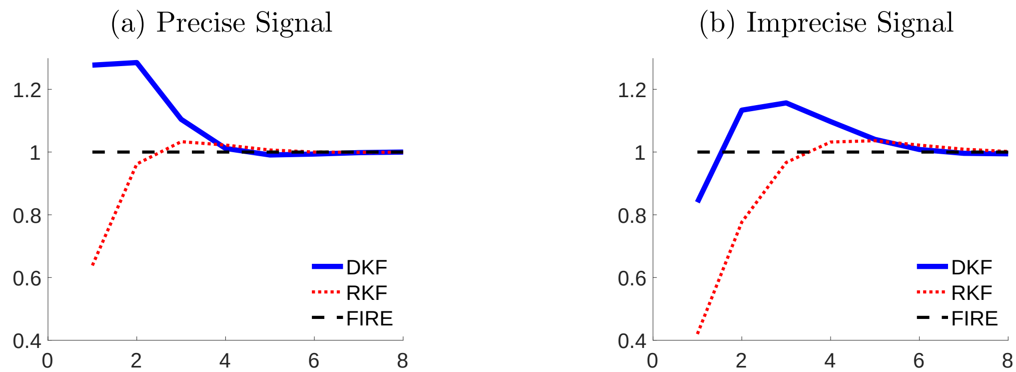
When $\rho_\zeta > 0$, delayed overreaction is possible. We offer a collection of numerical results using the following calibration. In order to capture the idea that the agent is forming beliefs about a very long-run object, we calibrate the persistence of the permanent component to a high value, $\rho_\zeta = 0.98$. We normalize the standard deviation of TFP to 1. We consider two values of the standard deviation of the signal: a relatively precise signal (of standard deviation 0.01), or a relatively imprecise signal (of standard deviation 0.03). Figure 2 presents the dynamics for beliefs about long-run productivity in response to a one standard deviation permanent shock. The left-hand side (LHS) panel presents the case of a precise signal, and the right-hand side (RHS) panel presents the case of an imprecise signal.

Under FIRE, long-run beliefs jump to 1 on impact and stay there. This is because the standard deviation of TFP innovations has been normalized to 1, and beliefs im-

[28] We take the limit $\phi_x \to 0$ and $\psi_p \to \infty$. For brevity we do not write down the equations more explicitly, but this conclusion can be reached by iterating forward the Euler equation.

[29] Even though the model does not explicitly have dispersed information as in Coibion and Gorodnichenko (2012), we follow Lorenzoni (2009) by using a simple representative agent model with aggregate noisy signals. The filter needs to be adapted to the particular information structure here, but the ideas are the same. For details of the model specification, see Blanchard et al. (2013), or Appendix F for a full specification in the context of the medium-scale DSGE.

Figure 2: Impulse Responses: Beliefs About the Long-Run



Notes: The panels depict the impulse responses of beliefs about long-run productivity to a one unit positive shock to the permanent component of TFP. The left-hand side panel presents the case of a precise signal ($\sigma_s = 0.01$ and $\theta = 1.0$); the right-hand side panel with the case of an imprecise signal ($\sigma_s = 0.03$ and $\theta = 1.0$).

mediately adjust to the long-run value of TFP after the shock. In the case of a precise signal (LHS panel), beliefs under the RKF underreact on impact, starting off at 0.70. As learning happens over time, these beliefs rise, gradually converging to 1 in the long run.[30] Instead, beliefs under the DKF strongly overreact on impact. This because the signal is so precise that diagnosticity overwhelms imperfect information.

Turning to the case of an imprecise signal (RHS panel), beliefs under the RKF underreact significantly, starting off at 0.41. Given that now imperfect information is more severe, DKF beliefs also slightly underreact on impact, starting off at 0.84. However, because agents receive a new signal every period, there is gradual learning. Therefore, as they gather more information, DKF implies a sizeable overreaction over periods 2 to 6, with a peak at 1.16. Notice, the RKF also slightly overreacts around period 5. This is due to a mechanical effect induced by the persistence of beliefs. However, diagnosticity induces overreaction above and beyond this mechanical effect.

We conclude by noting that we reported results only varying the precision of the signal. By varying the degree of diagnosticity one modifies the degree of overreaction independently. For instance, increasing θ to 1.5 (which is within the range of estimates reported by Bordalo, Gennaioli, Ma, and Shleifer 2020) can generate a slight overreaction in the short run and a stronger overreaction in the medium run, leading to a hump-shaped pattern of beliefs.

[30] There is a light overreaction in period 3 even in the case of the RKF. This is simply a mechanical implication of the persistence of beliefs inherited from the highly persistent permanent component.

4 Empirical Evaluation

The primary goal of this section is to ask the following question. Consider a baseline, medium scale, rational expectations DSGE model. Add diagnostic expectations to such a model. (The diagnostic model nests the rational expectations model via the diagnosticity parameter.) Is there clear evidence that diagnostic expectations improve the ability of the DSGE model to fit business cycle data?

With this formulation of the broad question that will guide our empirical investigation, three interrelated subquestions emerge: What is the estimated value of the diagnosticity parameter? Does the credible interval span the RE limit? And, ultimately, is there statistical evidence that diagnosticity provides an advantage when fitting business cycle data?

We highlight a number of aspects that discipline this exercise. First, we include a rich set of frictions and shocks in the baseline model. This includes the frictions introduced in the seminal work by Christiano, Eichenbaum, and Evans (2005). We include the exogenous driving processes introduced by Smets and Wouters (2007). In addition, we include information frictions, based on the specification by Blanchard, L'Huillier, and Lorenzoni (2013). By adding all these bells and whistles (nominal, real, and information frictions), and driving processes, we aim to perform a tough test of the usefulness of the behavioral friction embodied by diagnostic expectations. Indeed, we want to assess whether it provides a significant empirical advantage, *even when* all the other commonly used ingredients have already been included.

Second, our procedure is standard, because as we just explained, it employs the set of key frictions introduced by Christiano, Eichenbaum, and Evans (2005), subsequently used by Smets and Wouters (2007), in the context of Bayesian estimation. Also, we solely use the standard set of macroeconomic observables in Bayesian estimation.

Third, we note that the inclusion of information frictions leads to a diagnostic Kalman filter, as introduced by Bordalo, Gennaioli, Ma, and Shleifer (2020). Coibion and Gorodnichenko (2015b) have also emphasized the importance of expectation underreaction in the aggregate, which our information structure is able to account for.

4.1 Medium-Scale DSGE Model

Since the model is standard (Christiano, Eichenbaum, and Evans 2005), we describe here its main ingredients and relegate the details to the appendix. The preferences of the representative household feature habit formation and differentiated labor supply types. In addition to price rigidity, we introduce wage rigidity as in Erceg, Henderson, and Levin (2000). The capital stock is owned and rented by the representative household, and the capital accumulation features a quadratic adjustment cost in investment, as introduced by Christiano et al. (2005). The model features variable capacity utilization.

The final good is a Dixit-Stiglitz aggregate of a continuum of intermediate goods, produced by monopolistic competitive firms, with Rotemberg (1982) costs of price adjustment. Similarly, specialized labor services are supplied under monopolistic competition, with Rotemberg (1982) costs of nominal wage adjustment. The monetary authority sets the nominal interest rate following an inertial Taylor rule.

Following Smets and Wouters (2007) and Blanchard et al. (2013), the model is estimated based on U.S. time series for GDP, consumption, investment, employment, the federal funds rate, inflation, and wages, for the period 1954:III-2011:I. We set up a Kalman filter to get smoothed estimates of the permanent component of productivity and the associated agents' beliefs. We generate 1,500,000 draws using a Metropolis-Hastings algorithm and discard the first 40% as initial burn-in. We set a flat prior centered at 0.85 for the diagnosticity parameter.

4.2 Results

The parameter estimates are reported on Table 1. Estimation results report mean posterior estimates, along with 2.5% and 97.5% percentiles. The bottom row reports the marginal likelihood. Figure 3 plots the path of beliefs about the long-run level of TFP, in both DE and RE cases.[31]

Let us first look at the estimate for the diagnosticity parameter θ. The parameter is estimated at 0.9992. This is close to what is obtained in the previous empirical exercises reported by Bordalo, Gennaioli, Ma, and Shleifer (2020), and to the value used by Bordalo, Gennaioli, Shleifer, and Terry (2021). Figure 5 in the appendix shows that the posterior distribution of θ is unimodal.[32] The 95 percent credible interval covers values from 0.746 to 1.248, away from the RE limit of zero.

We use the Bayes factor to empirically evaluate the fit of the diagnostic model against the rational model. The log marginal likelihood of the data given the estimated diagnostic model is -1584.31. This statistic is -1590.66 in the case of the rational counterpart. Following the suggestion by Kass and Raftery (1995), we compute

[31] We have numerically verified that the parameter space we consider does not contain bifurcation values of θ (the only bifurcation value we have found is for $\theta > 10$.)

[32] We have checked the robustness of this finding to several variations of the prior distribution on θ, including unimodal distributions with a mean both above and below 1. This did not significantly change the resulting posterior distribution of θ.

Table 1: Estimated Parameters

	Parameter	Prior	Posterior	Conf. bands		Distribution	Prior SD
θ	Diagnosticity	0.85	0.9992	0.7462	1.2482	Uniform	0.4907
h	Habit	0.5	0.6970	0.6515	0.7436	Beta	0.1
α	Production function	0.3	0.1466	0.1361	0.1571	Normal	0.05
ν	Inv. Frisch elasticity	2	1.1073	0.5013	1.6057	Gamma	0.75
$\frac{\chi''(1)}{\chi'(1)}$	Capacity utilization cost	5	5.2601	3.6311	6.8841	Gamma	1
$S''(1)$	Investment adjustment cost	4	3.6204	2.6690	4.5390	Normal	1
ψ_p	Price adjustment	100	169.72	140.69	198.32	Normal	25
ψ_w	Wage adjustment	3000	18502.10	13275.99	23764.48	Normal	5000
ϕ_π	Taylor rule inflation	1.5	1.0383	1.0001	1.0735	Normal	0.3
ϕ_x	Taylor rule output	0.005	0.0157	0.0092	0.0223	Normal	0.005
	Technology Shocks and Noise						
ρ		0.6	0.9336	0.9175	0.9500	Beta	0.2
σ_a		0.5	1.1097	1.0176	1.2005	Inv. Gamma	1
σ_s		1	2.2576	1.1171	3.3484	Inv. Gamma	1
	Investment-Specific Shocks						
ρ_μ		0.6	0.7661	0.6778	0.8584	Beta	0.2
σ_μ		5	3.7301	2.5914	4.8255	Inv. Gamma	1.5
	Markup Shocks						
ρ_p		0.6	0.8260	0.7565	0.8965	Beta	0.2
ϕ_p		0.5	0.9611	0.9348	0.9900	Beta	0.2
σ_p		0.15	0.5583	0.4746	0.6407	Inv. Gamma	1
ρ_w		0.6	0.9545	0.9309	0.9797	Beta	0.2
ϕ_w		0.5	0.9661	0.9482	0.9849	Beta	0.2
σ_w		0.15	0.7512	0.6694	0.8286	Inv. Gamma	1
	Policy Shocks						
ρ_R		0.5	0.4743	0.4175	0.5315	Beta	0.2
ρ_{mp}		0.4	0.0353	0.0018	0.0670	Beta	0.2
σ_{mp}		0.15	0.3798	0.3431	0.4149	Inv. Gamma	1
ρ_g		0.6	0.9943	0.9902	0.9986	Beta	0.2
σ_g		0.5	0.3594	0.3311	0.3876	Inv. Gamma	1
	log Marg. Likelihood	-1584.31					

Notes: The table reports mean posterior estimates, along with 2.5% and 97.5% percentiles. We ran 1,500,000 MH draws, discarding the first 40% as initial burn-in. The observation equation is composed of U.S. time series for GDP, consumption, investment, employment, the federal funds rate, inflation, and wages.

$2\log(BF) = 12.70$, which represents very strong evidence in favor of the diagnostic model.[33]

In order to understand the empirical advantage of the DE model in terms of fit, we computed a collection of data moments (volatilities, cross-correlations and autocorrelations for all endogenous variables) and compared these moments to those predicted by both models. We find clear evidence that the DE model provides a superior fit of the first-order autocorrelation of output, consumption, and investment growth, the RE model consistently overestimating the degree of autocorrelation.[34]

We now focus on the amount of extra volatility afforded by DE according to the estimation. DE generate a sizable increase in the volatility of aggregate quantities, such as output, consumption, investment growth. Among these, the largest increase is the one of consumption, with a 36% increase volatility. This result is expected, since, as explained by Blanchard, L'Huillier, and Lorenzoni (2013), expectations matter the most for the determination of consumption in the present model. This volatility is propagated, in general equilibrium, to output and investment, with an observed 23% increase in the standard deviation of both variables. There is a slight decrease of inflation and interest rate volatility.[35]

Turning the attention to the path of beliefs in the DE model, we see in Figure 3 that diagnostic beliefs (dashed, in red) track their rational counterpart (solid, in blue) over medium-run horizons, but at the same time exhibit large short-run volatility. In particular, it is interesting to note that at turning points for rational beliefs (as, for instance, before the 2008 downturn, or the start of the sustained productivity pickup of the 1990s), diagnostic beliefs overshoot considerably, and then revert back to tracking the rational benchmark.[36]

5 Conclusion

In this paper, we argue that diagnostic expectations constitute a behavioral mechanism that can be fruitfully incorporated into New Keynesian macroeconomics. To this end, we first considered a set of challenges encountered by researchers working with this type of models, and revisited them analytically under diagnostic expectations. We

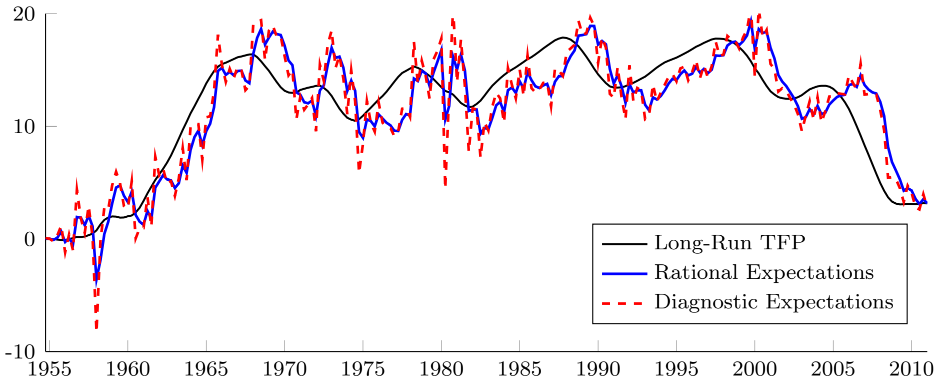
[33]The interpretation of this quantity is that the odds that the data was generated by the diagnostic model instead of the rational model are above 150:1.

[34]Specifically, in the data, the first-order autocorrelation of output, consumption, and investment growth are 0.33, 0.36, and 0.27, respectively; 0.39, 0.26, and 0.54 as predicted by the DE model, respectively; 0.58, 0.59, and 0.70 are predicted by the RE model.

[35]See Appendix F.

[36]For reasons of space, we present a number of complementary results in the appendix.

Figure 3: Long-Run TFP, and Agents' Real-Time Expectations



Notes: The figure plots estimated series using a Kalman smoother. The black solid line denotes the econometrician's smoothed series of the long-run level of TFP ($\hat{a}_{t+\infty|T}$), whereas the blue solid and red dashed lines correspond to the econometrician's smoothed series of the agents' real time beliefs about the long-run level of TFP under RE ($\hat{a}_{(t+\infty|t)|T}$) and DE ($\hat{a}^{\theta}_{(t+\infty|t)|T}$), respectively. Similar to Blanchard, L'Huillier, and Lorenzoni (2013), the aggregate TFP process $\hat{a}_t$ is composed of two components: a permanent component $\hat{\zeta}_t$ and a transitory component $\hat{\xi}_t$: $\hat{a}_t = \hat{\zeta}_t + \hat{\xi}_t$. The long-run level of TFP is given by $\hat{a}_{t+\infty} = (\hat{\zeta}_t - \rho\hat{\zeta}_{t-1})/(1-\rho)$.

concluded that the use of diagnostic expectations opens up avenues to make significant progress in the context of these challenges. We then asked if diagnostic expectations are validated empirically. Using a standard procedure, we conclude that the answer to this question is yes: The diagnostic model dominates the rational counterpart in terms of fit.

Our general solution method offers opportunities to explore and revisit a number of themes in macroeconomics and international macroeconomics in the context of diagnostic expectations. For example, a challenge in open economy models has been to account for the cyclicality of the current account in emerging countries, or to improve our understanding of exchange rate predictability. We leave these explorations to future work.

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A Linearity Results for the Diagnostic Expectation Operator

This appendix significantly expands Section 2 by providing extra, more detailed, material, and proofs. The theme of this appendix is to develop tools for handling the DE operator. We start by writing out some of the expressions in detail, and discussing some of the intuition for the results. The main goal is to build up to the proofs for two key results that allows us to handle linear stochastic recursive models: Proposition 1 (additivity of the DE operator) and Proposition 2 (RE representation). Standard matrix operations to obtain the solution, and associated proofs (needed once the RE representation has been obtained), are discussed in Appendix B.

Explicit Expression for Diagnostic Distribution Under the NNA, and Tractability Intuition. Given (realized) states $\check{x}_t$ and $\check{x}_{t-1}$, the diagnostic probability distribution function of x_{t+1} is

$$f_t^\theta(x_{t+1}) = f(x_{t+1}|x_t = \check{x}_t) \cdot \left[\frac{f(x_{t+1}|x_t = \check{x}_t)}{f(x_{t+1}|x_t = \rho_x \check{x}_{t-1})}\right]^\theta \cdot C \tag{17}$$

When looking at equation (17), it is important to notice that, generically, $\check{x}_t \neq \rho_x \check{x}_{t-1}$ (due to the realization of the shock ε_t.) However, since ε_t is fixed at 0 by the NNA, then

$$f(x_{t+1}|x_t = \rho_x \check{x}_{t-1}) \propto \varphi\left(\frac{x_{t+1} - \rho_x^2 \check{x}_{t-1}}{\sigma_\varepsilon}\right)$$

Thanks to the NNA, the variance of this pdf is σ_ε^2, which is the same as the variance of the true pdf of x_{t+1}. Thus, the true and the reference distributions have the same variance. This allows for tractability, implying that the diagnostic distribution is normally distributed.

A.1 Diagnostic Expectation of Future Variables

We now prove that the diagnostic expectation of a univariate variable can be expressed in terms of rational expectations.

Proof of Equation (4). The diagnostic expectation of x_{t+1} is given by

$$\mathbb{E}_t^\theta[x_{t+1}] = \int_{-\infty}^{\infty} x f_t^\theta(x) dx$$

The diagnostic pdf is given by

$$f_t^\theta(x) = \frac{\left[\frac{1}{\sigma_\varepsilon}\varphi\left(\frac{x-\rho_x\check{x}_t}{\sigma_\varepsilon}\right)\right]^{1+\theta}}{\left[\frac{1}{\sigma_\varepsilon}\varphi\left(\frac{x-\rho_x^2\check{x}_{t-1}}{\sigma_\varepsilon}\right)\right]^{\theta}} C$$

where C is a normalizing constant given by

$$\exp\left\{-\frac{1}{2}\left(\frac{\theta(1+\theta)\rho_x^2\check{x}_t^2 + \theta(\theta+1)\rho_x^4\check{x}_{t-1}^2 - 2(1+\theta)\theta\rho_x^3\check{x}_t\check{x}_{t-1}}{\sigma_\varepsilon^2}\right)\right\}$$

in which case

$$\begin{aligned}\mathbb{E}_t^\theta[x_{t+1}] &= \int_{-\infty}^{\infty} x f_t^\theta(x)dx \\ &= \int_{-\infty}^{\infty} x\frac{1}{\sigma_\varepsilon}\varphi\left(\frac{x-(\rho_x\check{x}_t+\theta(\rho_x\check{x}_t-\rho_x^2\check{x}_{t-1}))}{\sigma_\varepsilon}\right)dx\end{aligned}$$

Thus, the diagnostic distribution $f_t^\theta(x_{t+1})$ is normal with variance σ_ε^2 and mean

$$\mathbb{E}_t^\theta[x_{t+1}] = \mathbb{E}_t[x_{t+1}] + \theta(\mathbb{E}_t[x_{t+1}] - \mathbb{E}_{t-1}[x_{t+1}])$$

■

In this formula, the lagged expectation $\mathbb{E}_{t-1}[x_{t+1}]$ is the expectation conditional on information available at $t-1$, that is, conditional on $\check{x}_{t-1}$. Thus, $\mathbb{E}_t[x_{t+1}] = \rho_x\check{x}_t$ and $\mathbb{E}_{t-1}[x_{t+1}] = \rho_x^2\check{x}_{t-1}$. For the AR(1) process assumed in Section 2.1 and a given realized $\check{\varepsilon}_t$, this proof implies that:

$$\mathbb{E}_t^\theta[x_{t+1}] = E_t[x_{t+1}] + \theta\rho_x\check{\varepsilon}_t > \mathbb{E}_t[x_{t+1}]$$

if and only if $\check{\varepsilon}_t > 0$, that is diagnostic expectations indeed extrapolate the past shock into future beliefs.

Linearity of Diagnostic Expectation Operator. Assume that y_t also follows an AR(1) process:

$$y_t = \rho_y y_{t-1} + \eta_t \tag{18}$$

where $\eta_t \sim i.i.d.N(0, \sigma_\eta^2)$ and ρ_y is a persistence parameter satisfying $\rho_y \in [0,1)$, and σ_η^2 is the shock's variance.

Given this, and the assumption on the AR(1) processes made in Section 2.1, the sum $x_{t+1} + y_{t+1}$ is a normal random variable. The following is a corollary of (4).

Corollary 1 (Univariate RE Representation for a Sum of Random Variables)

$$\mathbb{E}_t^\theta[x_{t+1} + y_{t+1}] = \mathbb{E}_t[x_{t+1} + y_{t+1}] + \theta(\mathbb{E}_t[x_{t+1} + y_{t+1}] - \mathbb{E}_{t-1}[x_{t+1} + y_{t+1}])$$

Hence, we immediately see that, in this case, the DE operator is additive:

$$\mathbb{E}_t^\theta[x_{t+1} + y_{t+1}] = \mathbb{E}_t^\theta[x_{t+1}] + \mathbb{E}_t^\theta[y_{t+1}]$$

A.2 Diagnostic Expectation of Predetermined Variables

A technical challenge that arises in the context of DSGEs is the presence of predetermined variables, and how to handle them in the context of DE. In this appendix, we derive a series of results to maintain linearity. To the best of our knowledge, we are the first to formally address these issues.

Suppose the predetermined variable is y_t. In order to compute the DE of y_t, and of linear combinations of this variable with other variables following AR(1) processes, we will use the Dirac delta function, defined as follows. Suppose that $\check{y}_t$ is the realization of y_t. Since y_t is degenerate, it can be represented by a cumulative distribution function (cdf) with vanishing uncertainty:

$$Pr(y_t \leq \check{y} | y_t = \check{y}_t) = \lim_{\sigma_\eta \to 0^+} \frac{1}{\sigma_\eta} \Phi\left(\frac{\check{y} - \check{y}_t}{\sigma_\eta}\right)$$

This is the probability that y_t is below any given value $\check{y}$, where $\Phi(x)$ is the cumulative distribution function (cdf) of a standard normal random variable:

$$\Phi\left(\check{x}\right) = \int_{-\infty}^{\check{x}} \varphi(x) dx$$

This implies that $Pr(y_t = \check{y}_t) = 1$ and $Pr(y_t \neq \check{y}_t) = 0$, also denoted using the Dirac delta function $\delta(x)$:

$$\delta(x) = \lim_{a \to 0^+} \frac{1}{a} \varphi\left(\frac{x}{a}\right)$$

with the requirement that $\delta(x)$ is a pdf. Using this notation, $\delta(y_t - \check{y}_t)$ is the pdf of y_t, and thus

$$Pr(y_t \leq \check{y} | y_t = \check{y}_t) = \int_{-\infty}^{\check{y}} \delta(y - \check{y}_t) dy$$

is equal to 1 for $\check{y} \geq \check{y}_t$ and equal to 0 otherwise.

There are two equivalent approaches to compute the DE of linear combinations involving predetermined variables.

The first approach rests in the strong additivity result. To prove this result, we first need to compute the DE of a predetermined variable in isolation. Even though agents in the model do not compute the DE of predetermined variables in isolation, mathematically this is a necessary step.[37]

Lemma 1 computes the time-t diagnostic expectation of y_t under the NNA. In this case, the reference distribution of y_t is degenerate, with expectation $\rho_y \check{y}_{t-1}$, where $\check{y}_{t-1}$ is the past realization. We represent this reference distribution by a cdf with vanishing uncertainty, as follows

$$Pr(y_t \leq \check{y} | y_t = \rho_y \check{y}_{t-1}) = \lim_{\sigma_\eta \to 0^+} \frac{1}{\sigma_\eta} \Phi \left(\frac{\check{y} - \rho_y \check{y}_{t-1}}{\sigma_\eta} \right) \tag{19}$$

Lemma 1 (DE of a Degenerate Random Variable Under the NNA)

$$\mathbb{E}_t^\theta [y_t] = \check{y}_t + \theta(\check{y}_t - \rho_y \check{y}_{t-1})$$

Proof (Lemma 1). The diagnostic expectation of y_t is given by

$$\mathbb{E}_t^\theta [y_t] = \int_{-\infty}^{\infty} y f_t^\theta (y) dy$$

In order to get the diagnostic pdf of y_t, we start by looking at the diagnostic cdf, which by virtue of the NNA is

$$Pr_t^\theta (y_t \leq \check{y}) = \lim_{\sigma_\eta \to 0^+} \int_{-\infty}^{\check{y}} \frac{\left[\frac{1}{\sigma_\eta} \varphi \left(\frac{y - \check{y}_t}{\sigma_\eta} \right) \right]^{1+\theta}}{\left[\frac{1}{\sigma_\eta} \varphi \left(\frac{y - \rho_y \check{y}_{t-1}}{\sigma_\eta} \right) \right]^{\theta}} C \ dy$$

First, note that

$$\begin{aligned}
\frac{\left[\frac{1}{\sigma_\eta} \varphi \left(\frac{y - \check{y}_t}{\sigma_\eta} \right) \right]^{1+\theta}}{\left[\frac{1}{\sigma_\eta} \varphi \left(\frac{y - \rho_y \check{y}_{t-1}}{\sigma_\eta} \right) \right]^{\theta}} &= \frac{1}{\sqrt{2\pi}\sigma_\eta} \exp \left\{ -\frac{1}{2} \left[(1+\theta) \left(\frac{y - \check{y}_t}{\sigma_\eta} \right)^2 - \theta \left(\frac{y - \rho_y \check{y}_{t-1}}{\sigma_\eta} \right)^2 \right] \right\} \\
&= \frac{1}{\sqrt{2\pi}\sigma_\eta} \exp \left\{ -\frac{1}{2} \frac{\left[y - \left((1+\theta)\check{y}_t - \theta \rho_t \check{y}_{t-1} \right) \right]^2}{\sigma_\eta^2} \right\} \times \frac{1}{C}
\end{aligned}$$

[37] Hence, some readers will find it useful to think about this as an "as if" approach. That is, agents in the model behaved *as if* they computed the DE of predetermined variables directly.

where the value of C must be

$$C = \exp\left\{-\frac{1}{2}\left[\frac{\theta(1+\theta)\check{y}_t^2 + \theta(1+\theta)\rho_y^2\check{y}_{t-1}^2 - 2\theta(1+\theta)\rho_y\check{y}_t\check{y}_{t-1}}{\sigma_\eta^2}\right]\right\}$$

Hence, we can write

$$\begin{aligned}
\mathbb{E}_t^\theta[y_t] &= \lim_{\sigma_\eta\to 0^+}\lim_{u\to\infty}\int_{-\infty}^{u} y\ \frac{\left[\frac{1}{\sigma_\eta}\varphi\left(\frac{y-\check{y}_t}{\sigma_\eta}\right)\right]^{1+\theta}}{\left[\frac{1}{\sigma_\eta}\varphi\left(\frac{y-\rho_t\check{y}_{t-1}}{\sigma_\eta}\right)\right]^{\theta}}\ Cdy \\
&= \lim_{\sigma_\eta\to 0^+}\lim_{u\to\infty}\int_{-\infty}^{u} y\ \frac{1}{\sigma_\eta}\varphi\left(\frac{y-\big((1+\theta)\check{y}_t - \theta\rho_y\check{y}_{t-1}\big)}{\sigma_\eta}\right)dy \\
&= \lim_{\sigma_\eta\to 0^+}\lim_{u\to\infty}\left\{\int_{-\infty}^{u} \frac{y-\big((1+\theta)\check{y}_t - \theta\rho_y\check{y}_{t-1}\big)}{\sigma_\eta}\varphi\left(\frac{y-\big((1+\theta)\check{y}_t - \theta\rho_y\check{y}_{t-1}\big)}{\sigma_\eta}\right)dy\right. \\
&\left. + \big((1+\theta)\check{y}_t - \theta\rho_y\check{y}_{t-1}\big)\int_{-\infty}^{u}\frac{1}{\sigma_\eta}\varphi\left(\frac{y-\big((1+\theta)\check{y}_t - \theta\rho_y\check{y}_{t-1}\big)}{\sigma_\eta}\right)dy\right\}
\end{aligned}$$

We will evaluate the integral by change of variables. To this end, define $z \equiv \frac{y-((1+\theta)\check{y}_t - \theta\rho_y\check{y}_{t-1})}{\sigma_\eta}$ such that

$$\mathbb{E}_t^\theta[y_t] = \lim_{\sigma_\eta\to 0^+}\lim_{u\to\infty}\left\{\sigma_\eta\int_{-\infty}^{\frac{u-((1+\theta)\check{y}_t-\theta\rho_y\check{y}_{t-1})}{\sigma_\eta}} z\varphi(z)dz + ((1+\theta)\check{y}_t - \theta\rho_y\check{y}_{t-1})\int_{-\infty}^{\frac{u-((1+\theta)\check{y}_t-\theta\rho_y\check{y}_{t-1})}{\sigma_\eta}}\varphi(z)dz\right\}$$

Since $\lim_{\sigma_\eta\to 0^+}\frac{u-((1+\theta)\check{y}_t-\theta\rho_y\check{y}_{t-1})}{\sigma_\eta} = +\infty$ when $u > (1+\theta)\check{y}_t - \theta\rho_y\check{y}_{t-1}$, we have

$$\lim_{\sigma_\eta\to 0^+}\int_{-\infty}^{\frac{u-((1+\theta)\check{y}_t-\theta\rho_y\check{y}_{t-1})}{\sigma_\eta}} z\varphi(z)dz = 0 \quad\text{and}\quad \lim_{\sigma_\eta\to 0^+}\int_{-\infty}^{\frac{u-((1+\theta)\check{y}_t-\theta\rho_y\check{y}_{t-1})}{\sigma_\eta}}\varphi(z)dz = 1$$

and

$$Pr_t^\theta(y_t \le \check{y}) = \lim_{\sigma_\eta\to 0^+}\frac{1}{\sigma_\eta}\Phi\left(\frac{\check{y} - (\check{y}_t + \theta(\check{y}_t - \rho_y\check{y}_{t-1}))}{\sigma_\eta}\right)$$

Thus,

$$f_t^\theta(y_t) = \delta(y_t - (\check{y}_t + \theta(\check{y}_t - \rho_y\check{y}_{t-1})))$$

and

$$\mathbb{E}_t^\theta[y_t] = \check{y}_t + \theta(\check{y}_t - \rho_y\check{y}_{t-1})$$

■

This lemma generalizes the diagnostic expectation representation obtained in Equation (4) to degenerate variables. We highlight that the NNA is crucial for this result.

Later in this section, we show that alternative conditioning sets deliver a different result (cases in which one loses recursion in DSGE models when there are predetermined variables.)

We already mentioned above that agents in the model do not compute the DE of predetermined variables in isolation. Still, the expression given by Lemma 1 probably deserves some discussion. First, it is worth mentioning something we highlighted in Footnote 9 once again. In order to highlight the fact that the agent uses a reference distribution back to $t-1$ more apparent, one is tempted to denote the diagnostic expectation operator as $\mathbb{E}^{\theta}_{t,t-1}[\,\cdot\,]$, at the cost of making the notation heavier. This would also serve to clarify how the formula given by Lemma 1 arises: the formula is simply capturing the path-dependence property of beliefs, which impacts the expression for the expectation even for degenerate variables.

Also, the Lemma is solely presented to clarify what model consistency implies. In fact, notice that the reference distribution of y_t is *also* based on the no-news assumption.[38] Thus, the agent uses a reference distribution whereby the shock η_t is equal to 0 in expectation (equivalently, such that y_t is, in expectation, fully determined by the persistence of the process.)

Thus, the agent's computation of the diagnostic expectation is distorted by agents' memory of the past (which enters through the reference distribution.) Mathematically, the formula is an implication of both the true and the reference distributions of y_t being represented by a Dirac delta function, as formulated fully in Lemma 1. We emphasize that this step is crucial to achieve tractability in these models. Indeed, below we show two alternate approaches that deliver $\mathbb{E}^{\theta}_t[y_t] = \check{y}_t$ by breaking our consistency requirement. In those scenarios, a linear RE representation does not exist. Consequently, our consistency requirement is also valuable for delivering a linear RE representation in general settings.

Having established the results above, we are now in a position to provide the proof for Proposition 1.

Proof (Proposition 1). There are two cases:

- The case $s = r = 1$ follows from the fact that both x_{t+1} and y_{t+1} are normal and therefore Equation (4) and Corollary 1 apply.
- The case of $s = 0$ or $r = 0$ follows from Lemma 1.

■

[38] In the linear model below, this will amount to making the same assumption on shocks, both for future and predetermined variables.

The second approach to the computation of the DE of linear combination involving predetermined variables is to recognize that such linear combinations are also Gaussian, and extend the BGS formula to this case. We establish the validity of this approach as follows. Let us denote the predetermined random variable with y_t. It is useful to first record the following lemma, showing that the sum $x_{t+1} + y_t$ follows a normal distribution.

Lemma 2

$$x_{t+1} + y_t \sim N(\rho_x \check{x}_t + \check{y}_t, \sigma_\varepsilon^2)$$

Proof. We know that

$$x_{t+1} \sim N(\rho_x \check{x}_t, \sigma_\varepsilon^2)$$

To derive the pdf of $z_{t+1} \equiv x_{t+1} + y_t$, we evaluate the convolution

$$f_{z_{t+1}}(z) = \int_{-\infty}^{\infty} f_{x_{t+1}}(x) f_{y_t}(z - x) dx = \int_{-\infty}^{\infty} \frac{1}{\sigma_\varepsilon} \varphi \left(\frac{x - \rho_x \check{x}_t}{\sigma_\varepsilon} \right) \delta(z - x - \check{y}_t) dx$$

where $f_{z_{t+1}}$ is the pdf of z_{t+1}, $f_{x_{t+1}}$ is the pdf of x_{t+1}, and f_{y_t} is the pdf of y_t, and the second equality follows from the fact that x_{t+1} is normally distributed and y_t follows a Dirac delta distribution centered at $\check{y}_t$.

By the symmetry of the Dirac delta function,

$$f_{z_{t+1}}(z) = \int_{-\infty}^{\infty} \frac{1}{\sigma_\varepsilon} \varphi \left(\frac{x - \rho_x \check{x}_t}{\sigma_\varepsilon} \right) \delta(x - z + \check{y}_t) dx$$

and by the sifting property of the Dirac delta function:[39]

$$f_{z_{t+1}}(z) = \frac{1}{\sigma_\varepsilon} \varphi \left(\frac{z - \check{y}_t - \rho_x \check{x}_t}{\sigma_\varepsilon} \right)$$

which is what we wanted to show. ■

Armed with Lemma 2, we then obtain the following rational expectations representation for DE with predetermined variables.

Lemma 3 (Univariate RE Representation with Degenerate Random Variables)

[39] The Dirac delta function's sifting property is the following. For a continuous function $f(x)$ over $(-\infty, \infty)$,

$$\int_{-\infty}^{\infty} f(x) \delta(x - x_0) dx = f(x_0)$$

$$\mathbb{E}_t^\theta[x_{t+1} + y_t] = \mathbb{E}_t[x_{t+1} + y_t] + \theta(\mathbb{E}_t[x_{t+1} + y_t] - \mathbb{E}_{t-1}[x_{t+1} + y_t])$$

Proof. First, we need the reference distribution of $x_{t+1} + y_t$. Under no news, $\varepsilon_t = \eta_t = 0$ and so,

$$x_{t+1} + y_t = \rho_x^2 x_{t-1} + \rho_y y_{t-1} + \varepsilon_{t+1}$$

Then, by an easy extension of Lemma 2,

$$(x_{t+1} + y_t)|\varepsilon_t = \eta_t = 0 \;\sim\; N(\rho_x^2 \check{x}_{t-1} + \rho_y \check{y}_{t-1}, \sigma_\varepsilon^2)$$

It follows that both the reference and representative distributions are normal and have variance σ_ε^2. We then conclude that

$$\mathbb{E}_t^\theta[x_{t+1} + y_t] = \mathbb{E}_t[x_{t+1} + y_t] + \theta(\mathbb{E}_t[x_{t+1} + y_t] - \mathbb{E}_{t-1}[x_{t+1} + y_t])$$

■

We can in fact use this last proposition to compute the expectation of the linear combination for the processes presented in the body. The calculation is as follows:

$$\begin{aligned}
\mathbb{E}_t^\theta[x_{t+1} + y_t] &= \mathbb{E}_t[x_{t+1} + y_t] + \theta(\mathbb{E}_t[x_{t+1} + y_t] - \mathbb{E}_{t-1}[x_{t+1} + y_t]) \\
&= \rho_x \check{x}_t + \check{y}_t + \theta(\rho_x \check{x}_t + \check{y}_t - \rho_x^2 \check{x}_{t-1} - \rho_y \check{y}_{t-1}) \\
&= \rho_x \check{x}_t + \check{y}_t + \theta \rho_x \check{\varepsilon}_t + \theta \check{\eta}_t \\
&= \rho_x \check{x}_t + \check{y}_t + \theta(\rho_x \check{\varepsilon}_t + \check{\eta}_t)
\end{aligned}$$

A.3 Existence and Uniqueness of the Rational Expectations Representation for the General Linear Model

The proof for the existence and uniqueness of the RE representation for model (Proposition 2) now follows from the previous results.

Proof (Proposition 2). In order to prove this result, we first note that (7) implies that $\mathbf{y}_t$ is multivariate Gaussian random variable. $\mathbf{x}_t$ is also a multivariate Gaussian random variable.

As a consequence of this fact we can can evaluate the DE on the multivariate model. There are two equivalent ways of proceeding:

- Apply the strong additivity for DE, Proposition 1, and then use Lemma 1 to compute the DE of predetermined variables. Rearrange terms. Expression (8) follows.

- Apply Lemma 3 and rearrange the terms using the linearity of the RE operator. Expression (8) follows.

Uniqueness follows from the fact that the DE can only be evaluated in a unique way once NNA on the multivariate model (Assumption 2) has been assumed.

■

A.4 Alternative Assumptions for Predetermined Variables

We present two cases where the diagnostic expectation of a predetermined variable y_t at time t is equal to the observed time-t realization $\check{y}_t$, and the diagnostic expectation of a forward looking variable x_{t+1} is given by Equation (4). In these two alternate cases, we then show that one cannot obtain a linear recursive RE representation.

We maintain the no-news assumption for x_{t+1} in this section. That is,

$$\mathbb{E}_t^\theta[x_{t+1}] = \mathbb{E}_t[x_{t+1}] + \theta(\mathbb{E}_t[x_{t+1}] - \mathbb{E}_{t-1}[x_{t+1}]) \tag{20}$$

which is the same as equation (4).

Alternative Assumption 1 for Degenerate Variables. Instead of the NNA (Assumption 1), let us suppose that the reference distribution for the predetermined variable is same as the true distribution. In other words, there is no diagnosticity associated with computing DE for y_t at time t. Since y_t is degenerate, its true distribution can be represented by a cumulative distribution function (cdf) with vanishing uncertainty:

$$Pr(y_t \leq \check{y} | y_t = \check{y}_t) = \lim_{\sigma_\eta \to 0^+} \frac{1}{\sigma_\eta} \Phi\left(\frac{\check{y} - \check{y}_t}{\sigma_\eta}\right)$$

In order to get the diagnostic pdf of y_t, we start by looking at the diagnostic cdf, with the alternate reference distribution

$$Pr_t^\theta(y_t \leq \check{y} | y_t = \check{y}_t, y_t = \check{y}_t) = \lim_{\sigma_\eta \to 0^+} \int_{-\infty}^{\check{y}} \frac{\left[\frac{1}{\sigma_\eta}\varphi\left(\frac{y - \check{y}_t}{\sigma_\eta}\right)\right]^{1+\theta}}{\left[\frac{1}{\sigma_\eta}\varphi\left(\frac{y - \check{y}_t}{\sigma_\eta}\right)\right]^{\theta}} \, dy = Pr(y_t \leq \check{y} | y_t = \check{y}_t)$$

Thus,$f_t^\theta(y) = \delta(y_t - \check{y}_t)$. As a consequence,

$$\mathbb{E}_t^\theta[y_t] = \check{y}_t$$

In order to compute diagnostic expectation $x_{t+1} + y_t$, note that the true distribution for $x_{t+1} + y_t$ is same as shown in Lemma 2. Since we assume that the no-news assumption holds for x_{t+1}, the reference distribution for $x_{t+1} + y_t$ is $N(\rho_x^2 \check{x}_{t-1} + \check{y}_t, \sigma_\varepsilon^2)$. We then obtain the following representation for the diagnostic expectation:

$$
\begin{aligned}
\mathbb{E}_t^\theta[x_{t+1} + y_t] &= (1+\theta)(\rho \check{x}_t + \check{y}_t) - \theta(\rho^2 \check{x}_{t-1} + \check{y}_t) \\
&= (1+\theta)\rho \check{x}_t - \theta \rho^2 \check{x}_{t-1} + \check{y}_t \\
&= \mathbb{E}_t^\theta[x_{t+1}] + \mathbb{E}_t^\theta[y_t]
\end{aligned}
$$

Detailed steps available upon request.

While this alternate assumption seemingly provides a linear representation of DE, it is inconsistent with equation (20). To see this inconsistency, define $z_{t+1} = x_{t+1} + y_t$. Consequently,

$$
\begin{aligned}
\mathbb{E}_t^\theta[z_{t+1}] &= \mathbb{E}_t[z_{t+1}] + \theta(\mathbb{E}_t[z_{t+1}] - \mathbb{E}_{t-1}[z_{t+1}]) \qquad \text{(from equation 20)} \\
&= \mathbb{E}_t[x_{t+1} + y_t] + \theta(\mathbb{E}_t[x_{t+1} + y_t] - \mathbb{E}_{t-1}[x_{t+1} + y_t]) \\
&= (1+\theta)\mathbb{E}_t[x_{t+1}] - \theta \mathbb{E}_{t-1}[x_{t+1}] + (1+\theta)\check{y}_t - \theta \mathbb{E}_{t-1}[y_t] \\
&\neq \mathbb{E}_t^\theta[x_{t+1}] + \mathbb{E}_t^\theta[y_t] = \mathbb{E}_t^\theta[x_{t+1} + y_t]
\end{aligned}
$$

Hence, the linear RE representation does not exist in this case.

Alternative Assumption 2 for Degenerate Variables. Instead of the NNA (Assumption 1), let us suppose that the reference distribution of y_t is the (non-degenerate) normal distribution:

$$
f(y_t | y_{t-1} = \check{y}_{t-1}) \propto \varphi\left(\frac{y_t - \rho_y \check{y}_{t-1}}{\sigma_\eta}\right)
$$

which corresponds to replacing the NNA by the assumption that the conditioning set is $\{y_{t-1} = \check{y}_{t-1}\}$. This is the alternative discussed in Bordalo, Gennaioli, and Shleifer (2018), footnote 8. We highlight that this is an assumption about past y_{t-1} instead of current y_t. Indeed, the NNA embeds an assumption about the shock ε_t, on top of the conditioning on the realization $\check{y}_{t-1}$, resulting in the reference cdf (19) above. In this alternative case, the following lemma obtains.

Lemma 4 *Replace Assumption 1 by* $\{y_{t-1} = \check{y}_{t-1}\}$. *Then,*

$$
\mathbb{E}_t^\theta[y_t] = \check{y}_t
$$

Proof. The diagnostic expectation of y_t is given by

$$\mathbb{E}_t^\theta[y_t] = \int y f_t^\theta(y|y_t = \check{y}_t, y_{t-1} = \check{y}_{t-1})dy$$

Notice that in this notation, since we are not using the NNA, we explicitly write the two conditioning events $G = \{y_t = \check{y}_t\}$ and $-G = \{y_{t-1} = \check{y}_{t-1}\}$. In order to get the diagnostic pdf of y_t, we start by looking at the diagnostic cdf:

$$Pr_t^\theta(y_t \leq \check{y}|y_t = \check{y}_t, y_{t-1} = \check{y}_{t-1}) = \lim_{a\to 0^+} \int_{-\infty}^{\check{y}} \frac{\left[\frac{1}{a}\varphi\left(\frac{y-\check{y}_t}{a}\right)\right]^{1+\theta}}{\left[\frac{1}{\sigma_\eta}\varphi\left(\frac{y-\rho_y\check{y}_{t-1}}{\sigma_\eta}\right)\right]^\theta} C\ dy$$

Notice that this time it is only the uncertainty in the numerator that vanishes. First, note that

$$\frac{\left[\frac{1}{a}\varphi\left(\frac{y-\check{y}_t}{a}\right)\right]^{1+\theta}}{\left[\frac{1}{\sigma_\eta}\varphi\left(\frac{y-\rho_y\check{y}_{t-1}}{\sigma_\eta}\right)\right]^\theta} = \frac{1}{\sqrt{2\pi}\frac{a^{1+\theta}}{\sigma_\eta^\theta}} \exp\left\{-\frac{1}{2}\left[(1+\theta)\left(\frac{y-\check{y}_t}{a}\right)^2 - \theta\left(\frac{y-\rho_y\check{y}_{t-1}}{\sigma_\eta}\right)^2\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_a} \exp\left\{-\frac{1}{2}\frac{(y-\mu_a)^2}{\sigma_a^2}\right\} \times \frac{1}{C}$$

where

$$\mu_a = \frac{\sigma_\eta^2(1+\theta)\check{y}_t - a^2\theta\rho_y\check{y}_{t-1}}{\sigma_\eta^2(1+\theta) - a^2\theta}, \quad \sigma_a^2 = \frac{a^2\sigma_\eta^2}{\sigma_\eta^2(1+\theta) - a^2\theta}$$

and the value of C must be

$$C = \exp\left\{-\frac{1}{2}\left(\frac{\mu_a^2 - k_a}{\sigma_a^2}\right)\right\} \frac{a^{1+\theta}}{\sigma_a\sigma_\eta^\theta}$$

where

$$k_a = \frac{\sigma_\eta^2(1+\theta)\check{y}_t^2 - a^2\theta\rho_y^2\check{y}_{t-1}^2}{\sigma_\eta^2(1+\theta) - a^2\theta}$$

Hence, we can write

$$\begin{aligned}
\mathbb{E}_t^\theta[y_t] &= \lim_{a\to 0^+}\lim_{u\to\infty}\int_{-\infty}^{u} y\ \frac{\left[\frac{1}{a}\varphi\left(\frac{y-\check{y}_t}{a}\right)\right]^{1+\theta}}{\left[\frac{1}{\sigma_\eta}\varphi\left(\frac{y-\rho_y\check{y}_{t-1}}{\sigma_\eta}\right)\right]^{\theta}}\ C\ dy \\
&= \lim_{a\to 0^+}\lim_{u\to\infty}\int_{-\infty}^{u} y\ \frac{1}{\sigma_a}\varphi\bigg(\frac{y-\mu_a}{\sigma_a}\bigg)dy \\
&= \lim_{a\to 0^+}\lim_{u\to\infty}\left\{\int_{-\infty}^{u}\frac{y-\mu_a}{\sigma_a}\varphi\bigg(\frac{y-\mu_a}{\sigma_a}\bigg)dy+\mu_a\int_{-\infty}^{x}\frac{1}{\sigma_a}\varphi\bigg(\frac{y-\mu_a}{\sigma_a}\bigg)dy\right\}
\end{aligned}$$

We will evaluate the integral by change of variables. To this end, define $z\equiv\frac{y-\mu_a}{\sigma_a}$ such that

$$\mathbb{E}_t^\theta[y_t] = \lim_{a\to 0^+}\lim_{u\to\infty}\left\{\sigma_a\int_{-\infty}^{\frac{u-\mu_a}{\sigma_a}} z\varphi(z)dz+\mu_a\int_{-\infty}^{\frac{u-\mu_a}{\sigma_a}}\varphi(z)dz\right\}$$

Notice that

$$\lim_{a\to 0^+}\ \mu_a=\check{y}_t$$

and

$$\lim_{a\to 0^+}\ \sigma_a=0$$

Since $\lim_{a\to 0^+}\frac{u-\mu_a}{\sigma_a}=+\infty$ when $u>\mu_a$, we have

$$\lim_{a\to 0^+}\int_{-\infty}^{\frac{u-\mu_a}{\sigma_a}} z\varphi(z)dz=0\quad\text{and}\quad\lim_{a\to 0^+}\int_{-\infty}^{\frac{u-\mu_a}{\sigma_a}}\varphi(z)dz=1$$

and

$$Pr_t^\theta(y_t\leq\check{y}|y_t=\check{y}_t,y_{t-1}=\check{y}_{t-1})=\lim_{a\to 0^+}\frac{1}{a}\Phi\left(\frac{\check{y}-\check{y}_t}{a}\right)$$

Thus,

$$f_t^\theta(y)=\delta(y_t-\check{y}_t)$$

As a consequence,

$$\mathbb{E}_t^\theta[y_t]=\check{y}_t$$

as we wanted to show. ■

We can then show with some additional steps that the linearity of DE may fail under this alternate assumption on reference distribution for y_t. We obtain the following result

$$\begin{aligned}
\mathbb{E}_t^\theta[x_{t+1} + y_t] &= \frac{(1+\theta)2\mathbb{E}_t[x_{t+1}] - \theta\mathbb{E}_{t-1}[x_{t+1}] + (2+\theta)\check{y}_t}{2+\theta} \\
&\neq (1+\theta)\mathbb{E}_t[x_{t+1} + y_t] - \theta\mathbb{E}_{t-1}[x_{t+1} + y_t] \\
&\neq \mathbb{E}^\theta[x_{t+1}] + \mathbb{E}^\theta[y_t]
\end{aligned}$$

Detailed steps available upon request.

As in the previous case, we can show this RE representation leads to an inconsistency when solving DSGE models. Once again, define $z_{t+1} = x_{t+1} + y_t$. Consequently,

$$\begin{aligned}
\mathbb{E}_t^\theta[z_{t+1}] &= \mathbb{E}_t[z_{t+1}] + \theta(\mathbb{E}_t[z_{t+1}] - \mathbb{E}_{t-1}[z_{t+1}]) \qquad \text{(from equation 20)} \\
&= \mathbb{E}_t[x_{t+1} + y_t] + \theta(\mathbb{E}_t[x_{t+1} + y_t] - \mathbb{E}_{t-1}[x_{t+1} + y_t]) \\
&= (1+\theta)\mathbb{E}_t[x_{t+1}] - \theta\mathbb{E}_{t-1}[x_{t+1}] + (1+\theta)\check{y}_t - \theta\mathbb{E}_{t-1}[y_t] \\
&\neq \mathbb{E}_t^\theta[x_{t+1} + y_t]
\end{aligned}$$

[FOR ONLINE PUBLICATION]

Online Appendix For: "Incorporating Diagnostic Expectations into the New Keynesian Framework"

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August 2021

B Detailed Solution Procedure, Stability, and Boundedness of the Solution: Supplementary Materials and Proofs

Detailed Solution Procedure. We solve for the recursive equilibrium law of motion of a linear diagnostic-expectations DSGE model using the method of undetermined coefficients.

With the strong additivity result from Proposition 1, the class of forward-looking models of our interest is written in the following form:

$$\mathbf{F}\mathbb{E}_t^\theta[\mathbf{y}_{t+1}] + \mathbf{G_1}\mathbb{E}_t^\theta[\mathbf{y}_t] + \mathbf{G_2}\mathbf{y}_t + \mathbf{H}\mathbf{y}_{t-1} + \mathbf{M}\mathbb{E}_t^\theta[\mathbf{x}_{t+1}] + \mathbf{N_1}\mathbb{E}_t^\theta[\mathbf{x}_t] + \mathbf{N_2}\mathbf{x}_t = 0$$

Suppose that there is a unique stable solution of the model:

$$\mathbf{y}_t = \mathbf{P}\mathbf{y}_{t-1} + \mathbf{Q}\mathbf{x}_t + \mathbf{R}\mathbf{v}_t \tag{21}$$

we can rewrite the above stochastic difference equation as follows:

$$\begin{aligned}\mathbf{F}\mathbb{E}_t^\theta\big[\mathbf{P}\mathbf{y}_t + \mathbf{Q}\mathbf{x}_{t+1} + \mathbf{R}\mathbf{v}_{t+1}\big] + \mathbf{G_1}\mathbb{E}_t^\theta\big[\mathbf{P}\mathbf{y}_{t-1} + \mathbf{Q}\mathbf{x}_t + \mathbf{R}\mathbf{v}_t\big] + \mathbf{G_2}\mathbf{P}\mathbf{y}_{t-1} \\ + \mathbf{G_2}\mathbf{Q}\mathbf{x}_t + \mathbf{G_2}\mathbf{R}\mathbf{v}_t + \mathbf{M}\mathbb{E}_t^\theta\big[\mathbf{A}\mathbf{x}_t + \mathbf{v}_{t+1}\big] + \mathbf{N_1}\mathbb{E}_t^\theta\big[\mathbf{x}_t\big] + \mathbf{H}\mathbf{y}_{t-1} + \mathbf{N_2}\mathbf{x}_t = 0\end{aligned}$$

Diagnostic expectations can be represented as a linear combination of the rational expectations held at t and $t-1$:

$$\mathbf{F}\mathbb{E}_t^\theta\big[\mathbf{P}\mathbf{y}_t + \mathbf{Q}\mathbf{x}_{t+1} + \mathbf{R}\mathbf{v}_{t+1}\big] = (1+\theta)\mathbf{F}\mathbb{E}_t\Big[\mathbf{P}^2\mathbf{y}_{t-1} + \mathbf{P}\mathbf{Q}\mathbf{x}_t + \mathbf{P}\mathbf{R}\mathbf{v}_t + \mathbf{Q}\mathbf{A}\mathbf{x}_t + \mathbf{Q}\mathbf{v}_{t+1} + \mathbf{R}\mathbf{v}_{t+1}\Big]$$
$$-\theta\mathbf{F}\mathbb{E}_{t-1}\Big[\mathbf{P}^2\mathbf{y}_{t-1} + \mathbf{P}\mathbf{Q}\mathbf{A}\mathbf{x}_{t-1} + \mathbf{P}\mathbf{Q}\mathbf{v}_t + \mathbf{P}\mathbf{R}\mathbf{v}_t + \mathbf{Q}\mathbf{A}^2\mathbf{x}_{t-1} + \mathbf{Q}\mathbf{A}\mathbf{v}_t + \mathbf{Q}\mathbf{v}_{t+1} + \mathbf{R}\mathbf{v}_{t+1}\Big]$$
$$= \mathbf{F}\mathbf{P}^2\mathbf{y}_{t-1} + \mathbf{F}\mathbf{P}\mathbf{Q}\mathbf{x}_t + \theta\mathbf{F}\mathbf{P}\mathbf{Q}\mathbf{v}_t + (1+\theta)\mathbf{F}\mathbf{P}\mathbf{R}\mathbf{v}_t + \mathbf{F}\mathbf{Q}\mathbf{A}\mathbf{x}_t + \theta\mathbf{F}\mathbf{Q}\mathbf{A}\mathbf{v}_t$$

$$\mathbf{G_1}\mathbb{E}_t^\theta\big[\mathbf{P}\mathbf{y}_{t-1} + \mathbf{Q}\mathbf{x}_t + \mathbf{R}\mathbf{v}_t\big] = (1+\theta)\mathbf{G_1}\mathbb{E}_t[\mathbf{P}\mathbf{y}_{t-1} + \mathbf{Q}\mathbf{x}_t + \mathbf{R}\mathbf{v}_t] - \theta\mathbf{G_1}\mathbb{E}_{t-1}[\mathbf{P}\mathbf{y}_{t-1} + \mathbf{Q}\mathbf{x}_t + \mathbf{R}\mathbf{v}_t]$$
$$= \mathbf{G_1}\mathbf{P}\mathbf{y}_{t-1} + \mathbf{G_1}\mathbf{Q}\mathbf{x}_t + \theta\mathbf{G_1}\mathbf{Q}\mathbf{v}_t + (1+\theta)\mathbf{G_1}\mathbf{R}\mathbf{v}_t$$

$$\mathbf{M}\mathbb{E}_t^\theta\big[\mathbf{A}\mathbf{x}_t + \mathbf{v}_{t+1}\big] = (1+\theta)\mathbf{M}\mathbb{E}_t[\mathbf{A}\mathbf{x}_t + \mathbf{v}_{t+1}] - \theta\mathbf{M}\mathbb{E}_{t-1}[\mathbf{A}\mathbf{x}_t + \mathbf{v}_{t+1}]$$
$$= \mathbf{M}\mathbf{A}\mathbf{x}_t + \theta\mathbf{M}\mathbf{A}\mathbf{v}_t$$

$$\mathbf{N_1}\mathbb{E}_t^\theta\big[\mathbf{x}_t\big] = (1+\theta)\mathbf{N_1}\mathbb{E}_t[\mathbf{x}_t] - \theta\mathbf{N_1}\mathbb{E}_{t-1}[\mathbf{x}_t] = \mathbf{N_1}\mathbf{x}_t + \theta\mathbf{N_1}\mathbf{v}_t$$

We write the model in the rational expectations representation as

$$0 = \mathbf{F}\mathbf{P}^2\mathbf{y}_{t-1} + \mathbf{F}\mathbf{P}\mathbf{Q}\mathbf{x}_t + \theta\mathbf{F}\mathbf{P}\mathbf{Q}\mathbf{v}_t + (1+\theta)\mathbf{F}\mathbf{P}\mathbf{R}\mathbf{v}_t + \mathbf{F}\mathbf{Q}\mathbf{A}\mathbf{x}_t + \theta\mathbf{F}\mathbf{Q}\mathbf{A}\mathbf{v}_t + \mathbf{G_1}\mathbf{P}\mathbf{y}_{t-1} + ...$$
$$+ \mathbf{G_1}\mathbf{Q}\mathbf{x}_t + \theta\mathbf{G_1}\mathbf{Q}\mathbf{v}_t + (1+\theta)\mathbf{G_1}\mathbf{R}\mathbf{v}_t + \mathbf{G_2}\mathbf{P}\mathbf{y}_{t-1} + \mathbf{G_2}\mathbf{Q}\mathbf{x}_t + \mathbf{G_2}\mathbf{R}\mathbf{v}_t + \mathbf{M}\mathbf{A}\mathbf{x}_t + ...$$
$$+ \theta\mathbf{M}\mathbf{A}\mathbf{v}_t + \mathbf{N_1}\mathbf{x}_t + \theta\mathbf{N_1}\mathbf{v}_t + \mathbf{H}\mathbf{y}_{t-1} + \mathbf{N}_2\mathbf{x}_t$$

It is now straightforward to proceed by the method of undetermined coefficients to find a solution of the form (21), and the matrices $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ can be found solving the following matrix equations.

$$\mathbf{F}\mathbf{P}^2 + \mathbf{G}\mathbf{P} + \mathbf{H} = 0 \tag{22}$$

$$\mathbf{F}\mathbf{P}\mathbf{Q} + \mathbf{F}\mathbf{Q}\mathbf{A} + \mathbf{G}\mathbf{Q} + \mathbf{M}\mathbf{A} + \mathbf{N} = 0 \tag{23}$$

$$\theta\mathbf{F}\mathbf{P}\mathbf{Q} + (1+\theta)\mathbf{F}\mathbf{P}\mathbf{R} + \theta\mathbf{F}\mathbf{Q}\mathbf{A} + \theta\mathbf{G_1}\mathbf{Q} + \mathbf{G}\mathbf{R} + \theta\mathbf{G_1}\mathbf{R} + \theta\mathbf{M}\mathbf{A} + \theta\mathbf{N_1} = 0 \tag{24}$$

where $\mathbf{G} = \mathbf{G_1} + \mathbf{G_2}$ and $\mathbf{N} = \mathbf{N_1} + \mathbf{N_2}$.

We can use the techniques discussed in Uhlig (1995) to solve the quadratic matrix equation (22) in $\mathbf{P}$. The solution of the other two equations is straightforward as they are linear in $\mathbf{Q}$ and $\mathbf{R}$: After vectorization, equation (23) becomes

$$(\mathbf{I}_m \otimes \mathbf{FP})vec(\mathbf{Q}) + (\mathbf{A}^T \otimes \mathbf{F})vec(\mathbf{Q}) + (\mathbf{I}_m \otimes \mathbf{G})vec(\mathbf{Q}) + vec(\mathbf{MA}) + vec(\mathbf{N}) = 0$$

such that

$$vec(\mathbf{Q}) = -\Big((\mathbf{I}_m \otimes \mathbf{FP}) + (\mathbf{A}^T \otimes \mathbf{F}) + (\mathbf{I}_m \otimes \mathbf{G})\Big)^{-1} \times (vec(\mathbf{MA}) + vec(\mathbf{N}))$$

$\mathbf{R}$ can be found from (24):

$$\mathbf{R} = -\big((1+\theta)\mathbf{FP} + \mathbf{G} + \theta\mathbf{G_1}\big)^{-1}\big(\theta\mathbf{FPQ} + \theta\mathbf{FQA} + \theta\mathbf{G_1Q} + \theta\mathbf{MA} + \theta\mathbf{N_1}\big)$$

Observe that solution for matrices $\mathbf{P}$ and $\mathbf{Q}$ does not depend on diagnosticity parameter.

The Solution under Rational Expectations. Consider the model under rational expectations:

$$\mathbf{F}\mathbb{E}_t[\mathbf{y}_{t+1}] + \mathbf{G}\mathbf{y}_t + \mathbf{H}\mathbf{y}_{t-1} + \mathbf{M}\mathbb{E}_t[\mathbf{x}_{t+1}] + \mathbf{N}\mathbf{x}_t = 0 \tag{25}$$

where $\mathbf{G} = \mathbf{G_1} + \mathbf{G_2}$ and $\mathbf{N} = \mathbf{N_1} + \mathbf{N_2}$ and, as above, $\mathbf{y}_t$ and $\mathbf{x}_t$ denote vectors of endogenous variables (including controls and states) $(m \times 1)$ and of exogenous states $(n \times 1)$. $\mathbb{E}_t$ denotes the rational expectation operator, and the exogenous process is given by (5).

Suppose that there is a unique stable solution of the model:

$$\mathbf{y}_t = \widetilde{\mathbf{P}}\mathbf{y}_{t-1} + \widetilde{\mathbf{Q}}\mathbf{x}_t$$

then, we can rewrite the stochastic difference equation (25) as follows:

$$\mathbf{F}\mathbb{E}_t\big[\widetilde{\mathbf{P}}\mathbf{y}_t + \widetilde{\mathbf{Q}}\mathbf{x}_{t+1}\big] + \mathbf{G}\widetilde{\mathbf{P}}\mathbf{y}_{t-1} + \mathbf{G}\widetilde{\mathbf{Q}}\mathbf{x}_t + \mathbf{H}\mathbf{y}_{t-1} + \mathbf{MA}\mathbf{x}_t + \mathbf{N}\mathbf{x}_t = 0$$

We can simplify the above equation to

$$\mathbf{F}\widetilde{\mathbf{P}}^2\mathbf{y}_{t-1} + \mathbf{F}\widetilde{\mathbf{P}}\widetilde{\mathbf{Q}}\mathbf{x}_t + \mathbf{F}\widetilde{\mathbf{Q}}\mathbf{A}\mathbf{x}_t + \mathbf{G}\widetilde{\mathbf{P}}\mathbf{y}_{t-1} + \mathbf{G}\widetilde{\mathbf{Q}}\mathbf{x}_t + \mathbf{H}\mathbf{y}_{t-1} + \mathbf{MA}\mathbf{x}_t + \mathbf{N}\mathbf{x}_t = 0$$

and can solve similarly for the recursive equilibrium law of motion via the method of undetermined coefficients. Specifically, the matrices $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{Q}}$ can be found solving the following matrix equations.

$$\mathbf{F}\widetilde{\mathbf{P}}^2 + \mathbf{G}\widetilde{\mathbf{P}} + \mathbf{H} = 0$$

$$\mathbf{F}\widetilde{\mathbf{P}}\widetilde{\mathbf{Q}} + \mathbf{F}\widetilde{\mathbf{Q}}\mathbf{A} + \mathbf{G}\widetilde{\mathbf{Q}} + \mathbf{M}\mathbf{A} + \mathbf{N} = 0$$

Comparison of these equations with their counterpart under DE immediately shows that $\mathbf{P} = \widetilde{\mathbf{P}}$ and $\mathbf{Q} = \widetilde{\mathbf{Q}}$.

Stability Conditions. Given the quadratic matrix equation (22)

$$\mathbf{F}\mathbf{P}^2 + \mathbf{G}\mathbf{P} + \mathbf{H} = \mathbf{0}$$

for the $m \times m$ matrix $\mathbf{P}$ and $m \times m$ matrices $\mathbf{G}$ and $\mathbf{H}$, define the $2m \times 2m$ matrices $\mathbf{\Xi}$ and $\mathbf{\Delta}$:

$$\mathbf{\Xi} = \begin{bmatrix} -\mathbf{G} & -\mathbf{H} \\ \mathbf{I}_m & \mathbf{0}_m \end{bmatrix}$$

and

$$\mathbf{\Delta} = \begin{bmatrix} -\mathbf{F} & \mathbf{0}_m \\ \mathbf{0}_m & \mathbf{I}_m \end{bmatrix}$$

where $\mathbf{I}_m$ is the identity matrix of size m and $\mathbf{0}_m$ is the $m \times m$ matrix with only zero entries.

Uhlig (1995) shows that if (a) s is a generalized eigenvector and λ is the corresponding generalized eigenvalue of $\mathbf{\Xi}$ with respect to $\mathbf{\Delta}$, then s can be written as $s' = \left[\lambda x', x'\right]$ for some $x \in \mathsf{R}^m$, and (b) there are m generalized eigenvalues $\lambda_1, ..., \lambda_m$ together with generalized eigenvectors $s_1, ..., s_m$ of $\mathbf{\Xi}$ with respect to $\mathbf{\Delta}$, written as $s_i' = [\lambda_i x_i', x_i']$ for some $x_i \in \mathsf{R}^m$, and if $(x_1, ..., x_m)$ is linearly dependent, then

$$\mathbf{P} = \mathbf{\Omega}\mathbf{\Lambda}\mathbf{\Omega}'$$

is a solution to the matrix quadratic equation, where $\mathbf{\Omega} = [x_1, ..., x_m]$ and $\mathbf{\Lambda} = diag(\lambda_1, ..., \lambda_m)$.

The stability conditions are given as follows.[40]

Theorem 1 *The solution* $\mathbf{P}$ *is stable if* $|\lambda_i| < 1$ *for all* $i = 1, ..., m$.

Thus, we can easily show that the stability conditions for both models are the same.

[40] See Section 6.3 of Uhlig (1995) for a detailed discussion.

Proof (Proposition 3). The solutions $\mathbf{P}$ and $\widetilde{\mathbf{P}}$ are the same since they involve identical matrices $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$. Thus, the stability conditions stated in Theorem 1 are the same for both solutions. ■

Proof (Proposition 4). Let's consider the RE model presented in equation (25) where the exogenous variables are stacked in a $(n \times 1)$ vector $\mathbf{x}_t$ that is assumed to follow the AR(1) stochastic process

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{v}_t$$

where $\mathbf{v}_t$ is a $(k \times 1)$ vector of Gaussian and orthogonal exogenous shocks:

$$\mathbf{v}_t \sim N(0, \Sigma_{\mathbf{v}})$$

and A is a diagonal matrix of persistence parameters.

Suppose that there is a unique stable solution of the model:

$$\mathbf{y}_t = \mathbf{P}\mathbf{y}_{t-1} + \mathbf{Q}\mathbf{x}_t \tag{26}$$

Assume, without loss of generality, that any unanticipated shocks or news only hit the economy at date 1. The economy is in steady state at date 0 or before. Then, the solution of the DE model from date 2 onwards coincides with the RE model solution. We prove this statement by considering the RE representation of the DE model derived in equation (8), reproduced here:

$$\begin{aligned}
\mathbf{F}\mathbb{E}_t[\mathbf{y}_{t+1}] + \mathbf{G}\mathbf{y}_t + \mathbf{H}\mathbf{y}_{t-1} + \mathbf{M}\mathbb{E}_t[\mathbf{x}_{t+1}] + \mathbf{N}\mathbf{x}_t& \\
+\mathbf{F}\theta\big(\mathbb{E}_t[\mathbf{y}_{t+1}] - \mathbb{E}_{t-1}[\mathbf{y}_{t+1}]\big)& \\
+\mathbf{M}\theta\big(\mathbb{E}_t[\mathbf{x}_{t+1}] - \mathbb{E}_{t-1}[\mathbf{x}_{t+1}]\big)& \\
+\mathbf{G}_1\theta\big(\mathbf{y}_t - \mathbb{E}_{t-1}[\mathbf{y}_t]\big)& \\
+\mathbf{N}_1\theta\big(\mathbf{x}_t - \mathbb{E}_{t-1}[\mathbf{x}_t]\big) = 0&
\end{aligned}$$

Since no news or shocks are assumed to happen for $t \geq 2$, we get that

$$\mathbb{E}_t[\mathbf{y}_{t+1}] - \mathbb{E}_{t-1}[\mathbf{y}_{t+1}] = \mathbb{E}_t[\mathbf{x}_{t+1}] - \mathbb{E}_{t-1}[\mathbf{x}_{t+1}] = \mathbf{y}_t - \mathbb{E}_{t-1}[\mathbf{y}_t] = \mathbf{x}_t - \mathbb{E}_{t-1}[\mathbf{x}_t] = 0; \quad \forall t \geq 2$$

The system from date $t \geq 2$ then simplifies to the RE model, the solution of which is given by equation (26) for $t \geq 2$. Date 1 solution for the DE model can then be found from (note the assumption that the economy is in steady state before date 1):

$$\begin{aligned}&\mathbf{F}\mathbb{E}_1[\mathbf{y}_2] + \mathbf{G}\mathbf{y}_1 + \mathbf{M}\mathbb{E}_1[\mathbf{x}_2] + \mathbf{N}\mathbf{x}_1 \\ &+\theta\big(\mathbf{F}\mathbb{E}_1[\mathbf{y}_2] + \mathbf{M}\mathbb{E}_1[\mathbf{x}_2] + \mathbf{G}_1\mathbf{y}_1 + \mathbf{N}_1\mathbf{x}_1\big) = 0\end{aligned}$$

Notice that $\mathbb{E}_1[\mathbf{y}_2]$ and $\mathbb{E}_1[\mathbf{x}_2]$ are known at date 1 from the RE solution.

$$\mathbb{E}_1[\mathbf{y}_2] = \mathbf{P}\mathbf{y}_1 + \mathbf{Q}\mathbf{A}\mathbf{x}_1; \quad \mathbb{E}_1[\mathbf{x}_2] = \mathbf{A}\mathbf{x}_1$$

After substituting these values and rearranging, we get:

$$\big((1+\theta)\mathbf{F}\mathbf{P} + \mathbf{G} + \theta\mathbf{G}_1\big)\mathbf{y}_1 + \big((1+\theta)(\mathbf{F}\mathbf{Q} + \mathbf{M})\mathbf{A} + \mathbf{N} + \theta\mathbf{N}_1\big)\mathbf{x}_1 = 0$$

Then, it follows that a bounded solution for the DE model exists if $(1+\theta)\mathbf{F}\mathbf{P}+\mathbf{G}+\theta\mathbf{G}_1$ is full-rank. ■

General Condition for Extra Volatility We establish a general result about when DE generate extra volatility over RE. Specific examples are provided in Section 3. As a reminder, in the case of DE, the solution of a general linear model takes the form:

$$\mathbf{y}_t = \mathbf{P}\mathbf{y}_{t-1} + \mathbf{Q}\mathbf{x}_t + \mathbf{R}\mathbf{v}_t$$

Instead, in the case of RE, the solution of model takes the form:

$$\mathbf{y}_t = \widetilde{\mathbf{P}}\mathbf{y}_{t-1} + \widetilde{\mathbf{Q}}\mathbf{x}_t$$

Comparing these two immediately leads to conjecture that, under DE, there should be extra volatility due to the presence of the extra term $\mathbf{R}\mathbf{v}_t$. However, whether this conjecture is true for a given set of parameters will depend on the covariance of the matrix Q with the other matrices of parameters in the solution. This is what the following proposition makes precise.

Proposition 10 (Extra Volatility) *Let y_{it}^{DE} and y_{it}^{RE} respectively denote the i-th component of the vector of endogenous variables $\mathbf{y}_t^{DE}$ and $\mathbf{y}_t^{RE}$ and $Var(y_{it}^{DE})$ and $Var(y_{it}^{RE})$ denote the variance of the variable y_{it}^{DE} and of the variable y_{it}^{RE}. Then, $Var(y_{it}^{DE})$ is larger than $Var(y_{it}^{RE})$ if and only if:*

$$diag(\mathbf{R}\boldsymbol{\Sigma}_{\mathbf{v}}\mathbf{R}' + 2\mathbf{Q}\boldsymbol{\Sigma}_{\mathbf{v}}\mathbf{R}')_i > 0$$

where $\mathbf{\Sigma_v}$ *is the variance-covariance matrix of* $\mathbf{v}_t$*.*

Proof. We have already shown that $\mathbf{P}$ and $\widetilde{\mathbf{P}}$ are the same and that $\mathbf{Q}$ and $\widetilde{\mathbf{Q}}$ are the same. Thus, given the exogenous process $\mathbf{x}_t$, the solution for the model with diagnostic expectations and for the model with rational expectations can be formulated as

$$\mathbf{y}_t^{DE} = \mathbf{P}\mathbf{y}_{t-1} + \mathbf{Q}\mathbf{x}_t + \mathbf{R}\mathbf{v}_t$$
$$\mathbf{y}_t^{RE} = \mathbf{P}\mathbf{y}_{t-1} + \mathbf{Q}\mathbf{x}_t$$

such that the variance of the vector of endogenous variables under diagnostic expectations, $\mathbf{y}_t^{DE}$, is given by

$$\begin{aligned} Var(\mathbf{y}_t^{DE}) &= Var(\mathbf{P}\mathbf{y}_{t-1}) + Var(\mathbf{Q}\mathbf{x}_t) + Var(\mathbf{R}\mathbf{v}_t) \\ &\quad + 2\ Cov(\mathbf{P}\mathbf{y}_{t-1}, \mathbf{Q}\mathbf{x}_t) + 2\ Cov(\mathbf{P}\mathbf{y}_{t-1}, \mathbf{R}\mathbf{v}_t) + 2\ Cov(\mathbf{Q}\mathbf{x}_t, \mathbf{R}\mathbf{v}_t) \end{aligned} \tag{27}$$

Similarly, the variance of the vector of endogenous variables under rational expectations, $\mathbf{y}_t^{RE}$ is given by

$$Var(\mathbf{y}_t^{RE}) = Var(\mathbf{P}\mathbf{y}_{t-1}) + Var(\mathbf{Q}\mathbf{x}_t) + 2\ Cov(\mathbf{P}\mathbf{y}_{t-1}, \mathbf{Q}\mathbf{x}_t)$$

Since $cov(\mathbf{P}\mathbf{y}_{t-1}, \mathbf{R}\mathbf{v}_t) = 0$, (27) is simplified to

$$Var(\mathbf{y}_t^{DE}) = Var(\mathbf{P}\mathbf{y}_{t-1}) + Var(\mathbf{Q}\mathbf{x}_t) + Var(\mathbf{R}\mathbf{v}_t) + 2\ Cov(\mathbf{P}\mathbf{y}_{t-1}, \mathbf{Q}\mathbf{x}_t) + 2\ Cov(\mathbf{Q}\mathbf{x}_t, \mathbf{R}\mathbf{v}_t)$$

such that by taking the difference of the two variances, we have

$$\begin{aligned} Var(\mathbf{y}_t^{DE}) - Var(\mathbf{y}_t^{RE}) &= Var(\mathbf{R}\mathbf{v}_t) + 2\ Cov(\mathbf{Q}\mathbf{x}_t, \mathbf{R}\mathbf{v}_t) \\ &= Var(\mathbf{R}\mathbf{v}_t) + 2\ Cov(\mathbf{Q}\mathbf{A}\mathbf{x}_{t-1} + \mathbf{Q}\mathbf{v}_t, \mathbf{R}\mathbf{v}_t) \\ &= \mathbf{R}\mathbf{\Sigma_v}\mathbf{R}' + 2\mathbf{Q}\mathbf{\Sigma_v}\mathbf{R}' \end{aligned}$$

Thus, for an endogenous variable y_{it} to have extra volatility with diagnostic expectations, the i-th diagonal component of the matrix $\mathbf{R}\mathbf{\Sigma_v}\mathbf{R}' + 2\mathbf{Q}\mathbf{\Sigma_v}\mathbf{R}'$ must be greater than zero. ■

We conclude by making a parallel to the work by Matsuyama (2007), who highlights, in the context of financial frictions, that equilibrium properties change non-monotonically with parameter values in such models. Looking at the expression for the matrix $\mathbf{R}$ reveals that it is a non-linear function of θ. Hence, even values of θ close to zero have the potential to (discontinuously) induce large volatility in linear models.

C Diagnostic New Keynesian Model: Detailed Derivation

There are three sets of agents in the economy: households, firms and government. Total output produced is equal to consumption expenditure made by the households and adjustment costs spent in adjusting prices.

C.1 Households

Households have the following lifetime utility

$$\log C_t - \omega \frac{L_t^{1+\nu}}{1+\nu} + \mathbb{E}_t^\theta \left[\Sigma_{s=t+1}^{\infty} \beta^{s-t} \left[\log(C_s) - \frac{\omega}{1+\nu} L_s^{1+\nu} \right] \right]$$

subject to budget constraint:

$$P_t C_t + \frac{B_{t+1}}{(1+i_t)} = B_t + W_t L_t + D_t + T_t \,,$$

$P_t C_t$ is nominal expenditure on final consumption good, B_{t+1} denotes purchase of nominal bonds that pay off $1+i_t$ interest rate in the following period, $W_t L_t$ denotes labor income, D_t and T_t denote dividends from firm-ownership and lump-sum government transfers respectively. $\mathbb{E}_t^\theta$ is the diagnostic expectations operator with diagnosticity parameter θ.

Let $\log C_t \equiv u(C_t)$. The consumption Euler equation is given by:

$$\frac{u'(C_t)}{P_t} = \beta(1+i_t)\mathbb{E}_t^\theta \left[\frac{u'(C_{t+1})}{P_{t+1}} \right]$$

Multiplying with P_{t-1} on both sides:

$$\frac{u'(C_t)P_{t-1}}{P_t} = \beta(1+i_t)\mathbb{E}_t^\theta \left[\frac{u'(C_{t+1})P_{t-1}}{P_{t+1}} \right]$$

Let $\Pi_t = \frac{P_t}{P_{t-1}}$ be the gross inflation rate. We can rewrite the Euler equation as:

$$\frac{u'(C_t)}{\Pi_t} = \beta(1+i_t)\mathbb{E}_t^\theta \left[\frac{u'(C_{t+1})}{\Pi_t \; \Pi_{t+1}} \right]$$

substitute the functional form for $u(C_t)$ and log-linearize the equation around the deterministic steady state ($\Pi = 1$, $\beta(1+i) = 1$). Hat-variables in small-cases denote log-deviation from steady state.

$$-\hat{\pi}_t - \hat{c}_t = \hat{i}_t + \mathbb{E}_t^\theta\left[-\hat{c}_{t+1} - \hat{\pi}_t - \hat{\pi}_{t+1}\right]$$

Use the resource constraint: $\hat{y}_t = \hat{c}_t$, to get

$$\hat{y}_t = \mathbb{E}_t^\theta\left[\hat{y}_{t+1} + \hat{\pi}_{t+1} + \hat{\pi}_t\right] - \hat{\pi}_t - \hat{i}_t$$

Using additivity, rearranging, and using the fact that

$$\mathbb{E}_t^\theta[\hat{\pi}_t] = \mathbb{E}_t[\hat{\pi}_t] + \theta(\mathbb{E}_t[\hat{\pi}_t] - \mathbb{E}_{t-1}[\hat{\pi}_t]) = \hat{\pi}_t + \theta(\hat{\pi}_t - \mathbb{E}_{t-1}[\hat{\pi}_t])$$

which follows from Lemma 1, we obtain the equation in the body:

$$\hat{y}_t = \mathbb{E}_t^\theta\left[\hat{y}_{t+1}\right] - (\hat{i}_t - \mathbb{E}_t^\theta[\hat{\pi}_{t+1}]) + \theta(\hat{\pi}_t - \mathbb{E}_{t-1}[\hat{\pi}_t])$$

C.1.1 Alternate derivation of the log-linearized Euler equation

We briefly show an alternate proof to derive the log-linearized Euler equation without explicitly requiring Lemma 1. Consider the consumption Euler equation

$$\frac{u'(C_t)}{P_t} = \beta(1+i_t)\mathbb{E}_t^\theta\left[\frac{u'(C_{t+1})}{P_{t+1}}\right]$$

Loglinearizing:

$$\hat{y}_t = \mathbb{E}_t^\theta\left[\hat{y}_{t+1}\right] - (\hat{i}_t - (\mathbb{E}_t^\theta[\hat{p}_{t+1}] - \hat{p}_t))$$

where $\{\hat{y}_t, \hat{i}_t, \hat{p}_t\}$ denote loglinear deviations of output, inflation from their respective steady states, and of the price level from an initial price level, respectively. We can show that $(\mathbb{E}_t^\theta[\hat{p}_{t+1}] - \hat{p}_t)$ can be rewritten as $\mathbb{E}_t^\theta[\hat{\pi}_{t+1}] + \theta(\hat{\pi}_t - \mathbb{E}_{t-1}[\hat{\pi}_t])$. Using the BGS formula (4) presented in the main text, we can get:

$$\mathbb{E}_t^\theta[\hat{p}_{t+1}] - \hat{p}_t = (1+\theta)\mathbb{E}_t[\hat{p}_{t+1}] - \theta\mathbb{E}_{t-1}[\hat{p}_{t+1}] - \hat{p}_t$$

Adding and subtracting $(1+\theta)\hat{p}_t$, we get:

$$\mathbb{E}_t^\theta[\hat{p}_{t+1}] - \hat{p}_t = (1+\theta)\mathbb{E}_t[\hat{\pi}_{t+1}] - \theta\mathbb{E}_{t-1}[\hat{p}_{t+1}] + \theta\hat{p}_t$$

Adding and subtracting $\theta\mathbb{E}_{t-1}[\hat{p}_t]$, we get

$$\mathbb{E}_t^\theta[\hat{p}_{t+1}] - \hat{p}_t = (1+\theta)\mathbb{E}_t[\hat{\pi}_{t+1}] - \theta\mathbb{E}_{t-1}[\hat{\pi}_{t+1}] - \theta\mathbb{E}_{t-1}[\hat{p}_t] + \theta\hat{p}_t$$

Adding and subtracting $\theta\hat{p}_{t-1}$, we get

$$\mathbb{E}_t^\theta[\hat{p}_{t+1}] - \hat{p}_t = (1+\theta)\mathbb{E}_t[\hat{\pi}_{t+1}] - \theta\mathbb{E}_{t-1}[\hat{\pi}_{t+1}] + \theta(\hat{\pi}_t - \mathbb{E}_{t-1}[\hat{\pi}_t])$$

Recognize that $(1+\theta)\mathbb{E}_t[\hat{\pi}_{t+1}] - \theta\mathbb{E}_{t-1}[\hat{\pi}_{t+1}] \equiv \mathbb{E}_t^\theta[\hat{\pi}_{t+1}]$, we get that

$$\mathbb{E}_t^\theta[\hat{p}_{t+1}] - \hat{p}_t = \mathbb{E}_t^\theta[\hat{\pi}_{t+1}] + \theta(\hat{\pi}_t - \mathbb{E}_{t-1}[\hat{\pi}_t])$$

C.2 Firms

Monopolistically competitive firms, indexed by $j \in [0,1]$, produce a differentiated good, $Y_t(j)$. We assume a Dixit-Stiglitz aggregator that aggregates intermediate goods into a final good, Y_t. Intermediate goods demand given by:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t$$

where $\epsilon_p > 1$ is the elasticity of substitution across intermediate goods' varieties, $P_t(j)$ is price of intermediate good j, and P_t is the price of final good Y_t. Each intermediate good is produced using the technology:

$$Y_t(j) = A_t L_t(j)$$

where $\log(A_t)$ is an aggregate TFP process that follows an AR(1) process with persistence coefficient ρ_a:

$$\log A_t = \rho_a \log A_{t-1} + \varepsilon_{a,t}$$

where $\epsilon_{a,t} \sim iid\ N(0, \sigma_a^2)$. Firm pays a quadratic adjustment cost in units of final good (Rotemberg 1982) to adjust prices:

$$\frac{\psi_p}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)^2 P_t Y_t$$

Firm's per period profits are given by:

$$D_t \equiv P_t(j)Y_t(j) - W_t L_t(j) - \frac{\psi_p}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)^2 P_t Y_t$$

Firm's profit maximization problem

$$\max_{P_t(j)} \left\{ P_t(j)Y_t(j) - W_t L_t(j) - \frac{\psi_p}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)^2 P_t Y_t + \mathbb{E}_t^{\theta}\left[\sum_{s=1}^{\infty} \beta^s Q_{t,t+s} D_{t+s}\right]\right\}$$

where $Q_{t,t+s}$ is the nominal stochastic discount factor of the household. Substitute in the demand for intermediate goods to get:

$$\max_{P_t(j)} \left\{ P_t(j)\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t - \frac{W_t}{A_t}\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t - \frac{\psi_p}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)^2 P_t Y_t + \mathbb{E}_t^{\theta}\left[\sum_{s=1}^{\infty} \beta^s Q_{t,t+s} D_{t+s}\right]\right\}$$

Notice that $P_t(j)$ appears in period t profits and period $t+1$ adjustment costs. It doesn't appear anywhere else in the problem. So we can "ignore" the remaining terms as we take the first-order condition. The monopolistically competitive firm solves the following problem:

$$\max_{P_t(j)} \left\{ P_t(j)\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t - \frac{W_t}{A_t}\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t - \frac{\psi_p}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)^2 P_t Y_t - \mathbb{E}_t^{\theta}\left[\beta Q_{t,t+1}\frac{\psi_p}{2}\left(\frac{P_{t+1}(j)}{P_t(j)} - 1\right)^2 P_{t+1}Y_{t+1}\right]\right\}$$
$$+ \text{ other terms}$$

First order condition:

$$(1-\epsilon_p)\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t + \epsilon_p \frac{W_t}{A_t P_t}\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p - 1} Y_t - \psi_p\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)\frac{P_t}{P_{t-1}(j)} Y_t$$
$$-\psi_p \beta \mathbb{E}_t^{\theta}\left[\frac{u'(C_{t+1}}{u'(C_t)}\left(\frac{P_{t+1}(j)}{P_t(j)} - 1\right)\frac{P_{t+1}(j)}{P_t(j)}\frac{P_t}{P_t(j)} Y_{t+1}\right] = 0$$

Symmetry across all firms implies that reset price equals the aggregate price level. Define $\Pi_t = \frac{P_t}{P_{t-1}}$:

$$(1-\epsilon_p)Y_t + \epsilon_p \frac{W_t}{A_t P_t} Y_t - \psi_p(\Pi_t - 1)\Pi_t Y_t + \psi_p \beta \mathbb{E}_t^{\theta}\left[\frac{u'(C_{t+1}}{u'(C_t)}(\Pi_{t+1} - 1)\Pi_{t+1} Y_{t+1}\right] = 0$$

Divide by Y_t:

$$(1-\epsilon_p) + \epsilon_p \frac{W_t}{A_t P_t} - \psi_p(\Pi_t - 1)\Pi_t + \frac{\psi_p}{Y_t}\beta \mathbb{E}_t^{\theta}\left[\frac{u'(C_{t+1}}{u'(C_t)}(\Pi_{t+1} - 1)\Pi_{t+1} Y_{t+1}\right] = 0$$

Log-linearize around the deterministic steady state such that $A = 1$, $w = \frac{W}{P} =$

$\omega C Y^{\nu} = \frac{\epsilon_p - 1}{\epsilon_p}$, $\Pi = 1$, and $Y_t = Y$. Let $w_t = \frac{W_t}{P_t}$

$$\epsilon_p w(\hat{w}_t - \hat{a}_t) - \psi_p \hat{\pi}_t + \psi_p \beta \mathbb{E}_t^{\theta} \hat{\pi}_{t+1} = 0$$

Rearrange to get

$$\hat{\pi}_t = \beta \mathbb{E}_t^{\theta} \left[\hat{\pi}_{t+1} \right] + \frac{\epsilon_p \ w}{\psi_p} (\hat{w}_t - \hat{a}_t)$$

From the intra-temporal labor supply first order condition, we have:

$$\hat{w}_t = \hat{c}_t + \nu(\hat{y}_t - \hat{a}_t)$$

Use the resource constraint $\hat{c}_t = \hat{y}_t$, to rewrite the new Keynesian Phillips Curve (NKPC):

$$\hat{\pi}_t = \beta \mathbb{E}_t^{\theta} \left[\hat{\pi}_{t+1} \right] + \frac{\epsilon_p \ w}{\psi_p} (1 + \nu) \hat{y}_t$$

Note that $\frac{\epsilon_p \ w}{\psi_p} = \frac{\epsilon_p - 1}{\psi_p}$. Then, the NKPC is given by

$$\hat{\pi}_t = \beta \mathbb{E}_t^{\theta} \left[\hat{\pi}_{t+1} \right] + \kappa (\hat{y}_t - \hat{a}_t)$$

where $\kappa \equiv \frac{\epsilon_p - 1}{\psi_p}(1 + \nu)$.

C.3 Policy Rule

The government sets nominal interest rate with the following rule:

$$\frac{1 + i_t}{1 + i_{ss}} = \Pi_t^{\phi_\pi} \left(\frac{Y_t}{Y_t^*} \right)^{\phi_x}$$

where $Y_t^* = A_t$ is the natural rate allocation, $i_{ss} = \frac{1}{\beta} - 1$ is the steady state nominal interest rate, $\phi_\pi \geq 0$, $\phi_x \geq 0$, and steady state inflation $\Pi = 1$. Log-linearized policy rule is given by:

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_x (\hat{y}_t - \hat{a}_t)$$

We assume that nominal bonds are in net zero supply. Government spends G_t financed by lumpsum taxes.

C.4 Market Clearing

Total output produces is used for consumption and government expenditure.

$$Y_t = C_t + G_t$$

C.5 Equilibrium

The log-linearized equilibrium in the New Keynesian model with diagnostic expectations is given by following three equations in three unknowns $\{\hat{y}_t, \hat{\pi}_t, \hat{i}_t\}$ for a given shock process $\{\hat{a}_t\}$.

$$\hat{y}_t = \mathbb{E}_t^\theta [\hat{y}_{t+1}] - (\hat{i}_t - \mathbb{E}_t^\theta[\hat{\pi}_{t+1}]) + \theta(\hat{\pi}_t - E_{t-1}[\hat{\pi}_t]) + \hat{g}_t - \mathbb{E}_t^\theta \hat{g}_{t+1} \tag{28}$$

$$\hat{\pi}_t = \beta \mathbb{E}_t^\theta[\hat{\pi}_{t+1}] + \kappa(\hat{y}_t - \hat{a}_t) - \kappa\psi\hat{g}_t \tag{29}$$

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_x(\hat{y}_t - \hat{a}_t) \tag{30}$$

where $\kappa \equiv \frac{\epsilon_p - 1}{\psi_p}(1+\nu)$, and the shock processes are given by:

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{a,t} \tag{31}$$

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \varepsilon_{g,t} \tag{32}$$

where $\varepsilon_{a,t} \sim iid\ N(0, \sigma_a^2)$ and $\varepsilon_{g,t} \sim iid\ N(0, \sigma_g^2)$.[41]

C.6 Solution

C.6.1 Rational Expectations

Under RE, the solution of the model with TFP and government spending shocks is given by:

$$\hat{y}_t = \frac{(1-\beta\rho_g)(1-\rho_g) + \kappa\psi(\phi_\pi - \rho_g)}{(1-\beta\rho_g)(1-\rho_g+\phi_x) + \kappa(\phi_\pi - \rho_g)}\varepsilon_{g,t} + \frac{\phi_y(1-\beta\rho_a) + \kappa(\phi_\pi - \rho_a)}{(1-\beta\rho_a)(1-\rho_a+\phi_x) + \kappa(\phi_\pi - \rho_a)}\varepsilon_{a,t}$$

$$\hat{\pi}_t = \frac{\kappa(1-\psi)(1-\rho_g) - \kappa\psi\phi_x}{(1-\beta\rho_g)(1-\rho_g+\phi_x) + \kappa(\phi_\pi - \rho_g)}\varepsilon_{g,t} - \frac{\kappa(1-\rho_a)}{(1-\beta\rho_a)(1-\rho_a+\phi_x) + \kappa(\phi_\pi - \rho_a)}\varepsilon_{a,t}$$

Since $\psi < 1$, the fiscal multiplier is less than one in the NK model under rational expectations.

[41] We define $\hat{g}_t$ as percentage changes of government spending from its steady state as fraction of steady state output.

C.6.2 Diagnostic Expectations

Note that after a one-time unanticipated shock, the solution under DE and RE coincide at subsequent dates since there is no news. (This was shown formally in the context of the general linear model in the previous appendix, proof of Proposition 4.) At date $t = 1$, we can derive the solution under DE as follows. From the RE solution, we know the expectations of forward looking variables :

$$\mathbb{E}_1\hat{y}_2 = \rho_g \frac{(1-\beta\rho_g)(1-\rho_g)+\kappa\psi(\phi_\pi-\rho_g)}{(1-\beta\rho_g)(1-\rho_g+\phi_x)+\kappa(\phi_\pi-\rho)}\varepsilon_{g,t} + \rho_a \frac{\phi_y(1-\beta\rho_a)+\kappa(\phi_\pi-\rho_a)}{(1-\beta\rho_a)(1-\rho_a+\phi_x)+\kappa(\phi_\pi-\rho_a)}\varepsilon_{a,t};$$

$$\mathbb{E}_1\hat{\pi}_2 = \rho_g \frac{\kappa(1-\psi)(1-\rho_g)-\kappa\psi\phi_x}{(1-\beta\rho_g)(1-\rho_g+\phi_x)+\kappa(\phi_\pi-\rho_g)}\varepsilon_{g,t} - \rho_a \frac{\kappa(1-\rho_a)}{(1-\beta\rho_a)(1-\rho_a+\phi_x)+\kappa(\phi_\pi-\rho_a)}\varepsilon_{a,t}$$

$$\mathbb{E}_0\hat{y}_2 = \mathbb{E}_0\hat{\pi}_2 = \mathbb{E}_0\hat{\pi}_1 = 0$$

We can thus construct the diagnostic expectation terms that enter the DE model, and simplify the model to

$$\hat{y}_1 = (1+\theta)\mathbb{E}_1\left[\hat{y}_2 + \hat{\pi}_2 - \hat{g}_2\right] - \hat{i}_1 + \theta\hat{\pi}_1 + \hat{g}_1 \tag{33}$$

$$\hat{\pi}_1 = \beta(1+\theta)\mathbb{E}_1\left[\hat{\pi}_2\right] + \kappa(\hat{y}_1 - \hat{a}_1) - \kappa\psi\hat{g}_1 \tag{34}$$

$$\hat{i}_1 = \phi_\pi\hat{\pi}_1 + \phi_x(\hat{y}_1 - \hat{a}_1) \tag{35}$$

Substituting the latter two equations into the Euler equation, and rearranging we get

$$\hat{y}_1 = \frac{(1+\theta)\mathbb{E}_1\left[\hat{y}_2 + (1+\beta\theta-\beta\phi_\pi)\pi_2\right] + \left[1+(\phi_\pi-\theta)\kappa\psi-(1+\theta)\rho_g\right]\varepsilon_{g,1} + \left[\kappa(\phi_\pi-\theta)+\phi_x\right]\varepsilon_{a,1}}{1+\phi_x+(\phi_\pi-\theta)\kappa} \tag{36}$$

The corresponding RE solution can be seen with $\theta = 0$.

We study two scenarios with analytical results:

1. When the shocks are iid ($\rho_a = \rho_g = 0$), the solutions are:

$$\hat{y}_1 = \frac{1+(\phi_\pi-\theta)\kappa\psi}{1+\phi_x+(\phi_\pi-\theta)\kappa}\varepsilon_{g,1} + \frac{\phi_x+(\phi_\pi-\theta)\kappa}{1+\phi_x+(\phi_\pi-\theta)\kappa}\varepsilon_{a,1}$$

   For a bounded solution (and continuity with RE solution), we assume that $\theta < \phi_\pi + \kappa^{-1}(1+\phi_x)$. There are two cases for the fiscal multiplier:

   - $\phi_x < \nu$: The fiscal multiplier under DE is larger than under RE. The multiplier is increasing in θ, exceeds one for values of $\theta > \phi_\pi + \frac{\phi_x}{(1-\psi)\kappa}$. As $\theta \to \phi_\pi + \kappa^{-1}(1+\phi_y)$, the fiscal multiplier $\to \infty$.

- $\underline{\phi_x > \nu}$: The fiscal multiplier under DE is smaller than under RE.

The output gap $\hat{x}_1 = \frac{-\varepsilon_{a,1}}{1+\phi_x+(\phi_\pi-\theta)\kappa}$ always negatively co-moves with iid TFP shocks.

2. When prices are perfectly rigid, that is $\kappa \to 0$, the solution for output is given by:

$$\hat{y}_1 = \frac{(1-\rho_g)(1+\phi_x) - \theta\rho_g\phi_x}{(1+\phi_x)(1-\rho_g+\phi_x)}\varepsilon_{g,1} + \frac{\phi_x(1+\phi_x+\theta\rho_a)}{(1+\phi_x)(1-\rho_a+\phi_x)}\varepsilon_{a,1}$$

Fiscal multiplier under DE is smaller than under RE. Fiscal multiplier is decreasing in θ. For $\theta > \frac{(1-\rho_g)(1+\phi_x)}{\rho_g\phi_x}$, assuming $\rho_g > 0$, output falls under DE with increase in government spending.

Output gap $\hat{x}_t = \frac{\theta\rho_a\phi_x-(1-\rho_a)(1+\phi_x)}{(1+\phi_x)(1-\rho_a+\phi_x)}\varepsilon_{a,1}$ positively co-moves with TFP shock if and only if $\theta\rho_a\phi_x > (1-\rho_a)(1+\phi_x)$.

C.7 Proof of Propositions 5 and 7

Because there are no government shocks, $\hat{c}_t = \hat{y}_t$. The equilibrium with completely rigid prices, i.e. $\psi_p \to \infty$, given by:

$$\hat{y}_t = \mathbb{E}_t^\theta\left[\hat{y}_{t+1}\right] - \hat{i}_t \tag{37}$$

$$\hat{i}_t = \phi_x(\hat{y}_t - \hat{a}_t) \tag{38}$$

where $\hat{a}_t = \rho_a\hat{a}_{t-1} + \varepsilon_{a,t}$, $\rho_a \in [0,1)$, and $\epsilon_{a,t} \sim iid\ N(0,\sigma_a^2)$. Substituting the policy rule into the Euler equation, we get:

$$\hat{y}_t = \frac{1}{1+\phi_x}\mathbb{E}_t^\theta\left[\hat{y}_{t+1}\right] + \frac{\phi_x}{1+\phi_x}\hat{a}_t$$

By forward iteration, and using the law of iterated expectations under the no-news assumption,

$$\hat{y}_t = \lim_{T\to\infty}\frac{\mathbb{E}_t^\theta\left[\hat{y}_{T+1}\right]}{(1+\phi_x)^{T+1}} + \sum_{i=1}^{\infty}\frac{\phi_x\mathbb{E}_t^\theta\left[\hat{a}_{t+i}\right]}{(1+\phi_x)^{i+1}} + \frac{\phi_x}{1+\phi_x}\hat{a}_t$$

The system is locally determinate if and only if $\phi_x > 0$. Let $\phi_x > 0$. Then,

$$\hat{y}_t = \sum_{i=1}^{\infty}\frac{\phi_x\mathbb{E}_t^\theta\left[\hat{a}_{t+i}\right]}{(1+\phi_x)^{i+1}} + \frac{\phi_x}{1+\phi_x}\hat{a}_t$$

From the definition of the shock process, we know that, $\forall\ i > 0$

$$\mathbb{E}_t^\theta\left[\hat{a}_{t+i}\right] = \rho_a^i(1+\theta)\hat{a}_t - \theta\rho_a^{i+1}\hat{a}_{t-1} = \rho_a^i\left((1+\theta)\hat{a}_t - \theta\rho_a\hat{a}_{t-1}\right)$$

We can then derive the solution for output:

$$\hat{y}_t = \frac{\phi_x \rho_a (1+\theta) + \phi_x (1+\phi_x - \rho_a)}{(1+\phi_x)(1+\phi_x-\rho_a)} \hat{a}_t - \frac{\phi_x \theta \rho_a^2}{(1+\phi_x)(1+\phi_x-\rho_a)} \hat{a}_{t-1}$$

The solution for output gap $\hat{x}_t \equiv \hat{y}_t - \hat{a}_t$ is given by:

$$\hat{x}_t = \frac{-\rho_a(1-\rho_a)(1+\phi_x)}{(1+\phi_x)(1+\phi_x-\rho_a)} \hat{a}_{t-1} + \frac{\theta\phi_x\rho_a - (1-\rho_a)(1+\phi_x)}{(1+\phi_x)(1+\phi_x-\rho_a)} \varepsilon_{a,t}$$

In response to an unanticipated improvement in productivity, output gap can be positive on impact if and only

$$\theta\phi_x\rho_a - (1-\rho_a)(1+\phi_x) > 0$$

When $\theta = 0$, that is rational expectations, output gap negatively co-moves with productivity shock. Under diagnostic expectations, productivity improvements can be expansionary on impact.

Volatility of output gap is given by:

$$Var(\hat{x}_t) = \left(\frac{\rho_a(1-\rho_a)(1+\phi_x)}{(1+\phi_x)(1+\phi_x-\rho_a)} \right)^2 Var(\hat{a}_{t-1}) + \left(\frac{\theta\phi_x\rho_a - (1-\rho_a)(1+\phi_x)}{(1+\phi_x)(1+\phi_x-\rho_a)} \right)^2 \sigma_a^2$$

The first coefficient is same under rational and diagnostic expectations. Volatility is higher under diagnostic expectations relative to rational expectations if and only if

$$\begin{aligned}
& (\theta\phi_x\rho_a - (1-\rho_a)(1+\phi_x))^2 > (1-\rho_a)^2(1+\phi_x)^2 \\
\iff & (\theta\phi_x\rho_a)^2 + (1-\rho_a)^2(1+\phi_x)^2 - 2\theta\phi_x\rho_a(1-\rho_a)(1+\phi_x) > (1-\rho_a)^2(1+\phi_x)^2 \\
\iff & (\theta\phi_x\rho_a)^2 > 2\theta\phi_x\rho_a(1-\rho_a)(1+\phi_x) \\
\iff & \theta\phi_x\rho_a > 2(1-\rho_a)(1+\phi_x) \\
\iff & \theta > \frac{2(1-\rho_a)(1+\phi_x)}{\phi_x\rho_a}
\end{aligned}$$

If $\theta > \frac{2(1-\rho_a)(1+\phi_x)}{\phi_x\rho_a}$ is satisfied, then volatility of the output gap under DE is larger than under RE.

Note that output $\hat{y}_t = \hat{x}_t + \hat{a}_t$. Under DE, the volatility of output is given by $Var(\hat{y}_t^{DE}) = Var(\hat{x}_t^{DE}) + Var(\hat{a}_t) + 2Cov(\hat{x}_t^{DE}, \hat{a}_t)$. Under RE, the volatility of output is given by $Var(\hat{y}_t^{RE}) = Var(\hat{x}_t^{RE}) + Var(\hat{a}_t) + 2Cov(\hat{x}_t^{RE}, \hat{a}_t)$. When $\theta > \frac{2(1-\rho_a)(1+\phi_x)}{\phi_x\rho_a}$, we know that $Var(\hat{x}_t^{DE}) > Var(\hat{x}_t^{RE})$, and $Cov(\hat{x}_t^{DE}, \hat{a}_t) > 0 > Cov(\hat{x}_t^{RE}, \hat{a}_t)$. Hence output is also more volatile under DE relative to RE if $\theta > \frac{2(1-\rho_a)(1+\phi_x)}{\phi_x \rho_a}$.

C.8 Proof of Proposition 8

Assume iid government spending shocks. And assume that $\phi_x = 0$. Then from Section C.6, we obtain the solution for output under DE:

$$\hat{y}_1 = \frac{1 + (\phi_\pi - \theta)\kappa\psi}{1 + (\phi_\pi - \theta)\kappa} \varepsilon_{g,1}$$

For the solution to be continuous in the RE limit and bounded, we assume that $\theta < \phi_\pi + \kappa^{-1}$. Since $\psi = \frac{1}{1+\nu} < 1$, the fiscal multiplier is increasing in θ. Under the RE limit, $\theta = 0$, the fiscal multiplier is strictly less than one. For $\theta > \phi_\pi$, the multiplier is larger than one. Finally, the multiplier explodes to infinity as $\theta \to \phi_\pi + \kappa^{-1}$.

C.9 Proof of Proposition 9

1. When $\psi_p \to \infty$, $\phi_x = 0$, and $\rho_\zeta = 0$, beliefs about the long-run (BLR) under the DKF are given by

   $$\zeta^\theta_{t|t} \equiv \zeta_{t|t} + \theta(\zeta_{t|t} - \zeta_{t|t-1})$$

   where $\zeta_{t|t} \equiv \mathbb{E}_t[\zeta_t]$ and $\zeta_{t|t-1} \equiv \mathbb{E}_{t-1}[\zeta_t]$. Also, BLR under the RKF are given by $\zeta_{t|t}$. From the rational Kalman filter, we have

   $$\zeta_{t|t} = \zeta_{t|t-1} + Gain_t(s_t - s_{t|t-1})$$

   where $s_{t|t-1} \equiv \mathbb{E}_{t-1}[s_t]$ and $Gain_t$ is the Kalman gain. Thus, BLR under the DKF is simplified to

   $$\zeta^\theta_{t|t} \equiv \zeta_{t|t} + \theta \times Gain_t(s_t - s_{t|t-1}) \tag{39}$$

   and as $s_t - s_{t|t-1} > 0$ with a positive shock to ζ_t, BLR are greater under the DKF than under the RKF.

2. We can also rewrite (39) as

   $$\zeta^\theta_{t|t} = \zeta_{t-1} + (1 + \theta) \times Gain_t(s_t - s_{t|t-1})$$

   given that $\zeta_{t|t-1} = \zeta_{t-1}$. As BLR under FIRE are simply $\zeta_t = \zeta_{t-1} + \epsilon_{\zeta,t}$, BLR under the DKF are greater than under FIRE if $(1 + \theta) \times Gain_t(s_t - s_{t|t-1}) > \epsilon_{\zeta,t}$

where $\epsilon_{\zeta,t}$ is a shock to ζ_t. Thus, if

$$\theta \geq \frac{\epsilon_{\zeta,t}}{Gain_t(s_t - s_{t|t-1})} - 1$$

beliefs about the long-run under the DKF are greater than under FIRE.

D Real Business Cycle Model

We list the equilibrium conditions for a standard RBC model. Equilibrium is given by a sequence of seven unknowns $\{C_t, K_{t+1}, Y_t, I_t, N_t, R_t^k, \tilde{W}_t.\}$ that satisfy the following seven equations for a given exogenous process A_t and an initial value of capital stock K_0.

$$\frac{1}{C_t} = \beta \mathbb{E}_t^\theta \left[\frac{R_{t+1}^k + 1 - \delta}{C_{t+1}} \right] \tag{40}$$

$$\tilde{W}_t = \omega C_t N_t^\nu \tag{41}$$

$$K_{t+1} = (1-\delta)K_t + I_t \tag{42}$$

$$Y_t = K_t^\alpha \left(A_t N_t\right)^{1-\alpha} \tag{43}$$

$$Y_t = C_t + I_t \tag{44}$$

$$R_t^k = \alpha \frac{Y_t}{K_t} \tag{45}$$

$$\tilde{W}_t = (1-\alpha)\frac{Y_t}{N_t} \tag{46}$$

β is the discount rate, δ is depreciation rate, ν is inverse of the Frisch elasticity of labor supply, α is the capital share, and ω is a normalizing constant in the steady state. $\theta > 0$ is the diagnosticity parameter. The system of log-linearized equations is as follows (where the lower case letters denote the log-deviations form the respective steady state values)[42]:

$$\tilde{w}_t = c_t + \nu n_t \tag{47}$$

$$c_t = \mathbb{E}_t^\theta \left[c_{t+1} - \frac{R^k}{R^k + 1 - \delta} r_{t+1}^k \right] \tag{48}$$

$$k_{t+1} = \delta \hat{I}_t + (1-\delta)k_t \tag{49}$$

$$y_t = (1-\alpha)a_t + \alpha k_t + (1-\alpha)n_t \tag{50}$$

$$y_t = s_c c_t + (1-s_c)\hat{I}_t \tag{51}$$

$$r_t^k = y_t - k_t \tag{52}$$

$$\tilde{w}_t = y_t - n_t \tag{53}$$

where R^k is the steady state rental rate, and s_c is the steady state share of consumption in output. The economy starts in the steady state. There is a one-time unanticipated iid shock a_1 at time 1.

[42] $\hat{I}_t$ is also log-deviations of investment I_t from its steady state value.

D.1 Rational Expectations and full depreciation, $\delta = 1$

We derive analytical result assuming full depreciation, that is $\delta = 1$. The Euler equation under rational expectations and full depreciation is given by:

$$c_t - k_{t+1} = \mathbb{E}_t \left[c_{t+1} - y_{t+1}\right]$$

From the labor supply and labor demand conditions, we obtain

$$(1+\nu)n_t = y_t - c_t$$

When $\delta = 1$, $\hat{I}_t = k_{t+1}$. Use the above equation into the Euler equation, along with investment equation to get

$$\hat{I}_t - y_t + (1+\nu)n_t = (1+\nu)\mathbb{E}_t[n_{t+1}]$$

Substitute in the resource constraint,

$$\frac{1}{1-s_c}\left[y_t - s_c c_t\right] - y_t + (1+\nu)n_t = (1+\nu)\mathbb{E}_t[n_{t+1}]$$

$$\iff \frac{s_c}{1-s_c}\left[y_t - c_t\right] + (1+\nu)n_t = (1+\nu)\mathbb{E}_t[n_{t+1}]$$

$$\iff \left(1 + \frac{s_c}{1-s_c}\right) n_t = \mathbb{E}_t[n_{t+1}]$$

Solution for employment is

$$n_t = 0 \quad \forall t \geq 0$$

We can solve for the solution for other variables at dates 1 and 2:

$$c_1 = y_1 = \hat{I}_1 = k_2 = (1-\alpha)a_1;$$

$$c_2 = y_2 = \hat{I}_2 = k_3 = \alpha(1-\alpha)a_1$$

and so on.

D.2 Diagnostic Expectations and full depreciation, $\delta = 1$

The Euler equation is

$$c_t = \mathbb{E}_t^{\theta}\left[c_{t+1} - y_{t+1} + k_{t+1}\right]$$

As before, the economy starts in the steady state. There is a one-time unanticipated iid shock a_1 at time 1. From Date 2, the solution is same as rational expectations model. Since, we have iid shocks, the solution at date 1 is:

$$c_1 = (1+\theta)k_2$$

Substitute into the resource constraint to get

$$y_1 = (1+\theta s_c)k_2$$

From labor supply and labor demand,

$$(1+\nu)n_1 = y_1 - c_1 = -\theta(1-s_c)k_2$$

Finally, from the production function

$$\begin{aligned}
y_1 &= (1-\alpha)a_1 + (1-\alpha)n_1 \\
\iff (1+\theta s_c)k_2 &= (1-\alpha)a_1 + (1-\alpha)n_1 \\
\iff -\frac{(1+\theta s_c)}{\theta(1-s_c)}(1+\nu)n_1 &= (1-\alpha)a_1 + (1-\alpha)n_1 \\
n_1 &= -\frac{\theta(1-s_c)(1-\alpha)a_1}{(1+\theta s_c)(1+\nu)+(1-\alpha)\theta(1-s_c)}
\end{aligned}$$

Solution is

$$\begin{aligned}
n_1 &= -\frac{\theta(1-s_c)(1-\alpha)a_1}{(1+\theta s_c)(1+\nu)+(1-\alpha)\theta(1-s_c)} \\
k_2 &= \frac{(1-\alpha)(1+\nu)a_1}{(1+\theta s_c)(1+\nu)+(1-\alpha)\theta(1-s_c)} \\
c_1 &= \frac{(1+\theta)(1-\alpha)(1+\nu)a_1}{(1+\theta s_c)(1+\nu)+(1-\alpha)\theta(1-s_c)} \\
y_1 &= \frac{(1+\theta s_c)(1-\alpha)(1+\nu)a_1}{(1+\theta s_c)(1+\nu)+(1-\alpha)\theta(1-s_c)}
\end{aligned}$$

Date 2 solution is :

$$n_2 = 0; \quad y_2 = \alpha k_2 = \frac{\alpha(1-\alpha)(1+\nu)a_1}{(1+\theta s_c)(1+\nu)+(1-\alpha)\theta(1-s_c)}$$

D.3 Proof of Proposition 6

Volatility of output at date 1 is lower under DE compared to RE if and only if

$$\frac{(1+\theta s_c)(1+\nu)}{(1+\theta s_c)(1+\nu)+(1-\alpha)\theta(1-s_c)} < 1$$

which is true. Further, note that volatility of output at date 1 under DE is decreasing in ν. Similarly, we can show that volatility of output under diagnostic expectations is lower at all future horizons as well. For example, Volatility of output at date 2 is lower under DE compared to RE if and only if

$$\frac{(1+\nu)}{(1+\theta s_c)(1+\nu)+(1-\alpha)\theta(1-s_c)} < 1$$

which is true since $1+\theta s_c > 1$ and $(1-\alpha)\theta(1-s_c) > 0$.

E Numerical Results with NK and RBC model

To numerically demonstrate the excess volatility in the NK model, we use the calibration discussed in Table 2: Stationary TFP follows an AR(1) process with persistence 0.9 and standard deviation 0.02. We set the discount factor β to 0.99. For the RBC model, we set the capital share α to 0.2 and the capital depreciation rate δ to 0.025. For the NK model, we set $\phi_\pi = 1.5$, $\phi_x = 0.5$, and $\kappa = 0.05$. We also set the diagnosticity parameter θ to one.

Table 2: Parametrization: The NK and RBC models

	Parameter	Value
Common to Both Models		
θ	Diagnosticity	1
β	Discount factor	0.99
Simple NK model		
ν	Inv. Frisch elasticity	2
ϕ_π	Taylor rule inflation	1.5
ϕ_x	Taylor rule output gap	0.5
κ	Slope of the Phillips curve	0.05
RBC model		
α	Capital share	0.2
δ	Capital depreciation rate	0.025
Shock Process		
ρ_a	Shock persistence (stationary TFP)	0.9
σ_a	Standard dev. (stationary TFP)	0.02

Panel a) in Table 3 shows unconditional volatilities of output growth, and consumption growth under diagnostic and rational expectations. Since there is no government spending or investment, output growth and consumption growth are equivalent in the NK model. We find that the output gap under diagnostic expectations exhibits 63 percent higher standard deviation relative to the output gap under rational expectations.

Panel b) in Table 3 shows unconditional volatilities of output growth, consumption growth, and investment growth, both under diagnostic and rational expectations in the baseline RBC model. Consumption growth is twice as volatile under diagnostic expectations than under rational expectations. On the other hand, investment growth

Table 3: Model-Implied Volatilities with Stationary TFP Shocks

(a) New Keynesian Model

Variable	Rational Expectations	Diagnostic Expectations	Percentage Increase
Output	0.0182	0.0296	63%
Consumption	0.0182	0.0296	63%
Investment	–	–	–

(b) Real Business Cycle Model

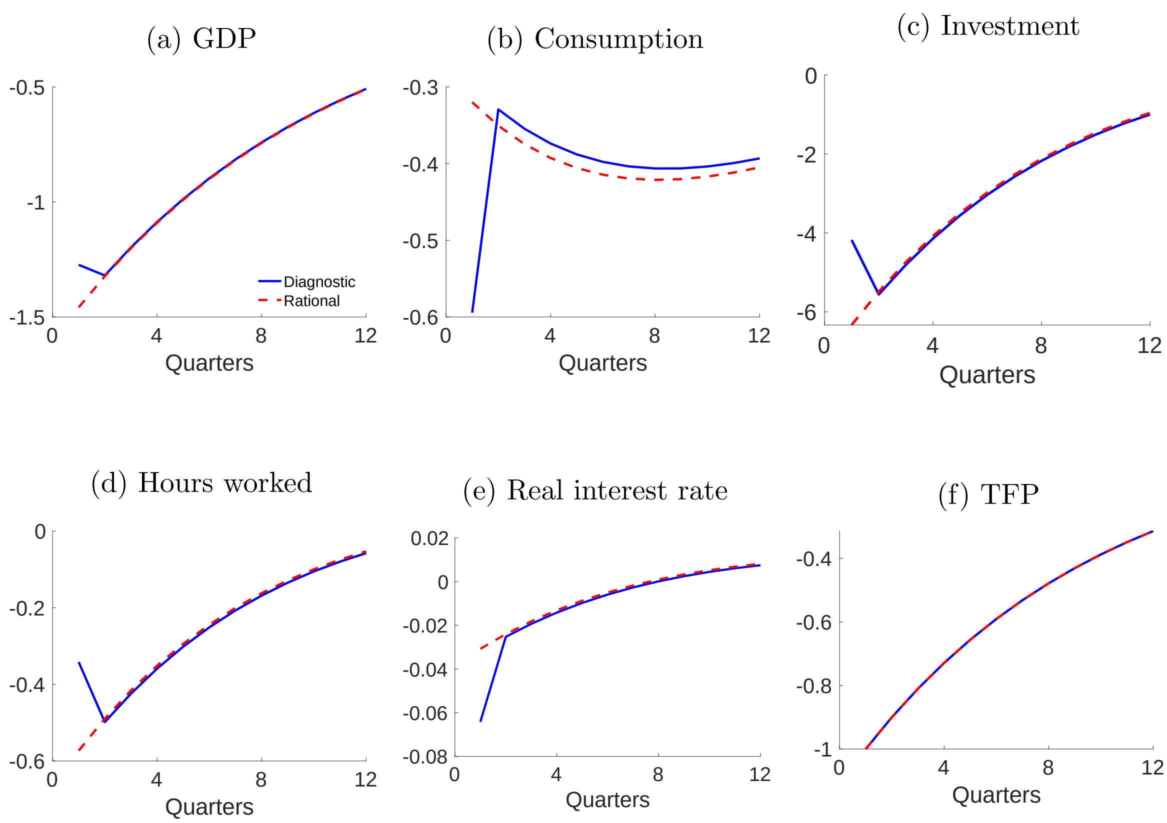
Variable	Rational Expectations	Diagnostic Expectations	Percentage Increase
Output	0.0204	0.0188	-8%
Consumption	0.0052	0.0103	98%
Investment	0.1147	0.0816	-29%

Notes: The table reports the standard deviations of output growth, consumption growth and investment growth in the New Keynesian (NK) model and the RBC model in Panels (a) and (b) respectively. Final column titled "Percentage Increase" shows the percentage increase in standard deviation under the diagnostic expectations model relative to the rational expectations benchmark. There is one shock process in the two models. See Table 2 for the parameters.

and output growth are dampened under diagnostic expectations due to the general equilibrium adjustment of the interest rate. Diagnosticity, therefore, does not always generate extra amplification.

In the main text, see Figure 1, we already discussed the impulse response of output gap to a TFP shock in the NK model. We conclude with a brief discussion of impulse responses to a stationary TFP shock in the RBC model. Figure 4 plots the impulse response of the exogenous TFP, consumption, output, investment, capital stock, and real interest rate to a one standard deviation shock to TFP. The greater reduction in real-interest rate under diagnostic expectations attenuates the fall in investment, and explains why there is lower volatility in output and investment with diagnostic beliefs, compared to the corresponding variables under rational expectations. Diagnosticity, therefore, does not always generate extra amplification.

Figure 4: Impulse responses to a stationary TFP shock in the RBC model



Notes: The panels depict the impulse responses of GDP, consumption, investment, hours worked, real interest rate and TFP shock ($\hat{a}$) to a unit shock to TFP, $\varepsilon_{a,t}$. The blue solid lines denote impulses responses with diagnostic expectations, whereas the red dashed lines denote responses with rational expectations. See Table 2 for parameters corresponding to the RBC model.

F A Medium-Scale DSGE model

F.1 Model Ingredients

The model follows the exposition in Blanchard, L'Huillier, and Lorenzoni (2013), henceforth referred to as BLL. The economy comprises of following agents: a continuum of households supplying differentiated labor, a continuum of firms producing differentiated goods, a perfectly competitive final goods firm, a perfectly competitive labor agency that provides the composite labor input demanded by firms, and a government in charge of fiscal and monetary policy.

F.1.1 Monopolistically Competitive Producers

Assume there is a continuum of differentiated intermediated good producers that sell the intermediate good Y_{jt}. A perfectly competitive firm aggregates intermediate goods into a final composite good $Y_t = \left[\int_0^1 Y_{jt}^{\frac{\epsilon_{p,t}-1}{\epsilon_{p,t}}} dj\right]^{\frac{\epsilon_{p,t}}{\epsilon_{p,t}-1}}$, where $\epsilon_p > 1$ is time-varying elasticity of demand. The iso-elastic demand for intermediate good j is given by: $Y_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\epsilon_{p,t}} Y_t$, where P_t is the aggregate price index and P_{jt} is the price of intermediate goods j. Each intermediate good j is produced by a price-setting monopolistically competitive firm using labor L_{jt} and physical capital K_{jt}:

$$Y_{jt} = (A_t L_{jt})^{1-\alpha} \; K_{jt}^{\alpha} \tag{54}$$

where the TFP process A_t is the sum of two components (in logs):

$$\log A_t = \log Z_t + \log \Xi_t \tag{55}$$

The variable Z_t denotes a non-stationary TFP series that evolves according to:

$$\frac{Z_t}{Z_{t-1}} = \left(\frac{Z_{t-1}}{Z_{t-2}}\right)^{\rho_\zeta} G_\zeta^{1-\rho_\zeta} \exp(\varepsilon_{\zeta,t}); \quad \varepsilon_{\zeta,t} \sim iid\ N(0, \sigma_\zeta^2)$$

where ρ_ζ is the persistence of the shock process, and $\varepsilon_{\zeta,t}$ is a random disturbance that causes deviations of the TFP growth from its balanced growth rate G_ζ. The stationary TFP evolves as follows:

$$\log \Xi_t = \rho_\xi \log \Xi_{t-1} + \varepsilon_{\xi,t}; \quad \varepsilon_{\xi,t} \sim iid\ N(0, \sigma_\xi^2)$$

where ρ_ξ is the persistence of the shock process, and $\varepsilon_{\xi,t}$ is an i.i.d shock with variance

σ_ξ^2. (We define $a_t \equiv \log A_t$, $\zeta_t \equiv \log Z_t$, $\xi_t \equiv \log \Xi_t$, $G_{a,t} \equiv A_t/A_{t-1}$, and $G_{\zeta,t} \equiv Z_t/Z_{t-1}$.)

Following BLL, we assume that

$$\rho_\zeta = \rho_\xi \equiv \rho$$

and that the variances satisfy the following restriction[43]

$$\rho\sigma_\zeta^2 = (1-\rho)^2\sigma_\xi^2$$

While agents observe the TFP process as a whole, they do not observe two components ζ_t and ξ_t separately. Considering the idea that agents have more information than merely about productivity, agents observe a noisy signal s_t about the permanent component of TFP:

$$s_t = \zeta_t + \varepsilon_{s,t}; \quad \varepsilon_{s,t} \sim iid\ N(0, \sigma_s^2) \tag{56}$$

where $\varepsilon_{s,t}$ is an i.i.d. normal shock, which affects agents' beliefs but is independent of fundamentals. This noisy signal relates to the additional informative signal that agents receive which is a straightforward interpretation of Equation (56). Ultimately, the presence of this noisy information helps the econometrician make inferences about the (unobserved) long-term productivity trend by looking at the behavior of consumption.

Firms choose inputs to minimize total cost each period. Marginal cost, independent of firm-specific variables, is given by $mc_t = \frac{1}{A_t^{1-\alpha}}\left(\frac{R_t^k/P_t}{\alpha}\right)^\alpha\left(\frac{W_t/P_t}{1-\alpha}\right)^{1-\alpha}$, where $\frac{R_t^k}{P_t}$ and $\frac{W_t}{P_t}$ denote aggregate rental rate of capital and real wage. A firm j pays a quadratic adjustment cost in units of final good (Rotemberg 1982) to adjust its price P_{jt}. The cost is given by $\frac{\psi_p}{2}\left(\frac{P_{jt}}{\tilde{\Pi}_{t-1}P_{jt-1}} - 1\right)^2 P_tY_t$, where $\psi_p \geq 0$ regulates the adjustment costs. Price change is indexed to $\tilde{\Pi}_{t-1} = \bar{\Pi}^{1-\iota_p}\Pi_{t-1}^{\iota_p}$, where ι_p governs indexation between previous period inflation rate Π_{t-1} and steady state inflation rate $\bar{\Pi}$. Firm's per period profits are given by: $D_{jt} \equiv P_{jt}Y_{jt} - P_tmc_tY_{jt} - \frac{\psi_p}{2}\left(\frac{P_{jt}}{\tilde{\Pi}_{t-1}P_{jt-1}} - 1\right)^2 P_tY_t$. Each period, the firm chooses P_{jt} to maximize present discounted value of real profits:

$$\max_{P_{jt}}\left\{\frac{\Lambda_t D_{jt}}{P_t} + \mathbb{E}_t^\theta\left[\sum_{s=1}^{\infty}\frac{\Lambda_{t+s}D_{jt+s}}{P_{t+s}}\right]\right\}$$

where Λ_t is the marginal utility of consumption in period t, and $\mathbb{E}_t^\theta[\ \cdot\ ]$ is the diag-

[43] As shown in BLL, these restrictions imply that the univariate process for a_t is a random walk with variance σ_a^2.

nostic expectation operator regulated by parameter θ. Notice that we write dynamic maximization problems by explicitly separating time t choice variables from the expectation of future choice variables. This separation is crucial for solving the model with diagnostic expectations, and is a consequence of the technical issues discussed in Section 2.

F.1.2 Households

There is a continuum of monopolistically competitive households, indexed by $i \in [0,1]$, supplying a differentiated labor input $L_{i,t}$. A perfectly competitive employment agency aggregates various labor types into a composite labor input L_t supplied to firms, in a Dixit-Stiglitz aggregator: $L_t = \left[\int_0^1 L_{i,t}^{\frac{\epsilon_{w,t}-1}{\epsilon_{w,t}}} di\right]^{\frac{\epsilon_{w,t}}{\epsilon_{w,t}-1}}$, where $\epsilon_{w,t} > 1$ is time-varying elasticity of demand. The iso-elastic demand for labor input i is given by: $L_{i,t} = \left(\frac{W_{i,t}}{W_t}\right)^{-\epsilon_{w,t}} L_t$, where $W_{i,t}$ is household i's wage rate, and W_t is the aggregate wage rate that the household takes as given.

The household i has following lifetime-utility at time t:

$$\left(\log(C_{i,t} - h\tilde{C}_{t-1}) - \frac{\omega}{1+\nu} L_{i,t}^{1+\nu} - \psi_{i,t}^w\right) + \mathbb{E}_t^\theta \left[\Sigma_{s=t+1}^\infty \beta^{s-t} \left(\log(C_{i,s} - h\tilde{C}_{s-1}) - \frac{\omega}{1+\nu} L_{i,s}^{1+\nu} - \psi_{i,s}^w\right)\right]$$

where h is the degree of habit formation on external habits over aggregate consumption $\tilde{C}_{t-1}$, which the household takes as given, $\nu > 0$ is inverse of the Frisch elasticity of labor supply, $\omega > 0$ is a parameter that pins down the steady-state level of hours, and the discount factor β satisfies $0 < \beta < 1$. $\psi_{i,t}^w$ is the loss in utility in adjusting wages. We assume a quadratic adjustment cost given by $\psi_{i,t}^w = \frac{\psi_w}{2}\left[\frac{W_{it}}{\tilde{\Pi}_{t-1}^w W_{it-1}} - 1\right]^2$, where $\psi_w \geq 0$ is a parameter, and wage contracts are indexed to productivity and price inflation. We assume $\tilde{\Pi}_{t-1}^w = G_a \bar{\Pi}^{1-\iota_w} \left(\exp(\varepsilon_{\zeta,t}\varepsilon_{\xi,t})\Pi_{t-1}\right)^{\iota_w}$ with $0 \leq \iota_w < 1$.

The household's budget constraint in period t is given by

$$P_t C_{i,t} + P_t I_{i,t} + \frac{B_{i,t+1}}{1+i_t} = B_{i,t} + W_{i,t} L_{i,t} + D_t + T_t + R_t^K u_{i,t} K_{i,t}^u - P_t a(u_{i,t}) K_{i,t}^u$$

where $I_{i,t}$ is investment, $W_{i,t}L_{i,t}$ is labor income, and $B_{i,t}$ is income from nominal bonds paying nominal interest rate i_t. Households own an equal share of all firms, and thus receive D_t dividends from profits. Finally, each household receives a lump-sum government transfer T_t.

The households own capital, $K_{i,t}^u$, and choose the utilization rate, $u_{i,t}$. The amount of effective capital, $K_{i,t}$, that the households rent to the firms at nominal rate R_t^K is given by $K_{i,t} = u_{i,t}K^u_{i,t}$. The (nominal) cost of capital utilization is $P_t\chi(u_{i,t})$ per unit of physical capital. As in the literature, we assume $\chi(1) = 0$ in the steady state and $\chi'' > 0$. Following GHLS, we assume investment adjustment costs, $S\left(\frac{I_{i,t}}{G_a I_{i,t-1}}\right)$, in the production of capital, where G_a is the steady state growth rate of A_t. Law of motion for capital is as follows:

$$K^u_{i,t+1} = \mu_t \left[1 - S\left(\frac{I_{i,t}}{G_a I_{i,t-1}}\right)\right] I_{i,t} + (1 - \delta_k)K^u_{i,t}$$

where δ_k denotes depreciation rate, and μ_t is an exogenous disturbance to the marginal efficiency of investment that follows:

$$\log \mu_t = \rho_\mu \log \mu_{t-1} + \varepsilon_{\mu,t}; \quad \varepsilon_{\mu,t} \sim iid\ N(0, \sigma^2_\mu)$$

As in the literature, we assume that $S(1) = S'(1) = 0$, and calibrate $S''(1) > 0$.

F.1.3 Government

The central bank follows a Taylor rule in setting the nominal interest rate i_t. It responds to deviations in (gross) inflation rate Π_t from its target rate $\bar{\Pi}$ and output.

$$\frac{1 + i_t}{1 + i_{ss}} = \left(\frac{1 + i_{t-1}}{1 + i_{ss}}\right)^{\rho_R} \left[\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi} Y_t^{\phi_y}\right]^{1-\rho_R} \exp(\lambda^{mp}_t) \tag{57}$$

with $0 < \rho_R < 1$, $\phi_\pi \geq 0$, and $\phi_y \geq 0$. i_{ss} is the steady state nominal interest rate, and λ^{mp}_t follows the process

$$\log \lambda^{mp}_t = \rho_{mp} \log \lambda^{mp}_{t-1} + \varepsilon_{mp,t}; \quad \varepsilon_{mp,t} \sim N(0, \sigma^2_{mp})$$

We assume government balances budget every period $P_t T_t = P_t G_t$, where G_t is the government spending. G_t is determined exogenously as as a fraction of GDP: $G_t = \left(1 - \frac{1}{\lambda^g_t}\right) Y_t$ where the government spending shock follows the process:

$$\log \lambda^g_t = (1 - \rho_g) \log \lambda^g + \rho_g \log \lambda^g_{t-1} + \varepsilon_{g,t}; \quad \varepsilon_{g,t} \sim N(0, \sigma^2_g)$$

λ^g is the steady state share of government spending in final output.

F.1.4 Market Clearing

We focus on a symmetric equilibrium where all intermediate goods producing firms and households make the same decisions. Therefore, we can drop subscripts i and j. The aggregate production function, in the symmetric equilibrium, is then given by: $Y_t = (A_t L_t)^{1-\alpha} K_t^{\alpha}$, since $K_t = K_{i,t} = K_{jt}$ and $N_t = N_{i,t} = N_{jt}$. The market clearing for the final good, in the symmetric equilibrium, requires that

$$Y_t = C_t + I_t + \chi(u_t)K_t^u + G_t + \frac{\psi_p}{2}\left[\frac{\Pi_t}{\tilde{\Pi}_{t-1}} - 1\right]^2 Y_t$$

This completes the presentation of the DSGE model.

F.2 Stationary Allocation

We normalize the following variables :

$$\begin{aligned}
y_t &= Y_t/A_t \\
c_t &= C_t/A_t \\
k_t &= K_t/A_t \\
k_t^u &= K_t^u/A_{t-1} \\
\mathbb{I}_t &= I_t/A_t \\
w_t &= W_t/(A_t P_t) \\
r_t^k &= R_t^k/P_t \\
\lambda_t &= \Lambda_t A_t
\end{aligned}$$

Definition 1 (Normalized Equilibrium) *18 endogenous variables* $\{\lambda_t, i_t, c_t, y_t, \Pi_t, mc_t, \tilde{\Pi}_{t-1}, \Pi_t^w, \tilde{\Pi}_{t-1}^w, w_t, L_t, k_{t+1}^u, r_t^K, \mathbb{I}_t, q_t, u_t, k_t, G_{a,t}\}$*, 8 endogenous shock processes* $\{G_{\zeta,t}, \Xi_t, s_t, \mu_t, \lambda_t^p, \lambda_t^w, \lambda_t^{mp}, \lambda_t^g\}$*, 8 exogenous shocks* $\{\varepsilon_{\zeta,t}, \varepsilon_{\xi,t}, \varepsilon_{s,t}, \varepsilon_{\mu,t}, \varepsilon_{p,t}, \varepsilon_{w,t}, \varepsilon_{mp,t}, \varepsilon_{g,t}\}$ *given initial values of* k_{t-1}^u.

Consumption Euler Equation

$$\frac{\lambda_t}{G_{a,t}\Pi_t} = \beta(1+i_t)\mathbb{E}_t^{\theta}\left[\frac{\lambda_{t+1}}{G_{a,t}G_{a,t+1}}\frac{1}{\Pi_t\Pi_{t+1}}\right] \tag{58}$$

$$\lambda_t = \frac{1}{c_t - \frac{hc_{t-1}}{G_{a,t}}} \tag{59}$$

Price-setting

$$(1-\epsilon_{p,t})+\epsilon_{p,t}\; mc_t-\psi_p\left(\frac{\Pi_t}{\tilde{\Pi}_{t-1}}-1\right)\frac{\Pi_t}{\tilde{\Pi}_{t-1}}+\psi_p\frac{\beta\Pi_t}{\lambda_t y_t}\mathbb{E}_t^{\theta}\left[\lambda_{t+1}\left(\frac{\Pi_{t+1}}{\tilde{\Pi}_t}-1\right)\frac{\Pi_{t+1}}{\tilde{\Pi}_t}\frac{y_{t+1}}{\Pi_t}\right]=0 \tag{60}$$

$$\tilde{\Pi}_{t-1}=\bar{\Pi}^{1-\iota_p}\Pi_{t-1}^{\iota_p} \tag{61}$$

Wage-setting

$$\psi_w\left[\frac{\Pi_t^w}{\tilde{\Pi}_{t-1}^w}-1\right]\frac{\Pi_t^w}{\tilde{\Pi}_{t-1}^w}=\psi_w\beta\mathbb{E}_t^{\theta}\left[\frac{\Pi_{t+1}^w}{\tilde{\Pi}_t^w}-1\right]\frac{\Pi_{t+1}^w}{\tilde{\Pi}_t^w}+L_t\lambda_t\epsilon_{w,t}\left[\omega\frac{L_t^{\nu}}{\lambda_t}-\frac{\epsilon_{w,t}-1}{\epsilon_{w,t}}w_t\right] \tag{62}$$

$$\tilde{\Pi}_{t-1}^w=G_a\bar{\Pi}^{1-\iota_w}(\exp(\varepsilon_{\zeta,t})\exp(\varepsilon_{\xi,t})\Pi_{t-1})^{\iota_w} \tag{63}$$

$$\Pi_t^w=\frac{w_t}{w_{t-1}}\Pi_t G_{a,t} \tag{64}$$

Capital Investment

$$k_{t+1}^u=\mu_t\left[1-S\left(\frac{\mathbb{I}_t}{\mathbb{I}_{t-1}}\frac{G_{a,t}}{G_a}\right)\right]\mathbb{I}_t+(1-\delta_k)\frac{k_t^u}{G_{a,t}} \tag{65}$$

$$q_t=\frac{\beta G_{a,t}}{\lambda_t}\mathbb{E}_t^{\theta}\left[\frac{\lambda_{t+1}}{G_{a,t}G_{a,t+1}}\left(r_{t+1}^K u_{t+1}-\chi(u_{t+1})+q_{t+1}(1-\delta_k)\right)\right] \tag{66}$$

$$\begin{aligned}&q_t\mu_t\left[1-S\left(\frac{\mathbb{I}_t}{\mathbb{I}_{t-1}}\frac{G_{a,t}}{G_a}\right)-S'\left(\frac{\mathbb{I}_t}{\mathbb{I}_{t-1}}\frac{G_{a,t}}{G_a}\right)\frac{\mathbb{I}_t}{\mathbb{I}_{t-1}}\frac{G_{a,t}}{G_a}\right]\\ &+\frac{\beta G_{a,t}}{\lambda_t}\mathbb{E}_t^{\theta}\left[\mu_{t+1}\frac{\lambda_{t+1}}{G_{a,t}}q_{t+1}\frac{G_{a,t+1}}{G_a}\left(\frac{\mathbb{I}_{t+1}}{\mathbb{I}_t}\right)^2 S'\left(\frac{\mathbb{I}_{t+1}}{\mathbb{I}_t}\frac{G_{a,t+1}}{G_a}\right)\right]=1\end{aligned} \tag{67}$$

Capital Utilization Rate

$$k_t=u_t\frac{k_t^u}{G_{a,t}} \tag{68}$$

$$r_t^K=\chi'(u_t) \tag{69}$$

Production Technologies

$$y_t=k_t^{\alpha}L_t^{1-\alpha} \tag{70}$$

$$\frac{k_t}{L_t}=\frac{w_t}{r_t^k}\frac{\alpha}{1-\alpha} \tag{71}$$

$$mc_t=\frac{(r_t^k)^{\alpha}w_t^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \tag{72}$$

Government

$$\frac{1+i_t}{1+i_{ss}} = \left(\frac{1+i_{t-1}}{1+i_{ss}}\right)^{\rho_R} \left[\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi} Y_t^{\phi_y}\right]^{1-\rho_R} \exp(\lambda_t^{mp}) \tag{73}$$

Market Clearing

$$y_t = c_t + \mathbb{I}_t + \chi(u_t)\frac{k_t^u}{G_{a,t}} + \left(1 - \frac{1}{\lambda_t^g}\right) y_t \tag{74}$$

TFP Growth Rate

$$\log G_{a,t} = \log G_{\zeta,t} + (\log \Xi_t - \log \Xi_{t-1}) \tag{75}$$

Law of Motion of Shocks

$$\log G_{\zeta,t} = (1-\rho)\log G_\zeta + \rho_\zeta \log G_{\zeta,t-1} + \varepsilon_{\zeta,t} \tag{76}$$

$$\log \Xi_t = \rho_\xi \log \Xi_{t-1} + \varepsilon_{\xi,t} \tag{77}$$

$$s_t = \log Z_t + \varepsilon_{s,t} \tag{78}$$

$$\log \mu_t = \rho_\mu \log(\mu_{t-1}) + \varepsilon_{\mu,t} \tag{79}$$

$$\log \lambda_t^{mp} = \rho_{mp} \log \lambda_{t-1}^{mp} + \varepsilon_{mp,t} \tag{80}$$

$$\log \lambda_t^g = \rho_g \log \lambda_{t-1}^g + \varepsilon_{g,t} \tag{81}$$

Disturbances

$$\text{TFP growth shock} \quad \varepsilon_{\zeta,t} \sim N(0, \sigma_\zeta^2) \tag{82}$$

$$\text{Stationary TFP shock} \quad \varepsilon_{\xi,t} \sim N(0, \sigma_\xi^2) \tag{83}$$

$$\text{Noise shock} \quad \varepsilon_{s,t} \sim N(0, \sigma_s^2) \tag{84}$$

$$\text{MEI shock} \quad \varepsilon_{\mu,t} \sim N(0, \sigma_\mu^2) \tag{85}$$

$$\text{Monetary policy shock} \quad \varepsilon_{mp,t} \sim N(0, \sigma_{mp}^2) \tag{86}$$

$$\text{Government spending shock} \quad \varepsilon_{g,t} \sim N(0, \sigma_g^2) \tag{87}$$

F.3 Steady State

$$1 = \beta \frac{1}{G_a} \frac{1+i}{\Pi}$$

$$\lambda = \frac{G_a}{c(G_a - h)}$$

$$mc = \frac{\epsilon_p}{\epsilon_p - 1}$$

$$\frac{\omega L^\nu}{\lambda} = \frac{\epsilon_w - 1}{\epsilon_w} w$$

$$\Pi^w = \Pi G_a$$

$$\Pi = \bar{\Pi}$$

$$q = 1$$

$$u = 1$$

$$(1 - \frac{1 - \delta_k}{G_a}) k^u = \mathbb{I}$$

$$1 = \beta \left[\frac{1}{G_a} \left(r^K + (1 - \delta_k) \right) \right]$$

$$k = \frac{k^u}{G_a}$$

$$r^K = \chi'(1)$$

$$y = k^\alpha L^{1-\alpha},$$

$$r^k = \frac{\epsilon_p}{\epsilon_p - 1} \alpha \frac{y}{k}$$

$$w = \frac{\epsilon_p}{\epsilon_p - 1} (1 - \alpha) \frac{y}{L}$$

$$y = c + \mathbb{I} + \left(1 - \frac{1}{\lambda^g} \right) y$$

$$S(1) = S'(1) = 0; S" > 0$$

$$G_a = G_\zeta$$

F.4 Log-linearized Model

Consumption Euler Equation

$$\hat{\lambda}_t - \hat{G}_{a,t} - \pi_t = \hat{i}_t + \mathbb{E}_t^\theta \left[\hat{\lambda}_{t+1} - \hat{G}_{a,t} - \hat{G}_{a,t+1} - \pi_t - \pi_{t+1} \right] \tag{88}$$

$$\hat{\lambda}_t + \frac{G_a}{G_a - h} \hat{c}_t - \frac{h}{G_a - h} \left(\hat{c}_{t-1} - \hat{G}_{a,t} \right) = 0 \tag{89}$$

Price-setting

$$\pi_t = \beta \mathbb{E}_t^\theta \pi_{t+1} - \iota_p \beta \mathbb{E}_t^\theta \pi_t + \iota_p \pi_{t-1} + \frac{\epsilon_p - 1}{\psi_p} \hat{mc}_t + \hat{\lambda}_t^{p,*} \tag{90}$$

where $\hat{\lambda}_t^{p,*}$ is the normalized price-markup shock process. Let the un-normalized process be denoted with $\hat{\lambda}_t^p$. Then $\hat{\lambda}_t^{p,*} = \frac{\epsilon_p - 1}{\psi_p} \hat{\lambda}_t^p$. In steady state $\lambda^p = \frac{\epsilon_p}{\epsilon_p - 1}$

Wage-setting

$$\pi_t^w = \beta \mathbb{E}_t^\theta \pi_{t+1}^w - \iota_w \beta \mathbb{E}_t^\theta \pi_t - \iota_w \beta \mathbb{E}_t^\theta \hat{G}_{a,t+1} + \iota_w \pi_{t-1} + \iota_w \hat{G}_{a,t} + \frac{\epsilon_w \omega L^{1+\nu}}{\psi_w} \left[\nu \hat{L}_t - \hat{w}_t - \hat{\lambda}_t \right] + \hat{\lambda}_t^{w,*} \tag{91}$$

where $\hat{\lambda}_t^{w,*}$ is the normalized wage-markup shock process. Let the un-normalized wage markup process be denoted with $\hat{\lambda}_t^w$. Then $\hat{\lambda}_t^{w,*} = \frac{\epsilon_w \omega L^{1+\nu}}{\psi_w} \hat{\lambda}_t^w$. In steady state $\lambda^w = \frac{\epsilon_w}{\epsilon_w - 1}$

$$\pi_t^w = \hat{w}_t - \hat{w}_{t-1} + \pi_t + \hat{G}_{a,t} \tag{92}$$

Capital Investment

$$\hat{k}_{t+1}^u = \frac{\mathbb{I}}{k^u} \left(\hat{I}_t + \hat{\mu}_t \right) + \frac{1 - \delta_k}{G_a} \left(\hat{k}_t^u - \hat{G}_{a,t} \right) \tag{93}$$

$$\hat{q}_t - \hat{G}_{a,t} + \hat{\lambda}_t = \mathbb{E}_t^\theta \left[\hat{\lambda}_{t+1} - \hat{G}_{a,t} - \hat{G}_{a,t+1} + \frac{r^K}{r^K + 1 - \delta_k} \hat{r}_{t+1}^K + \frac{1 - \delta_k}{r^K + 1 - \delta_k} \hat{q}_{t+1} \right] \tag{94}$$

$$\hat{q}_t + \hat{\mu}_t - S\text{''}(1) \left(\hat{I}_t - \hat{I}_{t-1} + \hat{G}_{a,t} \right) + \beta S\text{''}(1) \mathbb{E}_t^\theta \left(\hat{I}_{t+1} - \hat{I}_t + \hat{G}_{a,t+1} \right) = 0 \tag{95}$$

Capital Utilization Rate

$$\hat{k}_t = \hat{u}_t + \hat{k}_t^u - \hat{G}_{a,t} \tag{96}$$

$$\hat{r}_t^K = \frac{\chi\text{''}(1)}{\chi'(1)} \hat{u}_t \tag{97}$$

Production Technologies

$$\hat{y}_t = \alpha \hat{k}_t + (1 - \alpha) \hat{L}_t \tag{98}$$

$$\hat{r}_t^K = \hat{w}_t + \hat{L}_t - \hat{k}_t \tag{99}$$

$$\hat{mc}_t = \alpha \hat{r}_t^K + (1 - \alpha) \hat{w}_t \tag{100}$$

Government

$$\hat{i}_t = \rho_R \hat{i}_{t-1} + (1-\rho_R)(\phi_\pi \pi_t + \phi_y \hat{y}_t) + \varepsilon_{mp,t} \tag{101}$$

Market Clearing

$$\frac{1}{\lambda^g}\hat{y}_t = \frac{c}{y}\hat{c}_t + \frac{\mathbb{I}}{y}\hat{I}_t + \frac{\chi'(1)k}{y}\hat{u}_t + \frac{1}{\lambda^g}\hat{\lambda}_t^g \tag{102}$$

TFP Growth Rate

$$\hat{G}_{a,t} = \hat{G}_{\zeta,t} + \hat{\xi}_t - \hat{\xi}_{t-1} \tag{103}$$

$$\hat{a}_t = \hat{\zeta}_t + \hat{\xi}_t \tag{104}$$

where $\hat{a}_t$ and $\hat{\zeta}_t$ are defined as log deviations of A_t and Z_t from their initial values.

Law of Motion of Shocks

$$\hat{G}_{\zeta,t} = \rho_\zeta \hat{G}_{\zeta,t-1} + \varepsilon_{\zeta,t} \tag{105}$$

$$\hat{\xi}_t = \rho_\xi \hat{\xi}_{t-1} + \varepsilon_{\xi,t} \tag{106}$$

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + \varepsilon_{\mu,t} \tag{107}$$

$$\hat{\lambda}_t^{mp} = \rho_{mp} \hat{\lambda}_{t-1}^{mp} + \varepsilon_{mp,t} \tag{108}$$

$$\hat{\lambda}_t^{g} = \rho_{g} \hat{\lambda}_{t-1}^{g} + \varepsilon_{g,t} \tag{109}$$

$$\hat{\lambda}_t^{p,*} = \rho_p \hat{\lambda}_{t-1}^{p,*} + \varepsilon_{p,t} - \phi_p \varepsilon_{p,t-1} \tag{110}$$

$$\hat{\lambda}_t^{w,*} = \rho_w \hat{\lambda}_{t-1}^{w,*} + \varepsilon_{w,t} - \phi_w \varepsilon_{w,t-1} \tag{111}$$

Disturbances

$$\text{TFP growth shock} \quad \varepsilon_{\zeta,t} \sim N(0, \sigma_\zeta^2) \tag{112}$$

$$\text{Stationary TFP shock} \quad \varepsilon_{\xi,t} \sim N(0, \sigma_\xi^2) \tag{113}$$

$$\text{Noise shock} \quad \varepsilon_{s,t} \sim N(0, \sigma_s^2) \tag{114}$$

$$\text{MEI shock} \quad \varepsilon_{\mu,t} \sim N(0, \sigma_\mu^2) \tag{115}$$

$$\text{Monetary policy shock} \quad \varepsilon_{mp,t} \sim N(0, \sigma_{mp}^2) \tag{116}$$

$$\text{Government spending shock} \quad \varepsilon_{g,t} \sim N(0, \sigma_g^2) \tag{117}$$

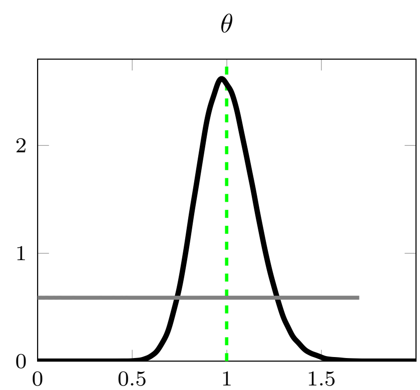
$$\text{Price markup shock} \quad \varepsilon_{p,t} \sim N(0, \sigma_p^2) \tag{118}$$

$$\text{Wage markup shock} \quad \varepsilon_{w,t} \sim N(0, \sigma_w^2) \tag{119}$$

F.5 Posterior Distribution of Diagnosticity Parameter

Figure 5 shows the prior and the posterior distribution of the diagnosticity parameter θ. The gray solid line denotes the prior distribution of θ. The diagnosticity parameter is assumed to follow a uniform distribution between 0 and 1.7. Our prior on θ is based on the estimated degree of diagnosticity in macro and financial variables.[44] The black solid line denotes the posterior distribution whereas the green dashed line denotes the posterior mode at 0.9918. (The estimated posterior mode is very close to the estimated posterior mean at 0.9992.) The data appear very informative, as indicated by the lower variance of the posterior distribution relative to the priorly assumed uniform distribution.

Figure 5: Posterior Distribution: Diagnosticity



Notes: The figure depicts the prior and posterior distribution of the diagnosticity parameter θ. The gray solid line denotes the prior distribution of θ, which follows a uniform distribution over 0 and 1.7. The black solid line (the green dashed) denotes the posterior distribution (the posterior mode) of θ.

[44]See Table 4 of Bordalo, Gennaioli, Ma, and Shleifer (2020).

F.6 Extra Volatility

Table 4 shows unconditional volatilities of output growth, consumption growth, investment growth, employment, capacity utilization, capital stock growth, inflation, and the nominal interest rate under diagnostic and rational expectations. We simulate the model using eight structural shocks with the estimated parameters in Table 1. Looking at the amount of extra volatility afforded by DE, measured as the percentage increase in unconditional volatility under DE from that under RE, DE generate a substantial volatility increase in quantities such as output (23%), consumption (36%), investment growth (23%). DE also generate a moderate volatility increase in employment (6%), capacity utilization (7%), and capital stock (8%). On the contrary, implied volatilities are lower for the nominal interest rate (−1%) and inflation (−10%) under DE.[45]

Table 4: Model-Implied Volatilities in the Medium-Scale DSGE Model

Variable	Rational Expectations	Diagnostic Expectations	Percentage Increase
Output	0.8886	1.0905	23%
Consumption	0.5616	0.7634	36%
Investment	3.6365	4.4720	23%
Employment	4.2786	4.5188	6%
Capacity Utilization	0.9766	1.0424	7%
Capital Stock	0.5194	0.5619	8%
Inflation	0.6316	0.5718	-10%
Nominal Interest Rate	0.6668	0.6610	-1%

Notes: The table reports the standard deviations of output growth, consumption growth, investment growth, employment, capacity utilization, capital stock growth, inflation, and the nominal interest rate in the medium-scale DSGE model. The final column entitled "Percentage Increase" shows the percentage increase in standard deviation under the DE model relative to the RE benchmark (setting $\theta = 0$ along with parameter estimates in Table 1). There are eight structural shocks in the model, as in Blanchard et al. (2013): monetary policy shocks, investment-specific shocks, government spending shocks, TFP growth rate shocks, stationary TFP shocks, price and wage mark-up shocks, and noise shocks.

[45] The full set of IRFs is available upon request.

F.7 Variance Decomposition

Turning to the variance decomposition, the contribution of each of the structural shocks to the forecast error variance of the endogenous variables at various horizons is shown in Table 5. We find that the noise and stationary TFP shocks are the main short-run driver of consumption, accounting for about two-thirds of consumption volatility in the very short run. A sizeable fraction of short-run consumption volatility is also explained by the monetary policy shock. Virtually all short-run investment volatility is due to the marginal efficiency of investment (MEI) shock. The responses of aggregate output follow from those of consumption and investment. Consequently, the three most important drivers of output are the stationary TFP, noise shock, and MEI shocks, with the latter explaining about 40 percent of volatility at a 1-year horizon. On the contrary, the TFP growth shock only explains 3 percent of output volatility at a 1-year horizon.

Table 5: Variance Decomposition

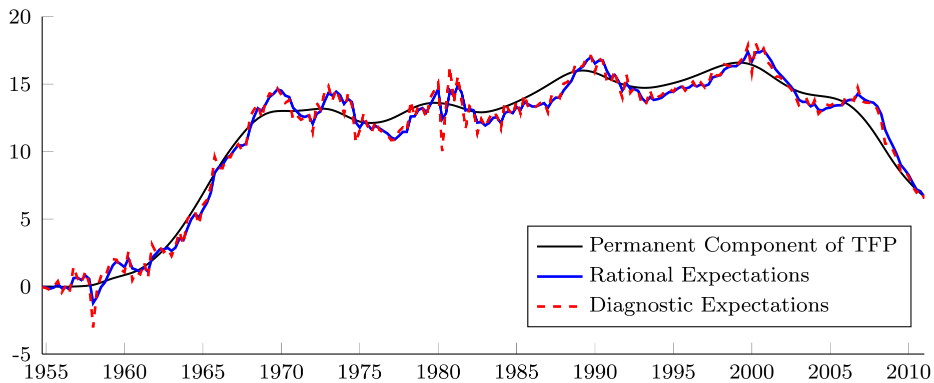
Quarter	TFP growth	Stationary TFP	Noise	MEI	Price markup	Wage markup	Monetary	Govt spending
Cons.								
1	0.003	0.314	0.378	0.000	0.001	0.000	0.230	0.074
4	0.048	0.401	0.271	0.000	0.015	0.003	0.164	0.099
8	0.223	0.358	0.167	0.001	0.022	0.011	0.095	0.122
12	0.439	0.249	0.101	0.004	0.015	0.017	0.058	0.118
Inv.								
1	0.000	0.048	0.027	0.884	0.025	0.000	0.016	0.000
4	0.004	0.052	0.018	0.883	0.033	0.000	0.009	0.000
8	0.023	0.062	0.014	0.853	0.038	0.002	0.007	0.001
12	0.073	0.068	0.012	0.798	0.038	0.004	0.006	0.001
Output								
1	0.002	0.216	0.250	0.316	0.005	0.002	0.150	0.059
4	0.027	0.245	0.150	0.421	0.038	0.003	0.087	0.029
8	0.131	0.237	0.095	0.405	0.051	0.008	0.051	0.023
12	0.310	0.197	0.066	0.316	0.040	0.013	0.036	0.021

Notes: The table reports the contribution of each of the structural shocks to the forecast error variance of the endogenous variables at various horizons (1, 4, 8, and 12 quarters). We consider eight structural shocks - the TFP growth shock, stationary TFP shock, noise shock, marginal efficiency of investment (MEI) shock, price markup shock, wage markup shock, monetary policy shock, and government spending shock.

F.8 Historical Decomposition of Agents' Beliefs

Using data from times 1 to T, we can recover the best estimates of states and shocks at any time $t \leq T$ with the Kalman smoother. Figure 6 plots the econometrician's smoothed estimate of the permanent, non-stationary component of TFP and the smoothed series for the agents' real time estimate regarding the permanent component of TFP under rational and diagnostic expectations.[46] The solid black line denotes the econometrician's smoothed estimate of this permanent component ($\hat{\zeta}_{t|T}$). The dashed blue and solid red lines correspond to the smoothed series for the agents' real-time rational estimate of this variable ($\hat{\zeta}_{(t|t)|T}$) and for the agents' real-time diagnostic estimate of the same variable ($\hat{\zeta}^{\theta}_{(t|t)|T}$).[47] The smoothed estimate of the permanent component of TFP under DE is 112% more volatile than under RE.

Figure 6: Smoothed Estimates of the Permanent Component of TFP and of Agents' Real Time Expectations



Notes: The black solid line denotes the econometrician's smoothed estimate of the permanent component of TFP ($\hat{\zeta}_{t|T}$) whereas the blue solid and red dashed lines correspond to the smoothed series for agents' real time expectations regarding the permanent component of TFP under rational ($\hat{\zeta}_{(t|t)|T}$) and diagnostic expectations ($\hat{\zeta}^{\theta}_{(t|t)|T}$), respectively. As in Blanchard, L'Huillier, and Lorenzoni (2013), an aggregate TFP process $\hat{a}_t$ is composed of two components, a permanent component $\hat{\zeta}_t$ and a transitory component $\hat{\xi}_t$: $\hat{a}_t = \hat{\zeta}_t + \hat{\xi}_t$.

[46]It is worth noting that the permanent component of TFP is different from the long-run level of TFP discussed in Figure 3. As in BLL, an aggregate TFP process $\hat{a}_t$ is composed of two components, a permanent component $\hat{\zeta}_t$ and a transitory component $\hat{\xi}_t$: $\hat{a}_t = \hat{\zeta}_t + \hat{\xi}_t$, and the long-run level of TFP is given by $\hat{a}_{t+\infty} = (\hat{\zeta}_t - \rho\hat{\zeta}_{t-1})/(1-\rho)$, where $\hat{\zeta}_t$ is the permanent component of TFP.

[47]Let $\mathcal{I}_t$ denotes data available from times 1 to T. The smoothed series for agents' real time estimate regarding the permanent component of TFP under diagnostic expectations is given by

$$\hat{\zeta}^{\theta}_{(t|t)|T} = \hat{\zeta}_{(t|t)|T} + \theta\big(\hat{\zeta}_{(t|t)|T} - \hat{\zeta}_{(t|t-1)|T}\big)$$

where $\hat{\zeta}_{(t|t)|T} \equiv \mathbb{E}_t[\hat{\zeta}_t|\mathcal{I}_T]$ and $\hat{\zeta}_{(t|t-1)|T} \equiv \mathbb{E}_{t-1}[\hat{\zeta}_t|\mathcal{I}_T]$.

F.9 Prior Distribution of the Parameters

The following parameters are fixed in the estimation procedure as shown in Table 6. The depreciation rate δ_k is fixed at 0.025, and the discount factor β is set to 0.99. The Dixit-Stiglitz aggregator for the goods (ϵ_p) and for labor services (ϵ_w) are fixed at 6. The parameter affecting the level of disutility from working (ω) is set to 1, and the steady-state share of government spending to final output is fixed at 1.2. The diagnosticity parameter θ is assumed to follow a uniform distribution between 0 and 1.7.

Table 6: Fixed Parameters

	Parameter	Value
β	Discount factor	0.99
δ_k	Capital depreciation rate	0.025
$1 - \frac{1}{\lambda_g}$	Government spending share	0.20
ω	Labor preference	1
ϵ_p	Elasticity of goods demand	6
ϵ_w	Elasticity of labor demand	6

Notes: The table reports parameters fixed in the estimation procedure for both DE and RE.

Table 7 reports the prior distribution of structural parameters.

Table 7: Prior Distribution of Structural Parameters

	Parameter	Distribution	Mean	St. Dev.
θ	Diagnosticity	Uniform	0.85	0.4907
h	Habit	Beta	0.5	0.1
α	Capital share	Normal	0.3	0.05
ν	Inv. Frisch elasticity	Gamma	2	0.75
$\frac{a''(1)}{a'(1)}$	Capital utilization cost	Gamma	5	1
$S''(1)$	Investment Adjustment cost	Normal	4	1
ψ_p	Price adjustment	Normal	100	25
ψ_w	Wage adjustment	Normal	3000	5000
ϕ_π	Taylor rule inflation	Normal	1.5	0.3
ϕ_x	Taylor rule output	Normal	0.005	0.005
	Technology and noise			
ρ		Beta	0.6	0.2
σ_a		Inv. Gamma	0.5	1
σ_s		Inv. Gamma	1	1
	Investment-specific			
ρ_μ		Beta	0.6	0.2
σ_μ		Inv. Gamma	5	1.5
	Markups			
ρ_p		Beta	0.6	0.2
ϕ_p		Beta	0.5	0.2
σ_p		Inv. Gamma	0.15	1
ρ_w		Beta	0.6	0.2
ϕ_w		Beta	0.5	0.2
σ_w		Inv. Gamma	0.15	1
	Policy			
ρ_R		Beta	0.5	0.2
ρ_{mp}		Beta	0.4	0.2
σ_{mp}		Inv. Gamma	0.15	1
ρ_g		Beta	0.6	0.2
σ_g		Inv. Gamma	0.5	1

Notes: The table reports the prior distribution of structural parameters in the estimation procedure. The diagnosticity parameter θ is fixed at 0 under RE.

F.10 Posterior Estimates of the Parameters under RE

Table 8 reports the parameter estimates under RE. We use the prior distribution of the parameters described in Tables 6 and 7.

Table 8: Imperfect Information RE: Estimated Parameters

	Parameter	Prior	Posterior	Conf. bands		Distribution	Prior SD
h	Habit	0.5	0.4397	0.3832	0.4948	Beta	0.1
α	Production function	0.3	0.1455	0.1353	0.1558	Normal	0.05
ν	Inv. Frisch elasticity	2	1.5194	0.6509	2.3048	Gamma	0.75
$\frac{a''(1)}{a'(1)}$	Capital utilization cost	5	5.1804	3.5247	6.7867	Gamma	1
$S''(1)$	Investment Adjustment cost	4	4.4618	3.1785	5.7109	Normal	1
ψ_p	Price adjustment	100	148.65	117.98	179.88	Normal	25
ψ_w	Wage adjustment	3000	12990.06	7802.70	17980.37	Normal	5000
ϕ_π	Taylor rule inflation	1.5	1.0347	1.0001	1.0751	Normal	0.3
ϕ_x	Taylor rule output	0.005	0.0080	0.0028	0.0130	Normal	0.005
	Technology and noise						
ρ		0.6	0.9304	0.9105	0.9503	Beta	0.2
σ_a		0.5	1.1669	1.0640	1.2668	Inv. Gamma	1
σ_s		1	1.4998	0.6487	2.3157	Inv. Gamma	1
	Investment-specific						
ρ_μ		0.6	0.5062	0.3738	0.6398	Beta	0.2
σ_μ		5	10.8984	6.6969	14.9452	Inv. Gamma	1.5
	Markups						
ρ_p		0.6	0.8343	0.7637	0.9083	Beta	0.2
ϕ_p		0.5	0.5549	0.3870	0.7272	Beta	0.2
σ_p		0.15	0.1916	0.1599	0.2232	Inv. Gamma	1
ρ_w		0.6	0.9415	0.9092	0.9758	Beta	0.2
ϕ_w		0.5	0.9621	0.9423	0.9830	Beta	0.2
σ_w		0.15	0.6135	0.5617	0.6651	Inv. Gamma	1
	Policy						
ρ_R		0.5	0.4827	0.4244	0.5402	Beta	0.2
ρ_{mp}		0.4	0.0346	0.0019	0.0662	Beta	0.2
σ_{mp}		0.15	0.3789	0.3427	0.4148	Inv. Gamma	1
ρ_g		0.6	0.9971	0.9942	0.9999	Beta	0.2
σ_g		0.5	0.3555	0.3276	0.3836	Inv. Gamma	1
	log Marg. Likelihood	-1590.66					

Notes: The table reports mean posterior estimates, along with 2.5% and 97.5% percentiles. We ran 1,500,000 MH draws, discarding the first 40% as initial burn-in. The observation equation is composed of U.S. time series for GDP, consumption, investment, employment, the federal funds rate, inflation, and wages.