Trade Dynamics in the Market for Federal Funds*

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February 2012

Abstract

We develop a model of the market for federal funds that explicitly accounts for its two distinctive features: banks have to search for a suitable counterparty, and once they have met, both parties negotiate the size of the loan and the repayment. The theory is used to answer a number of positive and normative questions: What are the determinants of the fed funds rate? How does the market reallocate funds? Is the market able to achieve an efficient reallocation of funds? We also use the model for theoretical and quantitative analyses of policy issues facing modern central banks.

Keywords: Fed funds market, search, bargaining, over-the-counter
JEL Classification: G1, C78, D83, E44

*We are especially grateful to Todd Keister for his generous feedback at various stages, and to Darrell Duffie for his discussion at the 2011 NBER Asset Pricing Meeting in Stanford. We also thank Tan Wang, our discussant at the 46th Annual Conference of the Western Finance Association. The views expressed in the paper are those of the authors and are not necessarily reflective of views at the Federal Reserve Bank of New York or the Federal Reserve System.

1 Introduction

In the United States, financial institutions keep reserve balances at the Federal Reserve Banks to meet requirements, earn interest, or to clear financial transactions. The *market for federal funds* is an interbank over-the-counter market for unsecured, mostly overnight loans of dollar reserves held at Federal Reserve Banks. This market allows institutions with excess reserve balances to lend reserves to institutions with reserve deficiencies. A particular average measure of the market interest rate on these loans is commonly referred to as the *fed funds rate*.

The fed funds market is primarily a mechanism that reallocates reserves among banks. As such, it is a crucial market from the standpoint of the economics of payments, and the branch of banking theory that studies the role of interbank markets in helping banks manage reserves and offset liquidity or payment shocks. The fed funds market is the setting where the interest rate on the shortest maturity, most liquid instrument in the term structure is determined. This makes it an important market from the standpoint of Finance. The fed funds rate affects commercial bank decisions concerning loans to businesses and individuals, and has important implications for the loan and investment policies of financial institutions more generally. This makes the fed funds market critical to macroeconomists. The fed funds market is the epicenter of monetary policy implementation: The Federal Open Market Committee (FOMC) communicates monetary policy by choosing the fed funds rate it wishes to prevail in this market, and implements monetary policy by instructing the trading desk at the Federal Reserve Bank of New York to "create conditions in reserve markets" that will encourage fed funds to trade at the target level. As such, the fed funds market is of first-order importance for economists interested in monetary theory and policy. For these reasons, we feel it is crucial to pry into the micro mechanics of trade in the market for federal funds, in order to understand the mechanism by which this market reallocates liquidity among banks, and the determination of the market price for this liquidity provision—the fed funds rate.

To this end, we develop a dynamic equilibrium model of trade in the fed funds market that explicitly accounts for the two distinctive features of the over-the-counter structure of the actual fed funds market: search for counterparties, and bilateral negotiations. In the theory, banks are required to hold a certain level of end-of-day reserve balances, and participate in the fed funds market to achieve this target. We model the fed funds market as an over-the-counter market in which banks randomly contact other banks, and once they meet, bargain over the terms of the loans. The model is presented in Section 2 and its key building blocks are related to the main institutional features of the market for federal funds in Section 3. In Sections 4 and 5 we define and characterize the equilibrium, and provide theoretical answers to a number of elementary positive and normative questions: What are the determinants of the fed funds rate? What accounts for the dispersion in observed rates? How does the market reallocate funds? Is the assumed market structure able to achieve an efficient reallocation of funds?

In Section 6 we use the theory to identify the determinants of commonly used empirical measures of trade volume, trading delays, and the fed funds rate. We also describe the equilibrium dynamics of the reserve balances of individual banks, and propose theory-based measures of the importance of bank-provided intermediation in the process of reallocation of reserves. Section 7 proves a number of propositions in the context of a small-dimensional version of the theory that can be analyzed using paper-and-pencil methods. In Section 8 we calibrate and simulate a large-scale version of the model to assess the ability of the theory to capture the salient empirical features of the market for federal funds in the United States, such as the intraday evolution of the distribution of reserve balances, the dispersion in loan sizes and fed funds rates, the skewness in the distribution of the number of transactions per bank, the intraday patterns of trade volume, and the skewness of the distribution of the proportion of traded funds that are intermediated by the banking sector. Finally, we use the large-scale calibrated model as a laboratory to study a key issue in modern central banking, namely the effectiveness of policies that use the interest rate on banks' reserves as a tool to manage the fed funds rate.

Appendix A contains all proofs. The baseline model has banks that only differ in their initial holdings of reserve balances, so in Appendix B we develop extensions that allow for ex-ante heterogeneity in bank types. Each extension is motivated by a particular aspect of the fed funds market that our baseline model has abstracted from. One extension allows banks to differ in their bargaining strengths. Another allows for heterogeneity in the rate at which banks contact potential trading partners. A third extension allows for the fact that policy may induce heterogeneity in the fed fund participants' payoffs from holding end-of-day balances. For example, the Federal Reserve remunerates the reserve balances of some participants, e.g., depository institutions, but not others, e.g., Government Sponsored Enterprises (GSEs). In Appendix C we compute the equilibrium of a small-scale example and carry out comparative dynamic experiments to illustrate and complement the analytical results of Section 7. Appendix D contains supplementary policy experiments.

Early research on the fed funds market includes the theoretical work of Poole (1968), Ho and Saunders (1985) and Coleman, Christian and Labadie (1996), and the empirical work of Hamilton (1996) and Hamilton and Jordà (2002). The over-the-counter nature of the fed funds market was stressed by Ashcraft and Duffie (2007) in their empirical investigation, and used by Bech and Klee (2011), Ennis and Weinberg (2009), and Furfine (2003), to try to explain certain aspects of interbank markets such as apparent limits to arbitrage, stigma, and banks' decisions to borrow from standing facilities. Relative to the existing literature on the fed funds market, our contribution is to model the intraday allocation of reserves and pricing of overnight loans using a dynamic equilibrium search-theoretic framework that captures the salient features of the decentralized interbank market in which these loans are traded. Recently, the search-theoretic techniques introduced in labor economics by Diamond (1982a, 1982b), Mortensen (1982) and Pissarides (1985) have been extended and applied to other fields. Our work is related to the young literature that studies search and bargaining frictions in financial markets. To date, this literature consists of two subfields: one that deals with macro issues, and another that focuses on micro considerations in the market microstructure tradition.

On the macro side, for instance, Lagos (2010a, 2010b, 2011) uses versions of the Lagos and Wright (2005) search-based model of exchange to study the effect of liquidity and monetary policy on asset prices. On the micro side, Duffie, Gârleanu and Pedersen (2005) employ search-theoretic techniques to model the trading frictions characteristic of real-world over-the-counter markets. Their work has been extended by Lagos and Rocheteau (2007, 2009) to allow for general preferences and unrestricted long positions, and by Vayanos and Wang (2007) and Weill (2008) to allow investors to trade multiple assets. Duffie, Gârleanu and Pedersen (2007) incorporate risk aversion and risk limits, and Afonso (2011) endogenizes investors' entry decision to the market. Relative to this particular micro branch of the literature, our contribution is twofold. First, our model of the fed funds market provides a theoretical framework to interpret and rationalize the findings of existing empirical investigations of this market, such as Furfine (1999), Ashcraft and Duffie (2007), Bech and Atalay (2008), and Afonso, Kovner and Schoar (2011). Our second contribution is methodological: we offer the first analytically tractable formulation of a search-based model of an over-the-counter market in which all trade is bilateral, and agents can hold essentially unrestricted asset positions.[1]

[1] In contrast, the tractability of the model of Lagos and Rocheteau (2009) (the only other tractable formulation of a search-based over-the-counter market with unrestricted asset holdings) relies on the assumption that all trade among investors is intermediated by dealers who have continuous access to a competitive interdealer market.

2 The model

There is a large population of agents that we refer to as *banks*, each represented by a point in the interval $[0, 1]$. Banks hold integer amounts of an asset that we interpret as *reserve balances*, and can negotiate these balances during a trading session set in continuous time that starts at time 0 and ends at time T. Let τ denote the time remaining until the end of the trading session, so $\tau = T - t$ if the current time is $t \in [0, T]$. The reserve balance that a bank holds (e.g., at its Federal Reserve account) at time $T - \tau$ is denoted by $k(\tau) \in \mathbb{K}$, with $\mathbb{K} = \{0, 1, ..., K\}$, where $K \in \mathbb{Z}$ and $1 \leq K$. The measure of banks with balance k at time $T - \tau$ is denoted $n_k(\tau)$. A bank starts the trading session with some balance $k(T) \in \mathbb{K}$. The initial distribution of balances, $\{n_k(T)\}_{k \in \mathbb{K}}$, is given. Let $u_k \in \mathbb{R}$ denote the flow payoff to a bank from holding k balances during the trading session, and let $U_k \in \mathbb{R}$ be the payoff from holding k balances at the end of the trading session. All banks discount payoffs at rate r.

Banks can trade balances with each other in an over-the-counter market where trading opportunities are bilateral and random, and represented by a Poisson process with arrival rate $\alpha > 0$. We model these bilateral transactions as loans of reserve balances. Once two banks have made contact, they bargain over the size of the loan and the quantity of reserve balances to be repaid by the borrower. After the terms of the transaction have been agreed upon, the banks part ways. We assume that (signed) loan sizes are elements of the set $\bar{\mathbb{K}} = \mathbb{K} \cup \{-K, ..., -1\}$, and that every loan gets repayed at time $T + \Delta$ in the following trading day, where $\Delta \in \mathbb{R}_+$. Let $x \in \mathbb{R}$ denote the net credit position (of federal funds due at $T + \Delta$) that has resulted from some history of trades. We assume that the payoff to a bank with a net credit position x who makes a new loan at time $T - \tau$ with repayment R at time $T + \Delta$, is equal to the post-transaction discounted net credit position, $e^{-r(\tau+\Delta)}(x + R)$.

3 Institutional features of the market for federal funds

The market for federal funds is a market for unsecured loans of reserve balances at the Federal Reserve Banks, that allows participants with excess reserve balances to lend (or *sell funds*) to those with reserve balance shortages. These unsecured loans, commonly referred to as *fed(eral) funds*, are delivered on the same day and their duration is typically overnight.[2] The interest rate

While there are several examples of such pure dealer markets, the market for federal funds is not one of them.

[2]There is a *term fed funds market* where maturities range from a few days to more than a year. This market has been estimated to be much smaller than the overnight market (Meulendyke, 1998, Kuo et al., 2010).

on these loans is known as the *fed funds rate*. Most fed funds transactions are settled through *Fedwire Funds Services* (Fedwire), a large-value real-time gross settlement system operated by the Federal Reserve Banks.[3] Participants include commercial banks, thrift institutions, agencies and branches of foreign banks in the United States, government securities dealers, government agencies such as federal or state governments, and GSEs (e.g., Freddie Mac, Fannie Mae, and Federal Home Loan Banks).[4] The market for fed funds is an over-the-counter market: in order to trade, a financial institution must first find a willing counterparty, and then bilaterally negotiate the size and rate of the loan. We use a search-based model to capture the over-the-counter nature of this market.[5]

In practice, there are two ways of trading federal funds. Two participants can contact each other directly and negotiate the terms of a loan, or they can be matched by a fed funds broker. Non-brokered transactions represent the bulk of the volume of fed funds loans, so we abstract from brokers in our baseline model.[6]

Fedwire operates 21.5 hours each business day, from 9:00 pm Eastern Time (ET) on the preceding calendar day to 6.30 pm ET. On a typical day, institutions receive the repayments

[3]Stigum and Crescenzi (2007) is a standard reference for institutional details and facts about the fed funds market. Most empirical investigations on the U.S. fed funds market use estimates constructed from a proprietary transactions-level data set that contains all transfers made through Fedwire. A Fedwire transaction is executed with an electronic request made by a financial institution (sent to the Federal Reserve Banks via Fedwire) to debit its reserve account by a stipulated amount in favor of the account of another institution. Such a transaction may occur for many reasons (e.g., to settle an asset purchase), so not all Fedwire transactions are associated to fed funds loans. In order to identify the likely fed funds transactions from the universe of Fedwire transactions, standard practice involves using an algorithm similar to the one proposed by Furfine (1999), supplied by the Money and Payments Studies Function at the Federal Reserve Bank of New York. (For a detailed description of the algorithm, and an analysis of the salient features of the estimated transactions, see Afonso and Lagos, 2012). In what follows, whenever we refer to empirical observations regarding fed funds transactions, we are actually referring to the subset of Fedwire transactions identified as fed funds loans by the Furfine algorithm. Despite its widespread use and general appeal, the algorithm may keep transactions that are not fed funds trades, or may discard transactions that are fed funds trades. For this reason the Federal Reserve Bank of New York is currently concerned about the accuracy of the Furfine estimates; efforts to assess how large are the type I and type II errors resulting from the application of this algorithm are currently underway.

[4]More than 7,000 Fedwire participants can potentially lend and borrow in the fed funds market. In 2008, the average daily number of borrowers and lenders were estimated to be 164 and 255, respectively (see Afonso, Kovner and Schoar, 2011).

[5]There is a growing search-theoretic literature on financial markets which includes Afonso (2011), Duffie, Gârleanu, and Pedersen (2005, 2007), Gârleanu (2009), Lagos and Rocheteau (2007, 2009), Lagos, Rocheteau, and Weill (2011), Miao (2006), Rust and Hall (2003), Spulber (1996), Vayanos and Wang (2007), Vayanos and Weill (2008), and Weill (2007, 2008), just to name a few. See Ashcraft and Duffie (2007) for more on the over-the-counter nature of the fed funds market.

[6]Ashcraft and Duffie (2007), for example, estimate that non-brokered transactions represented 73 percent of the volume of federal funds traded in 2005. Federal fund brokers do not take positions themselves; they only act as matchmakers, bringing buyers and sellers together.

corresponding to the fed funds sold the previous day, before they send out the new loans.[7] For simplicity, in our theory we take as given that every loan gets repaid after the end of the operating day, at a fixed time $T + \Delta$.

Fed funds activity is concentrated in the last two hours of the operating day.[8] For a typical bank, until mid afternoon transactions reflect its primary business activities. Later in the day the trading and payment activity is orchestrated by the fed funds trading desk and aimed at achieving a target balance of reserves. By around 4:00 pm, each bank would typically have a balance of reserves resulting from previous activities which is taken as given by the bank's fed funds trading desk.[9] We think of $t = 0$ as standing in for 4:00 pm and model the distribution of actual reserve balances given to the bank's fed funds trading desk at this time, with the initial condition $\{n_k(T)\}_{k \in \mathbb{K}}$. Fed funds transactions are usually made in round lots of over \$1 million.[10] To keep the analytics tractable, we assume discrete loan sizes in our model.

The motives for trading federal funds may vary across participants and their specific circumstances on any given day, but there are two main reasons in general. First, some institutions such as commercial banks use the fed funds market to offset the effects on their reserve balances of transactions (either initiated by their clients or by profit centers within the banks themselves) that would otherwise leave them with a reserve position that does not meet Federal Reserve regulations. Also, some participants regard fed funds as an investment vehicle; an interest-yielding asset that can be used to "deposit" balances overnight. In our model, all payoff-relevant policy and regulatory considerations are captured by the intraday and end-of-day payoffs, $\{u_k, U_k\}_{k \in \mathbb{K}}$.

[7] In 2006, the average value-weighted time of repayment was estimated to be 3:09 pm $\pm$ 9 minutes, and the average time of delivery, 4:30 pm $\pm$ 7 minutes. The estimated average duration of a loan was 22 hours and 39 minutes. (See Bech and Atalay, 2008.)

[8] In 2008, for example, Furfine estimates suggest that more than 75 percent of the value of fed funds traded among banks was traded after 4:00 pm. In line with this observation, Bartolini et al. (2005) and Bech and Atalay (2008) report very high fed funds loan activity during the latter part of the trading session. (See the illustrations of intraday loan networks for each half hour in a trading day in their Figure 6.)

[9] Ashcraft and Duffie (2007) and Duffie (2012) document useful institutional knowledge obtained from fed funds traders. They report that at some large banks, federal funds traders responsible for managing the bank's fed funds balance, ask other profit centers of their bank to avoid large unscheduled transactions near the end of the day. Around 4:00 pm, once the fed funds trading desk has a good estimate of the send and receive transactions pending until the end of the day, it begins adjusting its trading negotiations to push the bank's balances in the desired direction. In line with this observation, Bartolini et al. (2005) attribute the late afternoon rise in fed funds trading activity to the clustering of institutional deadlines, e.g., the settlement of securities transactions ends at 3:00 pm, causing some institutions to defer much of their money market trading until after that time, once their security-related balance sheet position becomes certain. Uncertainty about client transactions and other payment flows diminishes in the hour or two before Fedwire closes, which also contributes to the concentration of fed funds trading activity late in the day.

[10] See Furfine (1999) and Stigum (1990).

4 Equilibrium

Let $J_k(x,\tau)$ be the maximum attainable payoff to a bank that holds k units of reserve balances and whose net credit position is x, when the time until the end of the trading session is τ. Let $\boldsymbol{s} = (k,x) \in \mathbb{K} \times \mathbb{R}$ denote the bank's individual state, then

$$J_k(x,\tau) = \mathbb{E}\left\{\int_0^{\min(\tau_\alpha,\tau)} e^{-rz} u_k dz + \mathbb{I}_{\{\tau_\alpha > \tau\}} e^{-r\tau}\left(U_k + e^{-r\Delta}x\right)\right. \tag{1}$$
$$\left. + \mathbb{I}_{\{\tau_\alpha \leq \tau\}} e^{-r\tau_\alpha} \int J_{k-b_{\boldsymbol{ss}'}(\tau-\tau_\alpha)}\left(x + R_{\boldsymbol{s}'\boldsymbol{s}}(\tau-\tau_\alpha), \tau-\tau_\alpha\right)\mu\left(d\boldsymbol{s}', \tau-\tau_\alpha\right)\right\},$$

where $\mathbb{E}$ is an expectation operator over the exponentially distributed random time until the next trading opportunity, τ_α, and $\mathbb{I}_{\{\tau_\alpha \leq \tau\}}$ is an indicator function that equals 1 if $\tau_\alpha \leq \tau$ and 0 otherwise. For each time $\tau \in [0,T]$ until the end of the trading session, $\mu(\cdot,\tau)$ is a probability measure (on the Borel σ-field of the subsets of $\mathbb{K} \times \mathbb{R}$) that describes the heterogeneity of potential trading partners over individual states, $\boldsymbol{s}' = (k',x')$. The pair $(b_{\boldsymbol{ss}'}(\tau-\tau_\alpha), R_{\boldsymbol{s}'\boldsymbol{s}}(\tau-\tau_\alpha))$ denotes the bilateral terms of trade between a bank with state $\boldsymbol{s}$ and a (randomly drawn) bank with state $\boldsymbol{s}'$, when the remaining time is $\tau - \tau_\alpha$. That is, $b_{\boldsymbol{ss}'}(\tau-\tau_\alpha)$ is the amount of balances that the bank with state $\boldsymbol{s}$ lends to the bank with state $\boldsymbol{s}'$, and $R_{\boldsymbol{s}'\boldsymbol{s}}(\tau-\tau_\alpha)$ is the amount of balances that the latter commits to repay at time $T+\Delta$.

For all $\tau \in [0,T]$ and any $(\boldsymbol{s},\boldsymbol{s}')$ with $\boldsymbol{s},\boldsymbol{s}' \in \mathbb{K} \times \mathbb{R}$, we take $(b_{\boldsymbol{ss}'}(\tau), R_{\boldsymbol{s}'\boldsymbol{s}}(\tau))$ to be the outcome corresponding to the symmetric Nash solution to a bargaining problem.[11] For all $(k,k') \in \mathbb{K} \times \mathbb{K}$, the set

$$\Pi\left(k,k'\right) = \left\{\left(q,q'\right) \in \mathbb{K} \times \mathbb{K} : q + q' = k + k'\right\}$$

contains all feasible pairs of post-trade balances that could result from the bilateral bargaining between two banks with balances k and k'. This set embeds the restriction that an increase in one bank's balance must correspond to an equal decrease in the other bank's balance, and that no bank can transfer more balances than it currently holds. For every pair of banks that hold $(k,k') \in \mathbb{K} \times \mathbb{K}$, the set $\Pi(k,k')$ induces the set of all feasible (signed) loan sizes,

$$\Gamma\left(k,k'\right) = \left\{b \in \bar{\mathbb{K}} : \left(k-b, k'+b\right) \in \Pi\left(k,k'\right)\right\}.$$

[11] This axiomatic Nash solution can also be obtained from a strategic bargaining game in which, upon contact, Nature selects one of the banks with probability a half to make a take-it-or-leave-it offer which the other bank must either accept or reject on the spot. It is easy to verify that the expected equilibrium outcome of this game coincides with the solution to the Nash bargaining problem, subject to the obvious reinterpretation of $R_{\boldsymbol{s}'\boldsymbol{s}}(\tau)$ as an *expected* repayment, which is inconsequential. See Appendix C in Lagos and Rocheteau (2009).

Notice that $\Pi(k,k') = \Pi(k',k)$, and $\Gamma(k,k') = -\Gamma(k',k)$ for all $k,k' \in \mathbb{K}$. The bargaining outcome, $(b_{\boldsymbol{ss}'}(\tau), R_{\boldsymbol{s}'\boldsymbol{s}}(\tau))$, is the pair (b,R) that solves

$$\max_{b\in\Gamma(k,k'),R\in\mathbb{R}} \left[J_{k-b}(x+R,\tau) - J_k(x,\tau)\right]^{\frac{1}{2}} \left[J_{k'+b}\left(x'-R,\tau\right) - J_{k'}\left(x',\tau\right)\right]^{\frac{1}{2}}.$$

In Appendix A (Lemma 2) we show that

$$J_k(x,\tau) = V_k(\tau) + e^{-r(\tau+\Delta)}x \tag{2}$$

satisfies (1), if and only if $V_k(\tau) : \mathbb{K}\times[0,T] \to \mathbb{R}$ satisfies

$$\begin{aligned} V_k(\tau) = \mathbb{E}\Bigg\{ & \int_0^{\min(\tau_\alpha,\tau)} e^{-rz}u_k dz + \mathbb{I}_{\{\tau_\alpha>\tau\}}e^{-r\tau}U_k \\ & + \mathbb{I}_{\{\tau_\alpha\le\tau\}}e^{-r\tau_\alpha}\sum_{k'\in\mathbb{K}} n_{k'}(\tau-\tau_\alpha)\left[V_{k-b_{kk'}(\tau-\tau_\alpha)}(\tau-\tau_\alpha) + e^{-r(\tau+\Delta-\tau_\alpha)}R_{k'k}(\tau-\tau_\alpha)\right]\Bigg\}, \end{aligned} \tag{3}$$

for all $(k,\tau) \in \mathbb{K}\times[0,T]$, with

$$b_{kk'}(\tau) \in \arg\max_{b\in\Gamma(k,k')} \left[V_{k'+b}(\tau) + V_{k-b}(\tau) - V_{k'}(\tau) - V_k(\tau)\right] \tag{4}$$

$$e^{-r(\tau+\Delta)}R_{k'k}(\tau) = \frac{1}{2}\left[V_{k'+b_{kk'}(\tau)}(\tau) - V_{k'}(\tau)\right] + \frac{1}{2}\left[V_k(\tau) - V_{k-b_{kk'}(\tau)}(\tau)\right]. \tag{5}$$

In (4) and (5), we use $(b_{kk'}(\tau), R_{k'k}(\tau))$ (rather than $(b_{\boldsymbol{ss}'}(\tau), R_{\boldsymbol{s}'\boldsymbol{s}}(\tau))$) to denote the bargaining outcome between a bank with individual state $\boldsymbol{s} \in \mathbb{K}\times\mathbb{R}$ and a bank with individual state $\boldsymbol{s}' \in \mathbb{K}\times\mathbb{R}$, in order to stress that this outcome is independent of the banks' net credit positions, x and x'. Hereafter, we use $\boldsymbol{V} \equiv [\boldsymbol{V}(\tau)]_{\tau\in[0,T]}$, with $\boldsymbol{V}(\tau) \equiv \{V_k(\tau)\}_{k\in\mathbb{K}}$, to denote the value function in (3).

When a pair of banks meet, they jointly decide on the size of the loan and the size of the repayment. The loan size determines the gain from trade, and the repayment implements a division of this gain between the borrower and the lender. For example, suppose that a bank with $i \in \mathbb{K}$ balances and a bank with $j \in \mathbb{K}$ balances meet with time τ until the end of the trading session, and negotiate a loan of size $b_{ij}(\tau) = i - k = s - j \in \Gamma(i,j)$. Then the implied joint gain from trade, the (*match*) *surplus*, corresponding to this transaction is

$$S_{ij}^{ks}(\tau) \equiv V_k(\tau) + V_s(\tau) - V_i(\tau) - V_j(\tau). \tag{6}$$

Thus, according to (4), the bargaining outcome always involves a loan size that maximizes the surplus. According to (5), the size of the repayment is chosen such that each bank's individual gain from trade equals a fraction of the joint gain from trade, with that fraction being equal to the bank's bargaining power. To see this more clearly, note that (2), (4) and (5) imply that the gain from trade to a bank with balance k who trades with a bank with balance k' when the time remaining is τ, namely $J_{k-b_{kk'}(\tau)}(x+R_{k'k}(\tau),\tau)-J_k(x,\tau)$, equals

$$\begin{aligned}&V_{k-b_{kk'}(\tau)}(\tau)+e^{-r(\tau+\Delta)}R_{k'k}(\tau)-V_k(\tau)\\&=\frac{1}{2}\left[V_{k'+b_{kk'}(\tau)}(\tau)+V_{k-b_{kk'}(\tau)}(\tau)-V_{k'}(\tau)-V_k(\tau)\right].\end{aligned}\tag{7}$$

Consider a bank with i balances that contacts a bank with j balances when the time until the end of the trading session is τ. Let $\phi^{ks}_{ij}(\tau)$ be the probability that the former and the latter hold k and s balances after the meeting, respectively, i.e., $\phi^{ks}_{ij}(\tau)\in[0,1]$, with $\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}}\phi^{ks}_{ij}(\tau)=1$. Feasibility requires that $\phi^{ks}_{ij}(\tau)=0$ if $(k,s)\notin\Pi(i,j)$. Given any feasible path for the distribution of trading probabilities, $\boldsymbol{\phi}(\tau)=\{\phi^{ks}_{ij}(\tau)\}_{i,j,k,s\in\mathbb{K}}$, the distribution of balances at time $T-\tau$, i.e., $\boldsymbol{n}(\tau)=\{n_k(\tau)\}_{k\in\mathbb{K}}$, evolves according to

$$\dot{n}_k(\tau)=f\left[\boldsymbol{n}(\tau),\boldsymbol{\phi}(\tau)\right]\quad\text{for all } k\in\mathbb{K},\tag{8}$$

where

$$\begin{aligned}f\left[\boldsymbol{n}(\tau),\boldsymbol{\phi}(\tau)\right]\equiv{}&\alpha n_k(\tau)\sum_{i\in\mathbb{K}}\sum_{j\in\mathbb{K}}\sum_{s\in\mathbb{K}}n_i(\tau)\phi^{sj}_{ki}(\tau)\\&-\alpha\sum_{i\in\mathbb{K}}\sum_{j\in\mathbb{K}}\sum_{s\in\mathbb{K}}n_i(\tau)n_j(\tau)\phi^{ks}_{ij}(\tau).\end{aligned}\tag{9}$$

The first term on the right side of (9) contains the total flow of banks that leave state k between time $t=T-\tau$ and time $t'=T-(\tau-\varepsilon)$ for a small $\varepsilon>0$. The second term contains the total flow of banks into state k over the same interval of time.

The following proposition provides a sharper representation of the value function and the distribution of trading probabilities characterized in (3), (4) and (5).

Proposition 1 *The value function $\boldsymbol{V}$ satisfies (3), with (4) and (5), if and only if it satisfies*

$$\begin{aligned}V_i(\tau)={}&v_i(\tau)+\alpha\int_0^\tau V_i(z)e^{-(r+\alpha)(\tau-z)}dz\\&+\frac{\alpha}{2}\int_0^\tau\sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}}n_j(z)\phi^{ks}_{ij}(z)\left[V_k(z)+V_s(z)-V_i(z)-V_j(z)\right]e^{-(r+\alpha)(\tau-z)}dz,\end{aligned}\tag{10}$$

for all $(i, \tau) \in \mathbb{K} \times [0, T]$*, with*

$$v_i(\tau) = \left[1 - e^{-(r+\alpha)\tau}\right] \frac{u_i}{r+\alpha} + e^{-(r+\alpha)\tau} U_i, \tag{11}$$

for all $i \in \mathbb{K}$*, and*

$$\phi_{ij}^{ks}(\tau) = \begin{cases} \tilde{\phi}_{ij}^{ks}(\tau) & \textit{if } (k,s) \in \Omega_{ij}[\boldsymbol{V}(\tau)] \\ 0 & \textit{if } (k,s) \notin \Omega_{ij}[\boldsymbol{V}(\tau)], \end{cases} \tag{12}$$

for all $i, j, k, s \in \mathbb{K}$ *and all* $\tau \in [0, T]$*, where* $\tilde{\phi}_{ij}^{ks}(\tau) \geq 0$ *and* $\sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} \tilde{\phi}_{ij}^{ks}(\tau) = 1$*, with*

$$\Omega_{ij}[\boldsymbol{V}(\tau)] \equiv \arg \max_{(k',s') \in \Pi(i,j)} \left[V_{k'}(\tau) + V_{s'}(\tau) - V_i(\tau) - V_j(\tau)\right]. \tag{13}$$

The set $\Omega_{ij}[\boldsymbol{V}(\tau)]$ contains all the feasible pairs of post-trade balances that maximize the match surplus between a bank with i balances and a bank with j balances that is implied by the value function $\boldsymbol{V}(\tau)$ at time $T - \tau$. For any pair of banks with balances i and j, $\phi_{ij}^{ks}(\tau)$ defined in (12) is a probability distribution over the feasible pairs of post-trade portfolios that maximize the bilateral gain from trade.

Definition 1 *An equilibrium is a value function,* $\boldsymbol{V}$*, a path for the distribution of reserve balances,* $\boldsymbol{n}(\tau)$*, and a path for the distribution of trading probabilities,* $\boldsymbol{\phi}(\tau)$*, such that: (a) given the value function and the distribution of trading probabilities, the distribution of balances evolves according to (8); and (b) given the path for the distribution of balances, the value function and the distribution of trading probabilities satisfy (10) and (12).*

Assumption A. For any $i, j \in \mathbb{K}$, and all $(k, s) \in \Pi(i, j)$, the payoff functions satisfy:

$$u_{\lceil \frac{i+j}{2} \rceil} + u_{\lfloor \frac{i+j}{2} \rfloor} \geq u_k + u_s \tag{DMC}$$

$$U_{\lceil \frac{i+j}{2} \rceil} + U_{\lfloor \frac{i+j}{2} \rfloor} \geq U_k + U_s, \text{ “ > ” unless } k \in \left\{\left\lfloor \frac{i+j}{2} \right\rfloor, \left\lceil \frac{i+j}{2} \right\rceil\right\}, \tag{DMSC}$$

where $\lfloor x \rfloor \equiv \max\{k \in \mathbb{Z} : k \leq x\}$ and $\lceil x \rceil \equiv \min\{k \in \mathbb{Z} : x \leq k\}$ for any $x \in \mathbb{R}$.

In Appendix A (Lemma 3) we show that conditions (DMC) and (DMSC) are equivalent to requiring that the payoff functions $\{u_k\}_{k \in \mathbb{K}}$ and $\{U_k\}_{k \in \mathbb{K}}$ satisfy *discrete midpoint concavity*, and *discrete midpoint strict concavity*, respectively. These are the natural discrete approximations to the notions of *midpoint concavity* and *midpoint strict concavity* of ordinary functions defined on convex sets.[12]

The following result provides a full characterization of equilibrium under Assumption A.

Proposition 2 *Let the payoff functions satisfy Assumption A. Then:*

(i) *An equilibrium exists, and the equilibrium paths for the maximum attainable payoffs, $\boldsymbol{V}(\tau)$, and the distribution of reserve balances, $\boldsymbol{n}(\tau)$, are uniquely determined.*

(ii) *The equilibrium path for the distribution of trading probabilities, $\boldsymbol{\phi}(\tau) = \{\phi_{ij}^{ks}(\tau)\}_{i,j,k,s\in\mathbb{K}}$, is given by*

$$\phi_{ij}^{ks}(\tau) = \begin{cases} \tilde{\phi}_{ij}^{ks}(\tau) & \text{if } (k,s) \in \Omega_{ij}^{*} \\ 0 & \text{if } (k,s) \notin \Omega_{ij}^{*} \end{cases} \tag{14}$$

for all $i,j,k,s \in \mathbb{K}$ and all $\tau \in [0,T]$, where $\tilde{\phi}_{ij}^{ks}(\tau) \geq 0$ and $\sum_{(k,s)\in\Omega_{ij}^{*}} \tilde{\phi}_{ij}^{ks}(\tau) = 1$, where

$$\Omega_{ij}^{*} = \begin{cases} \left\{\left(\frac{i+j}{2}, \frac{i+j}{2}\right)\right\} & \text{if } i+j \text{ is even} \\ \left\{\left(\left\lfloor\frac{i+j}{2}\right\rfloor, \left\lceil\frac{i+j}{2}\right\rceil\right), \left(\left\lceil\frac{i+j}{2}\right\rceil, \left\lfloor\frac{i+j}{2}\right\rfloor\right)\right\} & \text{if } i+j \text{ is odd.} \end{cases} \tag{15}$$

(iii) *$\boldsymbol{V}$ is the unique bounded real-valued function that satisfies*

$$rV_i(\tau) + \dot{V}_i(\tau) = u_i + \frac{\alpha}{2}\sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} n_j(\tau)\,\phi_{ij}^{ks}(\tau)\left[V_k(\tau) + V_s(\tau) - V_i(\tau) - V_j(\tau)\right] \tag{16}$$

for all $(i,\tau) \in \mathbb{K}\times[0,T]$, with

$$V_i(0) = U_i \quad \text{for all } i \in \mathbb{K}, \tag{17}$$

and with the path for $\boldsymbol{\phi}(\tau)$ given by (14), and the path for $\boldsymbol{n}(\tau)$ given by $\dot{\boldsymbol{n}}(\tau) = f[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)]$.

(iv) *Suppose that at time $T-\tau$, a bank with balance j extends a loan of size $j-s=k-i$ to a bank with balance i. The present value of the equilibrium repayment from the latter to the former is*

$$e^{-r(\tau+\Delta)}R_{ij}^{ks}(\tau) = \frac{1}{2}\left[V_k(\tau) - V_i(\tau)\right] + \frac{1}{2}\left[V_j(\tau) - V_s(\tau)\right]. \tag{18}$$

[12] Let X be a convex subset of $\mathbb{R}^n$, then a function $g: X \to \mathbb{R}$ is said to be *concave* if $g(\epsilon x + (1-\epsilon)y) \geq \epsilon g(x) + (1-\epsilon)g(y)$ for all $x,y \in X$, and all $\epsilon \in [0,1]$. The function g is *midpoint concave* if $2g\left(\frac{x+y}{2}\right) \geq g(x) + g(y)$ for all $x,y \in X$. Clearly, if g is concave then it is midpoint concave. The converse is true provided g is continuous. The function $g: \mathbb{K} \to \mathbb{R}$ satisfies the *discrete midpoint concavity property* if $g\left(\left\lceil\frac{i+j}{2}\right\rceil\right) + g\left(\left\lfloor\frac{i+j}{2}\right\rfloor\right) \geq u_i + u_j$ for all $i,j \in \mathbb{K}$. See Murota (2003) for more on the midpoint concavity/convexity property and the role that it plays in the modern theory of discrete convex analysis.

The equilibrium distribution of trading probabilities (14) can be described intuitively as follows. At any point during the trading session, if a bank with balance i contacts a bank with balance j, then the post-transaction balance will necessarily be $\left\lfloor \frac{i+j}{2} \right\rfloor$ for one of the banks, and $\left\lceil \frac{i+j}{2} \right\rceil$ for the other. This property, and the uniqueness of the equilibrium paths for the distribution of reserve balances and maximum payoffs, hold under Assumption A. In Appendix A (Corollary 1) we show that if we instead assume that u satisfies *discrete midpoint strict concavity* and U satisfies *discrete midpoint concavity*, then the existence and uniqueness results in Proposition 2 still hold.

5 Efficiency

In this section we use our theory to characterize the optimal process of reallocation of reserve balances in the fed funds market. The spirit of the exercise is to take as given the market structure, including the contact rate α and the regulatory variables $\{u_k, U_k\}_{k\in\mathbb{K}}$, and to ask whether decentralized trade in the over-the-counter market structure reallocates reserve balances efficiently, given these institutions. To this end, we study the problem of a social planner who solves

$$\begin{aligned} &\max_{[\boldsymbol{\chi}(t)]_{t=0}^{T}} \left[\int_0^T \sum_{k\in\mathbb{K}} m_k(t)\, u_k e^{-rt} dt + e^{-rT} \sum_{k\in\mathbb{K}} m_k(T)\, U_k\right] \\ &\text{s.t. } \dot{m}_k(t) = -f\left[\boldsymbol{m}(t), \boldsymbol{\chi}(t)\right], \\ &\qquad \chi_{ij}^{ks}(t) \in [0,1], \text{ with } \chi_{ij}^{ks}(t) = 0 \text{ if } (k,s) \notin \Pi(i,j), \\ &\qquad \chi_{ij}^{ks}(t) = \chi_{ji}^{sk}(t), \text{ and } \sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} \chi_{ij}^{ks}(t) = 1, \end{aligned} \tag{19}$$

for all $t \in [0,T]$, and all $i,j,k,s \in \mathbb{K}$. We have formulated the planner's problem in chronological time, so $m_k(t)$ denotes the measure of banks with balance k at time t. Since $\tau \equiv T - t$, we have $m_k(t) = m_k(T-\tau) \equiv n_k(\tau)$, and therefore $\dot{m}_k(t) = -\dot{n}_k(\tau)$. Hence the flow constraint is the real-time law of motion for the distribution of balances implied by the bilateral stochastic trading process. The control variable, $\boldsymbol{\chi}(t) = \{\chi_{ij}^{ks}(t)\}_{i,j,k,s\in\mathbb{K}}$, represents the planner's choice of reallocation of balances between any pair of banks that have contacted each other at time t. The first, second, and fourth constraints on $\boldsymbol{\chi}(t)$ ensure that $\{\chi_{ij}^{ks}(t)\}_{k,s\in\mathbb{K}}$ is a probability distribution for each $i,j \in \mathbb{K}$, and that the planner only chooses among feasible reallocations of balances between a pair of banks. We look for a solution that does not depend on the identities or "names" of banks, so the third constraint on $\boldsymbol{\chi}(t)$ recognizes the fact that $\chi_{ij}^{ks}(t)$ and $\chi_{ji}^{sk}(t)$ represent the same decision for the planner. That is, $\chi_{ij}^{ks}(t)$ and $\chi_{ji}^{sk}(t)$ both represent the probability that a pair of banks with balances i and j who contact each other at time t, exit the meeting with balances k and s, respectively.

Proposition 3 *A solution to the planner's problem is a path for the distribution of balances, $\boldsymbol{n}(\tau)$, a path for the vector of co-states associated with the law of motion for the distribution of balances, $\boldsymbol{\lambda}(\tau) = \{\lambda_k(\tau)\}_{k\in\mathbb{K}}$, and a path for the distribution of trading probabilities, $\boldsymbol{\psi}(\tau) = \{\psi_{ij}^{ks}(\tau)\}_{i,j,k,s\in\mathbb{K}}$. The necessary conditions for optimality are,*

$$r\lambda_i(\tau) + \dot{\lambda}_i(\tau) = u_i + \alpha \sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} n_j(\tau)\,\psi_{ij}^{ks}(\tau)\left[\lambda_k(\tau) + \lambda_s(\tau) - \lambda_i(\tau) - \lambda_j(\tau)\right] \tag{20}$$

for all $(i,\tau) \in \mathbb{K}\times[0,T]$, with

$$\lambda_i(0) = U_i \quad \textit{for all } i \in \mathbb{K}, \tag{21}$$

with the path for $\boldsymbol{n}(\tau)$ given by $\dot{\boldsymbol{n}}(\tau) = f[\boldsymbol{n}(\tau), \boldsymbol{\psi}(\tau)]$, and with

$$\psi_{ij}^{ks}(\tau) = \begin{cases} \tilde{\psi}_{ij}^{ks}(\tau) & \textit{if } (k,s) \in \Omega_{ij}[\boldsymbol{\lambda}(\tau)] \\ 0 & \textit{if } (k,s) \notin \Omega_{ij}[\boldsymbol{\lambda}(\tau)], \end{cases} \tag{22}$$

for all $i,j,k,s \in \mathbb{K}$ and all $\tau \in [0,T]$, where $\tilde{\psi}_{ij}^{ks}(\tau) \geq 0$ and $\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} \tilde{\psi}_{ij}^{ks}(\tau) = 1$.

The following result provides a full characterization of solution to the planner's problem under Assumption A.

Proposition 4 *Let the payoff functions satisfy Assumption A. Then:*

(i) *The optimal path for the distribution of trading probabilities, $\boldsymbol{\psi}(\tau) = \{\psi_{ij}^{ks}(\tau)\}_{i,j,k,s\in\mathbb{K}}$, is given by*

$$\psi_{ij}^{ks}(\tau) = \begin{cases} \tilde{\psi}_{ij}^{ks}(\tau) & \textit{if } (k,s) \in \Omega_{ij}^{*} \\ 0 & \textit{if } (k,s) \notin \Omega_{ij}^{*} \end{cases} \tag{23}$$

for all $i,j,k,s \in \mathbb{K}$ and all $\tau \in [0,T]$, where $\tilde{\psi}_{ij}^{ks}(\tau) \geq 0$ and $\sum_{(k,s)\in\Omega_{ij}^{*}} \tilde{\psi}_{ij}^{ks}(\tau) = 1$.

(ii) *Along the optimal path, the shadow value of a bank with i reserve balances is given by (20) and (21), with the path for $\boldsymbol{\psi}(t)$ given by (23), and the path for $\boldsymbol{n}(\tau)$ given by $\dot{\boldsymbol{n}}(\tau) = f[\boldsymbol{n}(\tau), \boldsymbol{\psi}(\tau)]$.*

Notice the similarity between the equilibrium conditions and planner's optimality conditions. First, from (12) and (22), we see that the equilibrium loan sizes are *privately efficient.* That is, given the value function $\boldsymbol{V}$, the equilibrium distribution of trading probabilities is the one that would be chosen by the planner. Second, the path for the equilibrium values, $\boldsymbol{V}(\tau)$, satisfies (16) and (17), while the path for the planner's shadow prices satisfies (20) and (21). These pairs of conditions would be identical were it not for the fact that the planner imputes to each agent gains from trade with frequency 2α, rather α, which is the frequency with which the agent generates gains from trade for himself in the equilibrium. This reflects a composition externality typical of random matching environments. The planner's calculation of the value of a marginal agent in state i includes not only the expected gain from trade to this agent, but also the expected gains from trade that having this marginal agent in state i generates for all other agents, by increasing their contact rates with agents in state i. In the equilibrium, the individual agent in state i internalizes the former, but not the latter.[13]

Under Assumption A, however, condition (14) is identical to (23), so the equilibrium paths for the distribution of balances and trading probabilities coincide with the optimal paths. This observation is summarized in the following proposition.

Proposition 5 *Let the payoff functions satisfy Assumption A. Then, the equilibrium supports an efficient allocation of reserve balances.*

6 Positive implications

The performance of the fed funds market as a system that reallocates liquidity among banks, can be appraised by the behavior of empirical measures of the fed funds rate and of the effectiveness of the market to channel funds from banks with excess balances to those with shortages. In this section we derive the theoretical counterparts to these empirical measures, and argue that the theory is consistent with the most salient features of the actual fed funds market. We use the theory to identify the determinants of the fed funds rate, trade volume, and trading delays. We also describe the equilibrium dynamics of the fed fund balances of individual banks, and propose theory-based measures of the importance of bank-provided intermediation in the process of reallocation of reserve balances among banks.

[13] In a labor market context, a similar composition externality arises in the competitive matching equilibrium of Kiyotaki and Lagos (2007).

6.1 Trade volume and trading delays

The flow volume of trade at time $T-\tau$ is

$$\bar{\upsilon}(\tau) = \sum_{i\in\mathbb{K}}\sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} \upsilon_{ij}^{ks}(\tau),$$

where

$$\upsilon_{ij}^{ks}(\tau) = \alpha n_i(\tau) n_j(\tau) \phi_{ij}^{ks}(\tau) |k-i|,$$

and the total volume traded during the whole trading session is

$$\bar{\upsilon} = \int_0^T \bar{\upsilon}(\tau) d\tau.$$

Notice that the arrival rate of specific trading opportunities is endogenous, as it depends on the equilibrium distribution of balances. For example, $\alpha n_j(\tau) \phi_{ij}^{ks}(\tau)$ is the rate at which agents with balance i trade a balance equal to $k-i$ with agents with balance j at time $T-\tau$. Therefore, even though the contact rate, α, is exogenous in our baseline formulation, trading delays—a key distinctive feature of over-the-counter markets—are determined by agents' trading strategies.

6.2 Fed funds rate

In our baseline formulation, banks negotiate loans and the present value of the loan repayment. It is possible to reformulate the negotiation in terms of a loan size and an interest rate. For example, consider a transaction between a bank with i balances and a bank with j balances in which the former borrows $k-i=j-s$ from the latter. We can think of the corresponding repayment, $R_{ij}^{ks}(\tau)$ in (18), as composed of the principal of the loan, augmented by continuously compounded interest, ρ. That is, we can write $R_{ij}^{ks}(\tau) = e^{\rho(\tau+\Delta)}(k-i)$, and solve for the transaction-specific interest rate,

$$\rho_{ij}^{ks}(\tau) = \frac{\ln\left[\frac{R_{ij}^{ks}(\tau)}{k-i}\right]}{\tau+\Delta} = r + \frac{\ln\left[\frac{V_j(\tau)-V_s(\tau)}{j-s} + \frac{\frac{1}{2}S_{ij}^{ks}(\tau)}{j-s}\right]}{\tau+\Delta}. \tag{24}$$

According to (24), the interest on a loan of size $j-s$ extended by a lender with balance j to a borrower with balance i at time $T-\tau$, is equal to the discount rate, r, plus a premium, which increases with the size of the joint gain from trade, $S_{ij}^{ks}(\tau)$, and with the lender's bargaining power (here equal to 1/2). According to the theory, there is no such thing as *the* fed funds rate, rather there is a time-varying *distribution* of rates. That is, empirically, in order to "explain" the rate determination in over-the-counter fed fund transactions, one would have to control for the opportunity cost of funds (r), the duration of the loan ($\tau + \Delta$), the size of the loan ($j - s$ in (24)), the bargaining power of the borrower and the lender (1/2 each in (24)), the present discounted value of the loss to the lender from giving up the funds ($V_j(\tau) - V_s(\tau)$), and the present discounted value of the gain to the borrower from obtaining the funds ($V_k(\tau) - V_i(\tau)$), both of which depend on the time until the end of the trading session (τ).

In the theory,

$$\bar{\rho}(\tau) = \sum_{i \in \mathbb{K}} \sum_{j \in \mathbb{K}} \sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} \omega_{ij}^{ks}(\tau) \rho_{ij}^{ks}(\tau)$$

is a weighted average of rates at each point in time, and

$$\bar{\rho} = \frac{1}{T} \int_0^T \bar{\rho}(\tau) \, d\tau$$

is a daily average rate, where $\omega_{ij}^{ks}(\tau)$ is a weighting function with $\omega_{ij}^{ks}(\tau) \geq 0$ and $\sum_{i,j,k,s \in \mathbb{K}} \omega_{ij}^{ks}(\tau) = 1$. For example, if $\omega_{ij}^{ks}(\tau) = \upsilon_{ij}^{ks}(\tau) / \bar{\upsilon}(\tau)$, then $\bar{\rho}(\tau)$ is the value-weighted average fed funds rate at time $T - \tau$, and $\bar{\rho}$ is a value-weighted daily average fed funds rate akin to the *effective federal funds rate* published daily by the Federal Reserve.[14]

6.3 Equilibrium dynamics of fed funds balances

Consider a bank with balance $a(t_0) = i \in \mathbb{K}$ at time $t_0 \in [0, T)$, and let $t_1 \in (t_0, T)$ denote the time at which the bank receives its first trading opportunity on $[t_0, T]$. The probability distribution over post-trade balances at t_1, $a(t_1) \in \mathbb{K}$, is given by

$$\Pr[a(t_1) = j \mid a(t_0) = i] = \sum_{q \in \mathbb{K}} m_q(t_1) \phi_{iq}^{jq'}(t_1) \equiv \pi_{ij}(t_1),$$

where $q' \equiv q + i - j$ and $m_q(t_1)$ is the measure of banks with balance q at time t_1. Given a probability measure over $a(t_0) \in \mathbb{K}$, the $(K+1) \times (K+1)$ transition matrix $\mathbf{\Pi}(t_1) = [\pi_{ij}(t_1)]$ records the probabilities of making a transition from any balance $i \in \mathbb{K}$ to any balance $j \in \mathbb{K}$ at trading time t_1. More generally, consider a bank with balance $k_0 \in \mathbb{K}$ at t_0 that has N trading opportunities between time t_0 and time t, e.g., at times $\boldsymbol{t}^{(N)} = (t_1, t_2, \ldots, t_N)$, with $0 \leq t_0 < t_1 < t_2 < \cdots < t_N < t \leq T$. (We adopt the convention $\boldsymbol{t}^{(0)} = t_0$.) Then given

[14] The actual daily effective federal funds rate is a volume-weighted average of rates on trades arranged by major brokers. The Federal Reserve Bank of New York receives summary reports from the brokers, and every morning publishes the effective federal funds rate for the previous day.

the initial balance $k_0 \in \mathbb{K}$ and the realization of trading times $\boldsymbol{t}^{(N)} \in [t_0, t]^N$, the probability distribution over the sequence of post-trade balances at these trading times, i.e., $a(t_n) \in \mathbb{K}$ for all $n = 1, ..., N$, is given by

$$\Pr\left[a(t_1) = k_1, \ldots, a(t_N) = k_N \mid a(t_0) = k_0, \boldsymbol{t}^{(N)}\right] = \prod_{n=1}^{N} \pi_{k_{n-1}k_n}(t_n). \tag{25}$$

Given a probability measure over $a(t_0) \in \mathbb{K}$, the $(K+1) \times (K+1)$ transition matrix

$$\boldsymbol{\Pi}^{(N)}(\boldsymbol{t}^{(N)}) = \boldsymbol{\Pi}(t_1) \cdots \boldsymbol{\Pi}(t_N) \tag{26}$$

records the probabilities of making a transition from any balance $i \in \mathbb{K}$ to any other balance $j \in \mathbb{K}$ in N trades carried out at the realized trading times $\boldsymbol{t}^{(N)} = (t_1, ..., t_N)$. Notice that $\boldsymbol{\Pi}^{(1)}(\boldsymbol{t}^{(1)}) = \boldsymbol{\Pi}(t_1)$, and by convention, $\boldsymbol{\Pi}^{(0)}(\boldsymbol{t}^{(0)}) = \boldsymbol{I}$, where $\boldsymbol{I}$ denotes the $(K+1) \times (K+1)$ identity matrix. The following proposition provides a complete characterization of the stochastic process that rules the equilibrium dynamics of the balance held by an individual bank.

Proposition 6 *For any $t_0 \in [0, T)$, and any $t \in [t_0, T]$, the transition function for the stochastic process that rules the equilibrium dynamics of individual balances is*

$$P(t|t_0) = \sum_{N=0}^{\infty} \alpha^N e^{-\alpha(t-t_0)} \int_{\mathbb{T}^{(N)}} \boldsymbol{\Pi}^{(N)}(\boldsymbol{t}^{(N)}) d\boldsymbol{t}^{(N)}, \tag{27}$$

where $\mathbb{T}^{(N)} = \left\{\boldsymbol{t}^{(N)} \in [t_0, t]^N : t_0 < t_1 < \cdots < t_N < t\right\}$.

Let $p_{ij}(t|t_0)$ denote the (i, j) entry of the $(K+1) \times (K+1)$ matrix $P(t|t_0)$. Consider a bank with balance $i \in \mathbb{K}$ at time t_0, then $p_{ij}(t|t_0)$ is the probability the bank has balance $j \in \mathbb{K}$ at time t.

6.4 Intermediation and speculative trades

The equilibrium characterized in Proposition 2 (and by Proposition 5, the efficient allocation characterized in Proposition 4) exhibits endogenous intermediation in the sense that many banks act as dealers, buying and selling funds on their own account and channeling them from banks with larger balances to banks with smaller balances. To illustrate, consider a bank that starts the trading session with balance $a(0)$. Suppose, for example, that the bank in question only trades twice in the session, at times t_1 and t_2, with $0 < t_1 < t_2 < T$, first buying $a(t_1) - a(0)$, and then selling $a(t_1) - a(t_2)$, so that it ends the session with a balance $a(t_2)$, where $a(0) < a(t_2) < a(t_1)$. Throughout the trading session, this bank effectively intermediated a volume of funds equal to $a(t_1) - a(t_2)$, buying at time t_1 from a bank with some balance at least as large as $a(t_1)$, and then selling at a later time t_2 to a bank with some balance no larger than $a(t_2)$. This type of intermediation among participants is an important feature of the fed funds market.[15] Next, we propose several theory-based empirical measures of the importance of intermediation in the process of reallocation of reserves among banks.

Consider a bank with N trading opportunities between time t_0 and time t, e.g., at times $\boldsymbol{t}^{(N)} = (t_1, t_2, ..., t_N)$, with $0 \leq t_0 < t_1 < t_2 < \cdots < t_N < t \leq T$. Given the initial balance $k_0 \in \mathbb{K}$ and a realization $\boldsymbol{t}^{(N)} \in [t_0, t]^N$, the time-path of the bank's asset holdings during $[t_0, t]$ is described by a function $\boldsymbol{a}_{[t_0,t]} : [t_0, t] \to \mathbb{K}$ defined by

$$\boldsymbol{a}_{[t_0,t]}(x) = \begin{cases} k_0 & \text{for } t_0 \leq x < t_1 \\ k_1 & \text{for } t_1 \leq x < t_2 \\ \vdots & \qquad \vdots \\ k_N & \text{for } t_N \leq x \leq t, \end{cases}$$

where $k_n \in \mathbb{K}$ is the post-trade balance at time t_n for $n = 1, ..., N$. Given the initial balance k_0 at t_0, the realized path for a bank's balance during $[t_0, t]$ is completely described by the number of contacts, N, the vector of contact times, $\boldsymbol{t}^{(N)} \in [t_0, t]^N$, and the vector of post-trade balances at those contact times, $\boldsymbol{k}^{(N)} = (k_1, k_2, ..., k_N) \in \mathbb{K}^N$. Given k_0 and $\boldsymbol{k}^{(N)}$, define the bank's accumulated volume of purchases during $[t_0, t]$,

$$O^p(k_0, \boldsymbol{k}^{(N)}) = \sum_{n=1}^{N} \max\{k_n - k_{n-1}, 0\},$$

the accumulated volume of sales,

$$O^s(k_0, \boldsymbol{k}^{(N)}) = -\sum_{n=1}^{N} \min\{k_n - k_{n-1}, 0\},$$

and the *(signed) net trade*, $O^p(k_0, \boldsymbol{k}^{(N)}) - O^s(k_0, \boldsymbol{k}^{(N)}) = k_N - k_0$. Then

$$I(k_0, \boldsymbol{k}^{(N)}) = \min\left\{O^p(k_0, \boldsymbol{k}^{(N)}), O^s(k_0, \boldsymbol{k}^{(N)})\right\} \tag{28}$$

[15]This theoretical finding is consistent with a striking aspect of the fed funds market which was pointed out by Ashcraft and Duffie (2007): "A significant number of loans in our data are made by lenders in the lower deciles by relative balances. Many of these lenders are presumably themselves in relative need of funds but agree to lend at a sufficiently high rate, planning to borrow later in the day at a lower rate. In any OTC market, the borrower does not generally know the most attractive rates available from other counterparties, or which counterparties are offering them, and may have an incentive to accept the rate offered by such a lender."

measures the volume of funds intermediated by the bank during the time interval $[t_0, t]$. Alternatively, $O^p(k_0, \boldsymbol{k}^{(N)}) + O^s(k_0, \boldsymbol{k}^{(N)})$ is the gross volume of funds traded by the bank, and $|O^p(k_0, \boldsymbol{k}^{(N)}) - O^s(k_0, \boldsymbol{k}^{(N)})|$ is the size of the bank's net trade over the period $[t_0, t]$, so

$$X(k_0, \boldsymbol{k}^{(N)}) = O^p(k_0, \boldsymbol{k}^{(N)}) + O^s(k_0, \boldsymbol{k}^{(N)}) - \left|O^p(k_0, \boldsymbol{k}^{(N)}) - O^s(k_0, \boldsymbol{k}^{(N)})\right| \tag{29}$$

is a bank-level measure of *excess funds reallocation*, i.e., the volume of funds traded over and above what is required to accommodate the net trade. The measure $X(k_0, \boldsymbol{k}^{(N)})$ is an index of simultaneous buying and selling at the individual bank level during $[t_0, t]$. This leads to

$$\iota(k_0, \boldsymbol{k}^{(N)}) = \frac{X(k_0, \boldsymbol{k}^{(N)})}{O^p(k_0, \boldsymbol{k}^{(N)}) + O^s(k_0, \boldsymbol{k}^{(N)})}$$

as a natural measure of the proportion of the total volume of funds traded by a bank during $[0, t]$, that the bank intermediated during the same time period.

Having described the intermediation behavior of a single bank along a typical sample path, the next proposition shows how to calculate marketwide measures of intermediation.

Proposition 7 *Let $t_0 \in [0, T)$, and $t \in (t_0, T]$. During $[t_0, t]$:*

(i) *The aggregate cumulative volume of purchases (for $j = p$, sales, for $j = s$) is*

$$\bar{O}^j\left(t|t_0\right) = \sum_{k_0 \in \mathbb{K}} m_{k_0}\left(t_0\right) \sum_{N=0}^{\infty} \alpha^N e^{-\alpha(t-t_0)} \int_{\mathbb{T}^{(N)}} \tilde{O}^j(k_0, \boldsymbol{t}^{(N)}) d\boldsymbol{t}^{(N)}, \tag{30}$$

where

$$\tilde{O}^j(k_0, \boldsymbol{t}^{(N)}) = \sum_{\boldsymbol{k}^{(N)} \in \mathbb{K}^N} \left(\prod_{n=1}^{N} \pi_{k_{n-1}k_n}\left(t_n\right)\right) O^j(k_0, \boldsymbol{k}^{(N)}).$$

(ii) *The aggregate cumulative volume of intermediated funds is*

$$\bar{I}\left(t|t_0\right) = \frac{1}{2}\bar{X}\left(t|t_0\right), \tag{31}$$

and the proportion of intermediated funds in the aggregate volume of traded funds is

$$\bar{\iota}\left(t|t_0\right) = \frac{\bar{X}\left(t|t_0\right)}{\bar{O}^p\left(t|t_0\right) + \bar{O}^s\left(t|t_0\right)},$$

where

$$\bar{X}\left(t|t_0\right) = \sum_{k_0 \in \mathbb{K}} m_{k_0}\left(t_0\right) \sum_{N=0}^{\infty} \alpha^N e^{-\alpha(t-t_0)} \int_{\mathbb{T}^{(N)}} \tilde{X}(k_0, \boldsymbol{t}^{(N)}) d\boldsymbol{t}^{(N)} \tag{32}$$

is the aggregate excess reallocation of funds, with

$$\tilde{X}(k_0, \boldsymbol{t}^{(N)}) = \sum_{\boldsymbol{k}^{(N)} \in \mathbb{K}^N} \left(\prod_{n=1}^{N} \pi_{k_{n-1}k_n}(t_n) \right) X(k_0, \boldsymbol{k}^{(N)}).$$

Notice that our measure of *excess funds reallocation*, $\bar{X}(t|t_0)$, is a real-time analogue to the notion of *excess job reallocation* used in empirical studies of job creation and destruction (e.g., Davis, Haltiwanger and Schuh, 1996).

7 An analytical example

In this section we use the theory with $\mathbb{K} = \{0, 1, 2\}$ to study the effects that various institutional considerations and policies have on the performance of the market for federal funds. We interpret a bank with $k = 1$ as being "on target" (e.g., holding the level of required reserves), a bank with $k = 2$ as being "above target" (e.g., holding excess reserves), and a bank with $k = 0$ as being "below target" (e.g., unable to meet the level of required reserves). In this setting the quantity of reserves in the market, Q, equals $n_1(T) + 2n_2(T)$, so $Q \leq 1$ if and only if $n_2(T) \leq n_0(T)$. The feasible sets of post-trade balances are: $\Pi(0,2) = \{(0,2), (1,1), (2,0)\}$, $\Pi(1,j) = \{(1,j), (j,1)\}$ for $j = 0, 2$, and $\Pi(i,i) = \{(i,i)\}$ for $i = 0, 1, 2$. Hence,

$$\max_{(k,s)\in\Pi(2,0)} S_{20}^{ks}(\tau) = \max\left\{S_{20}^{11}(\tau), 0\right\}, \text{ and}$$
$$\max_{(k,s)\in\Pi(i,i)} S_{ii}^{ks}(\tau) = \max_{(k,s)\in\Pi(1,j)} S_{1j}^{ks}(\tau) = 0 \quad \text{for all } i \in \mathbb{K}, \text{ and } j = 0, 2.$$

That is, in this special case there can only be profitable trade between a bank with $i = 2$ and a bank with $j = 0$ balances.[16] To simplify the notation, let $S(\tau) \equiv S_{20}^{11}(\tau)$, and refer to a bank with $i = 2$ and a bank with $j = 0$ as a *lender*, and *borrower*, respectively. Let $\theta \in [0, 1]$ be the bargaining power of the borrower. We conjecture that $S(\tau) > 0$ for all $\tau \in [0, T]$, and will later verify that this is indeed the case. In this case, the flows (8) and (9) lead to

$$\dot{n}_0(\tau) = \alpha n_2(\tau) n_0(\tau)$$
$$\dot{n}_2(\tau) = \alpha n_2(\tau) n_0(\tau),$$

[16]Recall that from (6), we know that in general, $S_{ij}^{ks}(\tau) = S_{ji}^{ks}(\tau) = S_{ij}^{sk}(\tau) = S_{ji}^{sk}(\tau)$ for all $i, j, k, s \in \mathbb{K}$.

given the initial conditions $n_0(T)$ and $n_2(T)$. This implies

$$n_0(\tau) = \begin{cases} \frac{[n_2(T)-n_0(T)]n_0(T)}{n_2(T)e^{\alpha[n_2(T)-n_0(T)](T-\tau)}-n_0(T)} & \text{if } n_2(T) \neq n_0(T) \\ \frac{n_0(T)}{1+\alpha n_0(T)(T-\tau)} & \text{if } n_2(T) = n_0(T) \end{cases} \tag{33}$$

$$n_1(\tau) = 1 - n_0(\tau) - n_2(\tau) \tag{34}$$

$$n_2(\tau) = n_0(\tau) + n_2(T) - n_0(T). \tag{35}$$

The expression for the value function $\boldsymbol{V}$ in (16) and (17) (or (10)) leads to

$$rV_0(\tau) + \dot{V}_0(\tau) = u_0 + \alpha n_2(\tau)\theta S(\tau) \tag{36}$$

$$rV_1(\tau) + \dot{V}_1(\tau) = u_1 \tag{37}$$

$$rV_2(\tau) + \dot{V}_2(\tau) = u_2 + \alpha n_0(\tau)(1-\theta)S(\tau), \tag{38}$$

for all $\tau \in [0, T]$, given $V_i(0) = U_i$ for $i = 0, 1, 2$. Conditions (36), (37) and (38) imply

$$\dot{S}(\tau) + \delta(\tau)S(\tau) = \bar{u} \tag{39}$$

where $\bar{u} \equiv 2u_1 - u_2 - u_0$, and

$$\delta(\tau) \equiv \{r + \alpha[\theta n_2(\tau) + (1-\theta)n_0(\tau)]\}.$$

Given the boundary condition $S(0) = 2U_1 - U_2 - U_0$, the solution to (39) is

$$S(\tau) = \left(\int_0^\tau e^{-[\bar{\delta}(\tau)-\bar{\delta}(z)]}dz\right)\bar{u} + e^{-\bar{\delta}(\tau)}S(0), \tag{40}$$

where $\bar{\delta}(\tau) \equiv \int_0^\tau \delta(x)\,dx$.

Suppose that $\bar{u} \equiv 2u_1 - u_2 - u_0 \geq 0$ and $S(0) = 2U_1 - U_2 - U_0 > 0$, so Assumption A holds. Then it is clear from (40) that $S(\tau) > 0$ as conjectured. Then, with $S(\tau)$ given by (40), the unique equilibrium is simply the path for the distribution of reserve balances given by (33), (34) and (35), together with the distribution of trading probabilities given by $\phi_{ij}^{ks}(\tau) = 1$ if $(i, j, k, s) = (2, 0, 1, 1)$ or $(i, j, k, s) = (0, 2, 1, 1)$ and $\phi_{ij}^{ks}(\tau) = 0$ otherwise, and a value function $\boldsymbol{V}$ that satisfies the system of ordinary differential equations (36), (37), (38) with the boundary conditions $V_i(0) = U_i$ for $i = 0, 1, 2$. In equilibrium, the present value of the repayment is

$$e^{-r(\tau+\Delta)}R(\tau) = V_2(\tau) - V_1(\tau) + (1-\theta)S(\tau) = V_1(\tau) - V_0(\tau) - \theta S(\tau). \tag{41}$$

The interest rate implicit in the typical loan that promises to repay $R(\tau)$ at time $\tau+\Delta$ for one unit borrowed at time $T-\tau$ is

$$\rho(\tau)=\frac{\ln R(\tau)}{\tau+\Delta}=r+\frac{\ln\left[V_2(\tau)-V_1(\tau)+(1-\theta)S(\tau)\right]}{\tau+\Delta}. \tag{42}$$

The equilibrium in this example is a path for the distribution $\boldsymbol{n}(\tau)$, described explicitly by (33), (34) and (35); a path for the distribution of trading probabilities explicitly given by $\phi_{02}^{ks}(\tau)=\phi_{20}^{ks}(\tau)=\mathbb{I}_{\{(k,s)=(1,1)\}}$, $\phi_{ii}^{ks}(\tau)=0$ for all $(k,s)\in\Pi(i,i)$, for $i=0,1,2$, and $\phi_{1j}^{ks}(\tau)\in[0,1]$ for all $(k,s)\in\Pi(1,j)$, for $j=0,2$; and a path for the value function $V(\tau)$,

$$V_0(\tau)=\left(1-e^{-r\tau}\right)\frac{u_0}{r}+e^{-r\tau}U_0+\int_0^{\tau}e^{-r(\tau-z)}\alpha n_2(z)\,\theta S(z)\,dz \tag{43}$$

$$V_1(\tau)=\left(1-e^{-r\tau}\right)\frac{u_1}{r}+e^{-r\tau}U_1 \tag{44}$$

$$V_2(\tau)=\left(1-e^{-r\tau}\right)\frac{u_2}{r}+e^{-r\tau}U_2+\int_0^{\tau}e^{-r(\tau-z)}\alpha n_0(z)\,(1-\theta)\,S(z)\,dz, \tag{45}$$

which are given explicitly up to the path for the equilibrium surplus, $S(\tau)$. Some properties of the path for the equilibrium surplus are immediate from (40). For example, if $\bar{u}$ is small, then $\dot{S}(\tau)<0$ (the gain from trade is increasing in chronological time, i.e., as t approaches T). Conversely, $\dot{S}(\tau)>0$ will be the case in parametrizations with $\bar{u}$ large, and small enough α and r. The following proposition reports the analytical expressions for the equilibrium surplus and interest rate.

Proposition 8 *The surplus of a match at time $T-\tau$ between a bank with balance $i=2$ and a bank with balance $j=0$ is*

$$S(\tau)=\begin{cases}\frac{n_2(T)e^{\alpha[n_2(T)-n_0(T)](T-\tau)}-n_0(T)}{e^{\{r-\alpha(1-\theta)[n_2(T)-n_0(T)]\}\tau}[n_2(T)-n_0(T)]}\left[\frac{\xi(\tau)\bar{u}}{n_0(T)}+\frac{[n_2(T)-n_0(T)]S(0)}{n_2(T)e^{\alpha[n_2(T)-n_0(T)]T}-n_0(T)}\right] & \text{if } n_2(T)\neq n_0(T)\\ \frac{1+\alpha n_0(T)(T-\tau)}{e^{r\tau}}\left[\frac{\xi(\tau)\bar{u}}{n_0(T)}+\frac{S(0)}{1+\alpha n_0(T)T}\right] & \text{if } n_2(T)=n_0(T),\end{cases}$$

where

$$\xi(\tau)\equiv\begin{cases}\sum_{k=1}^{\infty}\frac{\left[\frac{n_2(T)}{n_0(T)}\right]^{k-1}}{\frac{r}{n_0(T)-n_2(T)}+\alpha(k-\theta)}\frac{e^{\{r+\alpha(k-\theta)[n_0(T)-n_2(T)]\}\tau}-1}{e^{\alpha(k-1)[n_0(T)-n_2(T)]T}} & \text{if } n_2(T)<n_0(T)\\ \sum_{k=0}^{\infty}\frac{(-r)^k\left\{\left[T+\frac{1}{\alpha n_0(T)}\right]^k-\left[T-\tau+\frac{1}{\alpha n_0(T)}\right]^k\right\}}{\alpha k k!}e^{r[T+\frac{1}{\alpha n_0(T)}]} & \text{if } n_2(T)=n_0(T)\\ \sum_{k=0}^{\infty}\frac{\left[\frac{n_0(T)}{n_2(T)}\right]^{k+1}}{\frac{r}{n_2(T)-n_0(T)}+\alpha(k+\theta)}\frac{e^{\{r+\alpha(k+\theta)[n_2(T)-n_0(T)]\}\tau}-1}{e^{\alpha(k+1)[n_2(T)-n_0(T)]T}} & \text{if } n_0(T)<n_2(T).\end{cases}$$

Given $S(\tau)$, the equilibrium repayment is given by (41), with

$$V_1(\tau)-V_0(\tau)=e^{-r\tau}(U_1-U_0)+\left(1-e^{-r\tau}\right)\frac{u_1-u_0}{r}-\theta\frac{e^{\alpha[n_2(T)-n_0(T)]T}n_2(T)}{n_0(T)}e^{-r\tau}\zeta\left[\tau,\bar{u},S(0)\right],$$

where $\zeta\left[\tau,\bar{u},S(0)\right]$ is a time-varying linear combination of $\bar{u}$ and $S(0)$.

7.1 Comparative dynamics

In this section we provide some analytical results on the effect that parameter changes have on the equilibrium paths for the trade surplus and the fed funds rate.

Proposition 9 describes the behavior of $S(\tau)$, namely the value of executing a trade (or the "value of a trade") between a borrower and a lender when the remaining time is τ. With $\bar{u}=0$, (40) specializes to $S(\tau)=e^{-\bar{\delta}(\tau)}S(0)$, so $S(\tau)$ is a discounted version of $S(0)$, with effective discount rate given by $\bar{\delta}(\tau)$. More generally, for $\bar{u}\geq 0$, $S(\tau)$ is a linear combination of $\bar{u}$ and $S(0)$. There are two reasons why $S(0)$ appears discounted in the expression for $S(\tau)$. First, the actual gains from trade accrue at the end of the trading session, so $S(0)$ is discounted by the pure rate of time preference, r. Second, consider a meeting between a borrower and a lender when the remaining time is $\tau>0$. The value $S(0)$ is discounted because both agents might meet alternative trading partners before the end of the session, and this increases their outside options. The borrower's outside option, $V_0(\tau)$, is increasing in the average rate at which he is able to contact a lender and reap gains from trade between time $T-\tau$ and T, i.e., $\alpha\theta\int_0^\tau n_2(s)\,ds$. Similarly, the lender's outside option, $V_2(\tau)$, is increasing in the average rate at which he is able to contact a borrower and reap gains from trade between time $T-\tau$ and T, i.e., $\alpha(1-\theta)\int_0^\tau n_0(s)\,ds$.

Proposition 9 *Assume $\bar{u}\geq 0$ and $S(0)>0$. Then:*

(i) *The surplus at each point in time is decreasing in the discount rate, i.e., for all $\tau>0$, $\frac{\partial S(\tau)}{\partial r}<0$.*

(ii) *If the initial population of lenders is larger (smaller) than that of borrowers, then the surplus at each point in time during the trading session is decreasing (increasing) in the borrower's bargaining power. If the initial populations of lenders and borrowers are equal, then changes in the bargaining power have no effect on the surplus, i.e., for all $\tau>0$, $\frac{\partial S(\tau)}{\partial\theta}$ is equal in sign to $n_0(T)-n_2(T)$.*

(iii) *The surplus at each point in time is increasing in the penalty for below-target end-of-day balances, increasing in the payoff for on-target end-of-day balances, and decreasing in the payoff for above-target end-of-day balances, i.e., for all* τ, $\frac{\partial S(\tau)}{\partial U_0} < 0$, $\frac{\partial S(\tau)}{\partial U_1} > 0$, *and* $\frac{\partial S(\tau)}{\partial U_2} < 0$.

Part *(i)* follows from the fact that a larger value of r increases the effective discount rate, $\bar{\delta}(\tau)$, and also results in a deeper discount of the "dividend-flow gain from trade," $\bar{u}$. The effect of θ on $S(\tau) = 2V_1(\tau) - V_0(\tau) - V_2(\tau)$ is more subtle because a higher θ tends to increase $V_0(\tau)$ (benefits borrowers) and at the same time it tends to decrease $V_2(\tau)$ (hurts lenders). In part *(ii)* we show that the former effect dominates if and only if $n_0(T) < n_2(T)$, and in this case, the effective discount rate decreases with θ, which implies $S(\tau)$ decreases with θ for all $\tau > 0$. Finally, making the penalty for below-target end-of-day balances more severe (lowering U_0), making the payoff for holding above-target end-of-day balances less attractive, or increasing the payoff for holding on-target end-of-day balances, increases the terminal surplus $S(0)$, and consequently increases every surplus along the trading session, which explains part *(iii)*.

The following proposition considers the case with $\bar{u} = 0$. For example, this would be the case when banks are not remunerated for holding intraday balances and have access to intraday credit from the central bank at no cost.

Proposition 10 *Assume* $\bar{u} = 0$ *and* $S(0) > 0$. *Then:*

(i) *The fed funds rate at each point in time is increasing in the discount rate, i.e., for all* τ, $\frac{\partial \rho(\tau)}{\partial r} > 0$.

(ii) *The fed funds rate at each point in time is decreasing in the borrower's bargaining power, i.e., for all* $\tau > 0$, $\frac{\partial \rho(\tau)}{\partial \theta} < 0$.

(iii) *The fed funds rate at each point in time is increasing in the penalty for below-target end-of-day balances, i.e., for all* τ, $\frac{\partial \rho(\tau)}{\partial U_0} < 0$.

Proposition 10 describes the behavior of the fed funds rate at each point in time along the trading session. Parts *(i)*–*(iii)* follow from (42) and the fact that the size of the loan repayment $R(\tau)$ increases with r and U_0, and decreases with the borrower's bargaining power, θ.

7.2 Efficiency

Under Assumption A, the equilibrium paths for the distribution of balances and the distribution of trading probabilities coincide with the efficient paths. The planner's co-states satisfy

$$r\lambda_0(\tau) + \dot{\lambda}_0(\tau) = u_0 + \alpha n_2(\tau) S^*(\tau) \tag{46}$$

$$r\lambda_1(\tau) + \dot{\lambda}_1(\tau) = u_1 \tag{47}$$

$$r\lambda_2(\tau) + \dot{\lambda}_2(\tau) = u_2 + \alpha n_0(\tau) S^*(\tau), \tag{48}$$

for all $\tau \in [0, T]$, given $\lambda_i(0) = U_i$ for $i = 0, 1, 2$, where $S^*(\tau) \equiv 2\lambda_1(\tau) - \lambda_2(\tau) - \lambda_0(\tau)$ satisfies

$$\dot{S}^*(\tau) + \delta^*(\tau) S^*(\tau) = \bar{u} \tag{49}$$

with

$$\delta^*(\tau) \equiv \{r + \alpha [n_2(\tau) + n_0(\tau)]\}.$$

Given the boundary condition $S^*(0) = 2U_1 - U_2 - U_0$, the solution to (49) is

$$S^*(\tau) = \left(\int_0^\tau e^{-\left[\bar{\delta}^*(\tau) - \bar{\delta}^*(z)\right]} dz \right) \bar{u} + e^{-\bar{\delta}^*(\tau)} S(0),$$

where $\bar{\delta}^*(\tau) \equiv \int_0^\tau \delta^*(x)\, dx$.

The comparison between (36), (37) and (38), and (46), (47) and (48), illustrates the composition externality discussed in Section 5. For instance, since in this example meetings involving at least one agent who holds one unit of reserves never entail gains from trade, (37) and (47) confirm that the equilibrium value of a bank with one unit of balances coincides with the shadow price it is assigned by the planner. In contrast, comparing (36) to (46), and (38) to (48), reveals that the equilibrium gains from trade as perceived by an individual borrower and lender at time $T - \tau$ are $\theta S(\tau)$ and $(1 - \theta) S(\tau)$, respectively, while according to the planner each of their marginal contributions equals $S^*(\tau)$.

Notice that $\delta^*(\tau) \geq \delta(\tau)$ for all $\tau \in [0, T]$, with "=" only for $\tau = 0$, so the planner effectively "discounts" more heavily than the equilibrium. It is easy to show that this implies $S(\tau) > S^*(\tau)$ for all $\tau \in (0, 1]$, with $S^*(0) = S(0) = 2U_1 - U_2 - U_0$. In words, due to the matching externality, the social value of a loan (loans are always of size 1 in this example) is smaller than the joint private value of a loan in the equilibrium. Intuitively, the planner internalizes the fact that borrowers and lenders who are searching make it easier for other lenders and borrowers to find trading partners, but these "liquidity provision services" to others receive no compensation in the equilibrium, so individual agents ignore them when calculating their equilibrium payoffs. Naturally, depending on the value of θ, the equilibrium payoff to lenders may be too high or too low relative to their shadow price in the planner's problem. It will be high if $(1-\theta) S(\tau) > S^*(\tau)$, as would be the case for example, if the borrower's bargaining power, θ, is small. As these considerations make clear, the efficiency proposition (Proposition 5) would typically become an *inefficiency* proposition in contexts where banks make some additional choices based on their private gains from trade (e.g., entry, search intensity decisions, etc.).

7.3 Frictionless limit

In this section we characterize the limit of the equilibrium as the contact rate, α, goes to infinity. From (33), (34) and (35), it is immediate that

$$\begin{aligned}
&\lim_{\alpha\to\infty} n_0(\tau) = \max\{n_0(T) - n_2(T), 0\} \\
&\lim_{\alpha\to\infty} n_1(\tau) = 1 - \max\{n_0(T) - n_2(T), n_2(T) - n_0(T)\} \\
&\lim_{\alpha\to\infty} n_2(\tau) = \max\{n_2(T) - n_0(T), 0\}.
\end{aligned}$$

The value function $V_1(\tau)$ is independent of α (see (44)), so $\lim_{\alpha\to\infty} V_1(\tau) = V_1(\tau)$. In Appendix A (proof of Proposition 11) we show that for $i, j = 0, 2$ (with $i \neq j$),

$$\lim_{\alpha\to\infty} V_i(\tau) = \begin{cases} \left(1-e^{-r\tau}\right)\frac{u_i+\bar{u}}{r} + e^{-r\tau}\left[U_i + S(0)\right] & \text{if } n_i(T) < n_j(T) \\ \left(1-e^{-r\tau}\right)\frac{u_i+\varpi(\tau)\theta_i\bar{u}}{r} + e^{-r\tau}\frac{TU_i+\theta_i\tau S(0)}{T} & \text{if } n_i(T) = n_j(T) \\ \left(1-e^{-r\tau}\right)\frac{u_i}{r} + e^{-r\tau}U_i & \text{if } n_j(T) < n_i(T), \end{cases} \tag{50}$$

with $1-\theta_2 = \theta_0 \equiv \theta$, and

$$\varpi(\tau) \equiv \frac{e^{r(T-\tau)}}{1-e^{-r\tau}} \sum_{k=0}^{\infty} \frac{r(-r)^k\left[\tau T^k - \frac{T^{k+1}-(T-\tau)^{k+1}}{k+1}\right]}{kk!}.$$

The following proposition summarizes the frictionless limits of the equilibrium surplus, $S^\infty(\tau) \equiv \lim_{\alpha\to\infty} S(\tau)$, and fed funds rate, $\rho^\infty(\tau) \equiv \lim_{\alpha\to\infty} \rho(\tau)$.

Proposition 11 *For $\tau \in (0, T]$,*

$$S^\infty(\tau) = \begin{cases} 0 & \text{if } n_2(T) \neq n_0(T) \\ (T-\tau)e^{-r\tau}\left[e^{rT}\sum_{k=0}^{\infty}\frac{(-r)^k\left[T^k-(T-\tau)^k\right]}{kk!}\bar{u} + \frac{1}{T}S(0)\right] & \text{if } n_2(T) = n_0(T). \end{cases}$$

For $\tau \in [0, T]$,

$$\rho^{\infty}(\tau) = \begin{cases} r + \frac{\ln\left[\left(1-e^{-r\tau}\right)\frac{u_1-u_0}{r}+e^{-r\tau}(U_1-U_0)\right]}{\tau+\Delta} & \text{if } n_2(T) < n_0(T) \\ r + \frac{\ln\left[\left(1-e^{-r\tau}\right)\frac{u_1-u_0-\theta\bar{u}}{r}+e^{-r\tau}(U_1-U_0-\theta S(0))\right]}{\tau+\Delta} & \text{if } n_2(T) = n_0(T) \\ r + \frac{\ln\left[\left(1-e^{-r\tau}\right)\frac{u_2-u_1}{r}+e^{-r\tau}(U_2-U_1)\right]}{\tau+\Delta} & \text{if } n_0(T) < n_2(T). \end{cases} \tag{51}$$

8 Quantitative analysis

In this section we calibrate a large-scale version of the model and simulate it to assess the ability of the theory to capture the salient features of the market for federal funds in the United States during normal times. We then use the model as a laboratory to conduct quantitative experiments to study a key issue in contemporary central banking, namely the effectiveness of policies that use the interest rate on banks' reserves as a tool to manage the overnight interbank rate.[17]

8.1 Calibration

The motives for trading, and the payoffs from holding reserve balances are different for different types of fed funds market participants. Since commercial banks account for the bulk of the trade volume in the fed funds market, we will adopt their trading motives and payoffs as the baseline for our quantitative implementation.[18] The Federal Reserve imposes a minimum level of reserves on commercial banks and other depository institutions, all of which we refer to as *banks*, for brevity.[19] End-of-day balances within a maintenance period may vary but remain in general positive as overnight overdrafts are considered unauthorized extensions of credit, and penalized. In practice, banks typically target an average daily level of end-of-day balances and try to avoid overnight overdrafts. On October 9, 2008, the Federal Reserve began

[17] In Appendix C we also compute the equilibrium of a small-scale example, and carry out comparative dynamic experiments to illustrate and complement the analytical results of Section 7.

[18] Ashcraft and Duffie (2007) report that commercial banks account for over 80 percent of the volume of federal funds traded in 2005, while 15 percent involves GSEs, and 5 percent corresponds to special situations involving nonbanks that hold reserve balances at the Federal Reserve. Their estimates are based on the Furfine algorithm, from a sample of the top 100 institutions ranked by monthly volume of fed funds sent, including commercial banks, GSEs, and excluding transactions involving accounts held by central banks, federal or state governments, or other settlement systems.

[19] The reserve balance requirement applies to the average level of a bank's end-of-day balances during a two-week maintenance period. For an explanation of how these required operating balances are calculated, see Bennett and Hilton (1997) and Federal Reserve (2009, 2010b).

remunerating banks' positive end-of-day balances. In the theory, all these policy considerations are represented by the end-of-day payoffs $\{U_k\}_{k\in\mathbb{K}}$. Currently, the Fed does not pay interest on intraday balances, but it charges interest on uncollateralized daylight overdrafts. In the theory, the flow payoff to a bank from holding intraday balances is captured by the vector $\{u_k\}_{k\in\mathbb{K}}$.

Let $i_f^r \geq 0$ denote the overnight interest rate that a bank earns on required reserves, and let $i_f^e \in [0, i_f^r]$ be the overnight interest rate on excess reserves. The overnight interest rate at which a bank can borrow from the Discount Window is denoted $i_f^w \geq 0$, and $P^w \geq 0$ represents the pecuniary value of the additional costs associated with Discount-Window borrowing (such as administrative costs and stigma). The deficiency charge for failing to meet the reserve requirement consists of an overnight interest rate charged on the shortfall, denoted $i_f^c > 0$. The parameter $P^c \geq 0$ represents the pecuniary value of additional penalties that the bank may suffer for failing to meet reserve requirements. The overnight overdraft penalty rate is $i_f^o \geq 0$, and $P^o \geq 0$ represents additional penalties resulting from the use of unauthorized overnight credit. The interest rate that a bank earns on positive intraday balances is $i_+^d \geq 0$, and $i_-^d \geq 0$ is the interest rate it pays on daylight overdraft.[20]

For the quantitative work we adopt the following formulation for banks' end-of-day payoffs:

$$U_k = e^{-r\Delta_f}\left(k - \bar{k}_0\right) + F_k \tag{52}$$

with

$$F_k = \begin{cases} F^e(k) & \text{if } \bar{k} \leq k - \bar{k}_0 \\ \max\left[F_{\bar{k}}^w(k), F^c(k)\right] & \text{if } 0 \leq k - \bar{k}_0 < \bar{k} \\ \max\left[F_{\bar{k}}^w(k), F_0^w(k), F^o(k)\right] & \text{if } k - \bar{k}_0 < 0, \end{cases} \tag{53}$$

where

$$\begin{aligned} F^e(k) &\equiv I_f^r \bar{k} + I_f^e\left(k - \bar{k}_0 - \bar{k}\right) \\ F_{\bar{k}}^w(k) &\equiv I_f^r \bar{k} - I_f^w[\bar{k} - (k - \bar{k}_0)] \\ F_0^w(k) &\equiv I_f^w(k - \bar{k}_0) - I_f^c \bar{k} \\ F^c(k) &\equiv I_f^r\left(k - \bar{k}_0\right) - I_f^c[\bar{k} - (k - \bar{k}_0)] \\ F^o(k) &\equiv I_f^o\left(k - \bar{k}_0\right) - I_f^c[\bar{k} - (k - \bar{k}_0)]. \end{aligned}$$

[20] In practice, when an institution has insufficient funds in its Federal Reserve account to cover its settlement obligations during the operating day, it can incur in a daylight overdraft up to an individual maximum amount known as *net debit cap*. (This cap is equal to zero for some institutions.) On March 24, 2011, the Federal Reserve Board implemented major revisions to the Payment System Risk policy, which include a zero fee for collateralized daylight overdrafts and an increased fee for uncollateralized daylight overdrafts to 50 basis points, annual rate (from 36 basis points) (see Federal Reserve, 2010a).

The parameter Δ_f represents the length of the period between the end of the trading session and the beginning of the following trading session, when the bank's reserves held overnight at the Federal Reserve become available (in practice, this period consists of the 2.5 hours between 6:30 pm and 9:00 pm ET). The parameter $\bar{k}_0 \in \{0, \ldots, K-1\}$ indexes translations of the set $\mathbb{K}$, which afford us a more a flexible interpretation of the elements of $\mathbb{K}$. Intuitively, $\bar{k}_0$ can be thought of as the overdraft threshold.[21] The parameter $\bar{k} \in \left\{1, ..., K-\bar{k}_0\right\}$ represents the reserve requirement imposed on every bank. The specification of end-of-day payoffs, (52) and (53), contemplates the fact that in practice, at the end of the trading day a bank with deficient balance has the option to borrow from the Federal Reserve Discount Window.[22] We use $I_f^r \equiv e^{-r\Delta_f^r} i_f^r$, $I_f^e \equiv e^{-r\Delta_f^r} i_f^e$, $I_f^c \equiv e^{-r\Delta_f^c}(i_f^c + P^c)$, $I_f^o \equiv e^{-r\Delta_f^o}(i_f^o + P^o)$, and $I_f^w \equiv e^{-r\Delta_f^w}(i_f^w + P^w)$ to denote the net discounted policy rates (cum penalties, if applicable) applied to a bank's required balances, excess balances, deficiency balances (relative to the reserve requirement), overdraft balances, and balances borrowed from the Discount Window, respectively. The parameter Δ_f^r represents the length of the period between the end of the trading session and the time when the actual interest payments on reserves are effectively made.[23] The parameters Δ_f^w, Δ_f^c, and Δ_f^o, represent the lengths of the periods between the end of the trading session and the times when the bank is required to pay the charges for the use of Discount-Window credit, for failing to meet reserve requirements, or for the use of unauthorized overnight credit, respectively. We assume $I_f^r < I_f^o$, $0 < I_f^c$, and $I_f^r < I_f^w$.[24]

[21]For example, in a parametrization with $\bar{k}_0 = 0$, $\mathbb{K}$ can be interpreted as the set of reserve balances that can be held by an individual bank. More generally, we can instead regard $k \in \mathbb{K}$ as an abstract index, and interpret $k' \equiv k - \bar{k}_0$ as a bank's actual reserve balance. Under this interpretation, reserve balances (i.e., k') held by banks are in the set $\mathbb{K}' \equiv \{k' : k' = k - \bar{k}_0 \text{ for some } k \in \mathbb{K}\}$. Then since $\mathbb{K}' = \left\{-\bar{k}_0, ..., K - \bar{k}_0\right\}$, this formulation allows the payoff functions to accomodate the possibility of negative reserve balances. In line with this more general interpretation, $\bar{k}$ represents the reserve requirement imposed on reserve balances, $k' \equiv k - \bar{k}_0$. (The reserve requirement stated in terms of the index k, would be $\bar{k} + \bar{k}_0$.)

[22]According to (53), a bank with end-of-day balance k such that $\bar{k} \leq k - \bar{k}_0$, chooses not to borrow from the Discount Window (and gets a payoff $e^{-r\Delta_f}(k - \bar{k}_0) + F^e(k)$). A bank with end-of-day balance k such that $0 \leq k - \bar{k}_0 < \bar{k}$, chooses whether to borrow from the Discount Window the amount needed to comply with the reserve requirement, $\bar{k} - (k - \bar{k}_0)$ (to secure a payoff $e^{-r\Delta_f}(k - \bar{k}_0) + F_{\bar{k}}^w(k)$), or not to resort to the Discount Window (and therefore face deficiency charges and get payoff $e^{-r\Delta_f}(k - \bar{k}_0) + F^c(k)$). Similarly, a bank that ends the day with balance k such that $k - \bar{k}_0 < 0$, can choose not to resort to the Discount Window (and face unauthorized overdraft and deficiency charges and get payoff $e^{-r\Delta_f}(k - \bar{k}_0) + F_{\bar{k}}^o$), or to borrow from the Discount Window, either an amount $\bar{k} - (k - \bar{k}_0)$ (to secure a payoff $e^{-r\Delta_f}(k - \bar{k}_0) + F_{\bar{k}}^w(k)$), or an amount $-(k - \bar{k}_0)$ to avoid an overnight overdraft (and secure payoff $e^{-r\Delta_f}(k - \bar{k}_0) + F_0^w(k)$). In Appendix A (Lemma 6) we pose the optimization problem of a bank with end-of-day balance $k \in \mathbb{K}$, and show that F_k is the maximum value of this problem.

[23]In practice, interest is credited to an institution's Federal Reserve account fifteen days after the close of a reserve maintenance period.

[24]The first condition says that at the margin, the present value of the penalty from unauthorized overnight

The flow payoff to a bank from holding intraday balances is given by

$$u_k = \begin{cases} e^{-r\Delta_f^d} i_+^d \left(k - \bar{k}_0\right)^{1-\epsilon} & \text{if } 0 \leq k - \bar{k}_0 \\ e^{-r\Delta_f^d} i_-^d \left(k - \bar{k}_0\right)^{1+\epsilon} & \text{if } k - \bar{k}_0 < 0, \end{cases} \tag{54}$$

where Δ_f^d represents the length of time between the moment when the interest on intraday balances is earned, and the moment when it is paid. The parameter $\epsilon \in [0, 1)$ will be set either to zero or to a negligible value.[25]

We measure time in days. The model is meant to capture trade dynamics in the last 2.5 hours of the daily trading session, so we set $T = 2.5/24$. Since most transactions are settled through Fedwire, and Fedwire does not operate between 6:30 pm and 9:00 pm ET, $\Delta_f = 2.5/24$. As for the other settlement lags, the baseline uses $\Delta_f^d = 0$, and $\Delta_f^r = \Delta_f^w = \Delta_f^c = \Delta_f^o = \Delta_f$. By setting $\Delta = 22/24$, we ensure that all interbank loans in the model have a maturity between 22 and 24.5 hours. The values of the policy rates i_-^d, i_+^d, i_f^r, i_f^e, i_f^w, i_f^c and i_f^o, are chosen to mimic policies in the United States during 2007 prior to the financial crisis. The interest rate charged on daylight overdrafts, i_-^d, is set to 0.0036/360, and the interest rate paid on positive intraday balances, i_+^d, is set to $10^{-7}/360$ (one thousandth of a basis point, annualized).[26] The Federal Reserve did not pay interest on reserves prior to October 9, 2008, so $i_f^r = i_f^e = 0$. The interest rate on Discount-Window loans under the Primary Credit Facility was $i_f^w = 0.0625/360$ (i.e., 6.25 percent per annum). In practice, the penalty rate charged for reserve deficiencies is 100 basis points above the Primary Credit Facility Discount Window lending rate on the first day of the calendar month in which the deficiency occurred, so we set $i_f^c = 0.0725/360$. The interest penalty on overnight overdrafts is generally 400 basis points over the effective fed funds rate. The average daily effective fed funds rate during the second quarter of 2007 was 5.25 percent (annualized), so we let $i_f^o = 0.0925/360$. The discount rate, r, is set to $0.0001/365$.

The parametrization of the initial condition $\{n_k(T)\}_{k\in\mathbb{K}}$ is guided by identifying $n_k(T)$

overdraft is larger than the present value of the interest on required reserves. The second condition implies that there is a penalty for failing to comply with the reserve requirement. The third condition ensures that the Discount Window does not create arbitrage opportunities. These three conditions on policy parameters are used in the proof of Lemma 6, and as will be clear from the calibration that follows, they are currently satisfied in the United States.

[25] By setting ϵ to a negligible positive value, and i_-^d large enough relative to i_+^d, we can ensure that $\{u_k\}_{k\in\mathbb{K}}$ satisfies the discrete midpoint strict concavity property.

[26] The 360-day year is customary for interest rate calculation in money markets. The interest that a bank receives for holding positive intraday reserves has actually been zero in the United States. We set i_+^d to a small positive number (and $\epsilon = 10^{-6}$, a negligible positive number) only to ensure that $\{u_k\}_{k\in\mathbb{K}}$ satisfies the discrete midpoint strict concavity property, which significantly simplifies our solution algorithm. A negligible i_+^d only has a negligible effect on the equilibrium rates.

with the empirical proportion of commercial banks whose balances at the beginning of the trading session are $k/\bar{k}$ times larger than their average daily reserve requirement (over the two-week holding period). Specifically, the initial distribution of balances, $\{n_k(T)\}_{k\in\mathbb{K}}$, was estimated from data using the following procedure. First, we applied the Furfine algorithm to Fedwire interbank payments data for the second quarter of 2007.[27] Second, we identified the 144 commercial banks that traded fed funds at least once during that quarter (according to the output obtained from the Furfine algorithm), and for which we have been able to obtain information on their required operating balance. Third, we obtained data on the cross-sectional distribution of reserve balances across these 144 banks at 6:30 pm, for each day of a two-week maintenance period in the same quarter. Fourth, for every day in the sample, we constructed a measure of each bank's *imputed reserve balance* at 4:00 pm, as follows. Given each bank's end-of-day balance on a given day, we subtracted the bank's net fed funds activity during the last 2.5 hours of the trading day, as estimated by the Furfine algorithm. Fifth, for each bank i, we calculated the average (over each day in the two-week maintenance period) imputed reserve balance at 4:00 pm, and normalized it by dividing it by bank i's daily average required operating balance over the same maintenance period. Let this average normalized imputed reserve balance for bank i be denoted by $\hat{k}^i$. We then removed outliers with $\hat{k}^i < -50$ or with $\hat{k}^i > 200$ and for the sample of the remaining 142 banks, computed maximum likelihood estimates of the parameters of a Gaussian mixture model with 2 components. The estimated parameters are: $\hat{\mu}_1 = 12.98$ and $\hat{\mu}_2 = 0.09$ (the means), $\sigma_1 = 25.45$ and $\sigma_2 = 2.1$ (the standard deviations), and $p_1 = 1 - p_2 = 0.34$ (the probability of drawing from the first component).[28]

Notice that the mean of the estimated distribution of average normalized imputed reserve balances for the 142 banks in the sample is $p_1\hat{\mu}_1 + p_2\hat{\mu}_2$. In order for the calibrated model to capture typical overall market conditions during the second quarter of 2007, we shift the estimated Gaussian mixture by choosing its mean to match the empirical mean (during the second quarter of 2007) of the ratio of *total* seasonally adjusted reserves of depository institutions to *total* required reserves reported in the H.3 Federal Reserve Statistical Release, which equals 1.04. This is done by considering a Gaussian mixture with the same p_1, p_2, σ_1, and σ_2 that were

[27] See Afonso and Lagos (2012) for a detailed description of the Furfine algorithm.

[28] The Kolmogorov-Smirnov goodness-of-fit test does not reject the null hypothesis that the data have been drawn from the Gaussian mixture with 2 components, at the 1% confidence level. We have also attempted to fit a Gaussian, a Logistic, and a Generalized Extreme Value distribution, but the null hypothesis is rejected by the Kolmogorov-Smirnov goodness-of-fit test at the 10% confidence level. At the 1% confidence level the test does not reject the null hypothesis that the data have been drawn from a t-Location Scale distribution.

estimated from the sample of 142 banks, but replacing the estimated means, $\hat{\mu}_1$ and $\hat{\mu}_2$, with $\mu_i = 1.04\hat{\mu}_i/(p_1\hat{\mu}_1 + p_2\hat{\mu}_2)$, for $i = 1, 2$.[29] Let Φ denote the cumulative distribution function of the Gaussian mixture with parameters $(\mu_1, \mu_2, \sigma_1, \sigma_1, p_1)$, we then let $\mathbb{K} = \{0, \ldots, 250\}$ and $\bar{k}_0 = 50$, and set $n_k(T) = \Phi(k - \bar{k}_0 + 1) - \Phi(k - \bar{k}_0)$ for $k = 1, ..., 249$.[30] Throughout, we normalize $\bar{k} = 1$, so k can be interpreted as a multiple of the theoretical reserve requirement. Notice that the mean of the distribution of normalized imputed reserves is $p_1\mu_1 + p_2\mu_2 = 1.04$, and that $Q \equiv \sum_{k=0}^{250}(k - \bar{k}_0)n_k(T) \approx 1.04$.

In the baseline we set $P^c = P^o = P^w$.[31] We calibrate the pecuniary value of the additional costs for borrowing from the Discount Window, P^w, and the contact rate, α, so that the equilibrium of the model is consistent with the following two calibration targets: (a) the target fed funds rate during the second quarter of 2007, which was 0.0525 per annum, and (b) the standard deviation of the empirical end-of-day distribution of average normalized reserve balances (for the two-week holding period used to estimate the initial distribution), which was 1.15. The parameter values implied by this calibration strategy are: $P^w = 0.0525/360$ and $\alpha = 100$.[32] Throughout, all banks have bargaining power equal to 1/2.

[29] The standard deviation of the Gaussian mixture is a function of the means of the two components so changes in μ_1 affect the variance of the mixture. As a robustness check, we have also conducted experiments changing σ_1 along with μ_1 so as to keep the variance constant, and found no significant difference in our results.

[30] In order to work with a finite grid, we truncate by setting $n_0(T) = \Phi(-50)$ and $n_{250}(T) = 1 - \Phi(200)$. The maximum likelihood procedure is useful because it delivers a parametric estimate of the initial distribution of average normalized inputed reserve balances. This allows us to perform an array of quantative experiments, including changes in the initial distribution of balances to simulate various scenarios regarding the relative abundance or scarcity of reserve balances in the interbank market on a given day. Alternatively, we could side-step the estimation procedure and simply construct the discretized version of the empirical distribution of average normalized imputed reserve balances by setting $n_k(T) = \frac{1}{144}\sum_{i=1}^{144}\mathbb{I}_{\{\hat{k}^i \in [k-\bar{k}_0, k-\bar{k}_0+1)\}}$ for each $k \in \mathbb{K}$.

[31] None of the quantitative results are sensitive to the specific values of P^c or P^o, provided that they are not too small relative to P^w. Notice that the baseline parametrization satisfies our maintained assumptions, $I^e_f \leq I^r_f$, $I^r_f < I^o_f$, $0 < I^c_f$, and $I^r_f < I^w_f$. In addition, it implies $I^w_f < I^c_f + I^r_f$, so according to Lemma 6, a bank with a deficient end-of-day balance k (i.e., such that $k - \bar{k}_0 < \bar{k}$) always chooses to borrow from the Discount Window the amount needed to comply with the reserve requirement, namely $\bar{k} - (k - \bar{k}_0)$, and secures a payoff $e^{-r\Delta_f}(k - \bar{k}_0) + F^w_{\bar{k}}(k)$.

[32] With these values, the equilibrium value-weighted daily average fed funds rate implied by the model ($\bar{\rho}$ as defined in Section 6.2) is 0.0527 per annum, and the standard deviation of the end-of-day distribution of balances implied by the model is 1.2. The value of $P^w = 0.0525/360$ implies that the pecuniary value of the additional cost associated with borrowing from the Discount Window (e.g., stigma to the bank, reputational cost for the trader in charge of managing the bank's fed funds desk, etc.) is equivalent to a 5.25 percent annual rate of interest per dollar borrowed. The choice of $\alpha = 100$ implies that banks have an average of about 10 meetings during the trading session, i.e., a trading opportunity every 15 minutes, on average. The implied equilibrium mean and median numbers of trading partners per bank during the session are 6.89 and 7, respectively. In the actual market for federal funds during 2007, the mean and median numbers of fed funds counterparties that a commercial bank traded with between 4:00 pm and 6:30 pm ET have been estimated to equal 4.5 and 2, respectively. The implied equilibrium proportion of intermediated funds in the theory (i.e., $\bar{\iota}(T|0)$ as defined in

8.2 Simulation results

With the parameter values reported in Section 8.1, we simulated the equilibrium paths of one million banks. In this section we report the quantitative performance of the model. The results are presented in three figures. Figure 1 displays the equilibrium behavior of the distribution of reserve balances and of the fed funds rate. Figure 2 reports several dimensions of trade volume, such as the distribution of transactions per bank, the distribution of loan sizes, and the intraday time path of the volume of trade. Figure 3 focuses on intermediation.

In Figure 1, the top row describes the evolution of the distribution of balances. The left panel shows the opening and the end-of-day distribution of balances across banks. The middle panel describes the intraday evolution of the distribution of balances by depicting box plots of the distribution at fifteen-minute intervals throughout the trading session. The distribution of banks' reserve balances follows a clear pattern of convergence.[33] The right panel shows that the standard deviation of the cross-sectional distribution of balances falls over time—another indication that the market is continuously reallocating balances from banks with larger reserves to banks with smaller reserves. The bottom row describes the behavior of the (distribution of) fed funds rate(s). The left panel plots in chronological time, $t = T - \tau$, at each minute t during the trading session, the value-weighted average of $\{\rho_{ij}^{ks}(\tau)\}$ for each τ, i.e., $\bar{\rho}(\tau)$. The middle panel shows the histogram of $\{[\rho_{ij}^{ks}(\tau)]_{\tau\in[0,T]}\}_{i,j,k,s\in\mathbb{K}^4}$. The right panel exhibits a box plot every 15 minutes of the spread between the theoretical rates on loans traded at minute $t = T - \tau$ (measured by $\rho_{ij}^{ks}(\tau)$) and the value-weighted average of these rates on all transactions traded in that minute, $\bar{\rho}(\tau)$.[34]

In Figure 2, the top left panel shows the proportion of the daily volume (the solid line) and the proportion of the daily number of loans (the dashed line) traded by time $t = T - \tau$. Notice that neither the volume of trade nor the number of trades are distributed uniformly throughout the day; rather, trading activity tends to be higher earlier in the session. The top middle panel shows the daily distribution of loan sizes, and the top right panel uses box plots every

Proposition 7) is 0.65, while the empirical average of the proportion of all fed funds traded by commercial banks between 4:00 pm and 6:30 pm ET which were intermediated by commercial banks has been estimated to equal 0.43 for the second quarter of 2007. (All empirical estimates are based on the Furfine algorithm, for details see Afonso and Lagos, 2012.)

[33] The data display a similar pattern of convergence during the last 2.5 hours of the typical trading day. Empirical analogues of these box plots of the intraday distributions of reserve balances can be found in Afonso and Lagos (2012) and Ashcraft and Duffie (2007).

[34] See Afonso and Lagos (2012) for an empirical analogue of this figure (based on Furfine estimates), for an average "synthetic day" corresponding to each year in 2005-2010.

15 minutes to describe the evolution of the distribution of loan sizes during the day. On the bottom row, from left to right, are the distribution of the number of counterparties per bank, the distribution of the number of borrowers that a bank lends to, and the distribution of the number of lenders that a bank borrows from. As in the Furfine estimates, the distribution of loan sizes is skewed, with a few large trades and many small trades, and so are the distributions of counterparties, with a few banks that have many and many banks that have a few.[35]

In Figure 3, the top left panel shows the distributions (and the corresponding boxplots) of fed funds purchased throughout the trading day every 15 minutes by banks whose *adjusted balances*, $k - \bar{k}_0 - \bar{k}$, at the time of the trade are in the top 70 percent of the distribution of nonnegative adjusted balances. The top right panel shows the distributions (and the corresponding boxplots) of fed funds sold throughout the trading day (every 15 minutes) by banks whose *adjusted balances*, $k - \bar{k}_0 - \bar{k}$, at the time of the trade are in the bottom 70 percent of the distribution of negative adjusted balances. The figure shows that it is common for banks with relatively large balances to borrow, as well as for banks with relatively low balances to lend, which can be interpreted as *prima facie* evidence of the presence of over-the-counter trading frictions in the fed funds market.[36] The bottom panels show the distribution of excess funds reallocation and the distribution of the proportion of intermediated funds—the two measures of intermediation introduced in Section 6.4.[37]

8.3 Policy evaluation

During the five years prior to the onset of the 2008-2009 financial crisis, total reserve balances held by depository institutions in the United States fluctuated between \$38 billion and \$56 billion, and required reserves stood between 80 percent and 99 percent of total reserves. The quantity of reserves increased dramatically from about \$41.5 billion in the months prior to September 2008 to more than \$900 billion in January 2009.[38] Most of the increase was accounted for by a sharp rise in excess reserves, which represented more than 93 percent of total reserves in January 2009 (up from less than 3 percent in the months prior to September 2008). This

[35] Empirical versions of these figures based on Furfine estimates can be found in Afonso and Lagos (2012).

[36] Ashcraft and Duffie (2007) were the first to point out that this type of trading activity is present in the loan estimates obtained with the Furfine algorithm. Afonso and Lagos (2012) report similar findings.

[37] See Afonso and Lagos (2012) for versions of these figures constructed using Furfine estimates. This theoretical formulation with homogeneous banks is unable to replicate the extreme skewness in the distribution of intermediation observed in the Furfine estimates.

[38] Lehman Brothers filed for bankruptcy on September 15, 2008.

situation persisted throughout 2010, with required reserves accounting for less than 7 percent of total reserves, which typically remained above \$1 trillion.[39] On the policy front, the Emergency Economic Stabilization Act of 2008 authorized the Federal Reserve to begin paying interest on reserve balances held by or on behalf of depository institutions beginning October 1, 2008. With this authority, the Federal Reserve Board approved a rule to amend its Regulation D (Reserve Requirements of Depository Institutions) to direct the Federal Reserve Banks to pay interest on reserves.[40]

The unprecedented scale of excess reserve balances and the new policy instruments at the disposal of the Federal Reserve raise important questions regarding the Federal Reserve's ability to adjust its policy stance. For example, how large an open market operation would be necessary to increase the fed funds rate by 25 basis points in a market with excess reserves standing at about \$930 billion—i.e., more than 93 percent of total reserves? Is it possible to uncouple the quantity of reserves from the implementation of the interest rate target? And if so, what will be the elasticity of the fed funds rate to changes in the interest on reserves? These issues are crucial for the conduct of monetary policy, and as such they are receiving much attention in policy circles.[41] Consequently, there is a growing need for quantitative models that can be used to explore the effectiveness of the interest rate on reserves (or the Discount-Window rate) as a tool to manage the fed funds rate.[42] In this section we take steps toward meeting this demand.

For the policy experiments that follow, we recalibrate the model so that the equilibrium is in line with market conditions on a typical day in 2011.[43] The values of the policy rates i^d_-, i^w_f, i^c_f, i^o_f, i^r_f and i^e_f, are all chosen to mimic the policies in place in the United States during the first quarter of 2011. Specifically, $i^d_- = 0.0036/360$, $i^w_f = 0.0075/360$, $i^c_f = i^w_f + 0.01/360$,

[39] An analysis of the macro reasons why banks are holding so many excess reserves is beyond the scope of this paper (but see Keister and McAndrews, 2009).

[40] The Financial Services Regulatory Relief Act of 2006 had originally authorized the Federal Reserve to begin paying interest on balances held by or on behalf of depository institutions beginning October 1, 2011. The Emergency Economic Stabilization Act of 2008 accelerated the effective date to October 1, 2008. The Federal Reserve began paying interest on reserve balances held by depository institutions on October 9, 2008.

[41] See Ennis and Wolman (2010), Goodfriend (2002), and Keister, Martin and McAndrews (2008).

[42] Keister, Martin and McAndrews (2008), for example, conclude that "While the floor system has received a fair amount of attention in policy circles recently, there are important open questions about how well such a system will work in practice. Going forward, it will be useful to develop theoretical models of the monetary policy implementation process that can adress these questions...". Ennis and Wolman (2010) point out that "In contrast to the predictions of simple theories, the interest on reserves (IOR) rate has not acted as a floor on the federal funds rate. It is now well-understood why certain institutional features of the fed funds market and the IOR program should prevent the IOR rate from acting as a floor, but the precise determination of the fed funds rate in this environment remains poorly understood."

[43] We report the results of similar policy experiments for the 2007 calibration in Appendix D.

and $i^r_f = i^e_f \equiv i_f = 0.0025/360$.[44] The effective fed funds rate for the first quarter of 2011 was about 15 basis points, and the overnight overdraft rate, i^o_f, was set at 400 basis points above the effective fed funds rate during 2011, so $i^o_f = 0.0415/360$. The initial distribution of balances, $\{n_k(T)\}_{k\in\mathbb{K}}$, was estimated from data using the procedure described in Section 8.1, but from a sample of the 137 commercial banks that traded fed funds at least once during the first quarter of 2011 (according to the output obtained from the Furfine algorithm), and for which we have been able to obtain information on their required operating balance for a two-week maintenance period during the same quarter. We eliminated four outliers (any bank i with $\hat{k}^i < -50$ or with $\hat{k}^i > 200$). The maximum likelihood estimates of the parameters of a Gaussian mixture model with 2 components are: $\mu_1 = 50.21$, $\mu_2 = 4.50$, $\sigma_1 = 40.38$, $\sigma_2 = 4.36$, and $p_1 = 1 - p_2 = 0.32$.[45] Notice that the mean of the estimated distribution of average normalized imputed reserve balances for the 133 banks in the sample is $\bar{\mu} \equiv p_1\mu_1 + p_2\mu_2 = 19$, which is roughly equal to the empirical mean (during the first quarter of 2011) of the ratio of *total* seasonally adjusted reserves of depository institutions to *total* required reserves reported in the H.3 Federal Reserve Statistical Release.[46] As before, we let $\mathbb{K} = \{0, \ldots, 250\}$, $\bar{k}_0 = 50$ and $\bar{k} = 1$, and set $n_k(T) = \Phi(k - \bar{k}_0 + 1) - \Phi(k - \bar{k}_0)$ for $k = 1, ..., 249$, $n_0(T) = \Phi(-50)$, and $n_{250}(T) = 1 - \Phi(200)$, where Φ denotes the cumulative distribution function of the Gaussian mixture with parameters $(\mu_1, \mu_2, \sigma_1, \sigma_1, p_1)$. We maintain the assumption that $P^c = P^o = P^w$, but now set $P^w = 0$ and $\alpha = 1$ so that the equilibrium of the model is consistent with: (a) the upper band of the fed funds rate target during the second quarter of 2011, which was 0.0025 per annum, and (b) the standard deviation of the empirical end-of-day distribution of average normalized reserve balances (for the two-week holding period used to estimate the initial distribution), which was 31.[47] All other parameter values, i.e., r, T, Δ, Δ_f, Δ^d_f, Δ^r_f, Δ^w_f,

[44] As it turns out, the value of i^r_f does not matter for our experiments because it does not affect the equilibrium fed funds rate (provided that i^r_f satisfies the maintained assumptions). The reason is that as explained in footnote 31, under the maintained assumptions, banks with deficient end-of-day balances choose to borrow from the Discount Window the amount needed to comply with the reserve requirement. Therefore banks can *always* meet their reserve requirement, and earn i^r_f per unit of required reserves.

[45] Again, the Kolmogorov-Smirnov goodness-of-fit test does not reject the null hypothesis that the data have been drawn from the Gaussian mixture with 2 components, at the 1% confidence level. We have also attempted to fit a Gaussian, a Logistic, and a t-Location Scale distribution, but the the null hypothesis is rejected by the Kolmogorov-Smirnov goodness-of-fit test at the 10% confidence level. The test does not reject the null hypothesis that the data have been drawn from a Generalized Extreme Value distribution at the 1% confidence level.

[46] The consolidated banking sector currently holds reserve balances that amount to about twenty times its reserve requirement.

[47] With this parametrization, the equilibrium of the model delivers a value-weighted daily average fed funds rate of 0.0029 per annum, and a standard deviation of the end-of-day distribution of balances equal to 30.6.

Δ_f^c, Δ_f^o, and i_+^d are set as in the calibration of Section 8.1. Figure 4 illustrates the performance of the model calibrated to 2011, in terms of the reallocation of reserve balances and the implied distribution of fed funds rates.

The policy experiments consist of varying either i_f or i_f^w for different values of Q. In the theory, $Q \equiv \sum_{k=0}^{250}(k - \bar{k}_0)n_k(T)$ is the quantity of reserves held by the banking system as a whole, while $\bar{k}$ is the reserve requirement of the consolidated banking system. Hence $Q/\bar{k}$ indicates whether total reserve balances are scarce or abundant relative to the total amount of required reserves on a given day, and we can represent different market conditions by varying Q.[48] For example, a situation in which $Q/\bar{k}$ is small may result from an open market sale at the onset of the trading session, or from some other portfolio decisions made by banks. We conduct three types of policy experiments, and for each we consider seven scenarios depending on the value of $Q/\bar{k}$, namely 0.1, 0.5, 1, 5, 10, 15, and 30.

The first experiment consists of increasing i_f by 25 basis points from 0 to 75 basis points, while leaving i_f^w fixed at its baseline value (75 basis points). The second experiment consists of increasing i_f^w by 25 basis points from 25 to 150 basis points, while leaving i_f at its baseline value (25 basis points). The implied values of the equilibrium (value-weighted) daily average fed funds rate, $\bar{\rho}_f$, for the first and second experiments are summarized in Table 1 and Table 2, respectively. The first scenario, $Q/\bar{k} = 0.1$, represents a day in which reserves are very scarce in the sense that the consolidated banking system holds reserves that are only one tenth of the (average) required reserves. In the seventh scenario, $Q/\bar{k} = 30$, the quantity of reserves in the system is large relative to the quantity of required reserves, similar to what is the case on a typical day nowadays. In this case, the equilibrium fed funds rate essentially varies one-for-one with the interest on reserves, i_f, and is insensitive to the Discount-Window rate, i_f^w. The fed funds rate is sensitive to both policy rates when market conditions are less extreme (in terms of the size of the total reserves relative to required reserves). For example, if the market is "balanced", e.g., if $Q/\bar{k} = 1$, then a 25 basis point increase in either policy rate, increases the

While $P^w = 0$ allows the model to replicate the much lower fed funds rate prevailing in 2011 relative to 2007, a lower value of P^w for 2011 than for 2007 is also in line with recent efforts by the Federal Reserve to make the Discount Window more accessible and less stigmatic.

[48]Since for a large enough grid, $\mathbb{K}$, our procedure ensures $Q \approx \bar{\mu}$, we vary Q by varying $\bar{\mu}$. The desired value of Q for each experiment is achieved by using a Gaussian mixture with parameters $(\mu_1(Q), \mu_2(Q), \sigma_1, \sigma_1, p_1)$, where $\mu_i(Q) \equiv Q\mu_i/\bar{\mu}$, and cumulative distribution function denoted by $\Phi(\cdot; Q)$, and then setting the initial distribution of balances to $n_k(T) = \Phi(k - \bar{k}_0 + 1; Q) - \Phi(k - \bar{k}_0; Q)$ for $k = 1, ..., 249$, $n_0(T) = \Phi(-50; Q)$, and $n_{250}(T) = 1 - \Phi(200; Q)$.

fed funds rate roughly by 0.5×25 basis points.[49] Other intermediate market conditions give different intermediate results, for example, if $Q/\bar{k} = 10$, then a 25 basis-point increase in the interest rate paid on reserves, increases the fed funds rate by about 20 basis points, while a 25 basis-point increase in the Discount-Window rate would increase the fed funds rate by about 5 basis points. Notice that the responsiveness of the equilibrium fed funds rate with respect to i_f increases with $Q/\bar{k}$, while the opposite is true for i_f^w.

For a given policy, the equilibrium fed funds rate is decreasing in the overall quantity of funds in the system, $Q/\bar{k}$, as can be seen by following from left to right any of the rows in Table 1 or Table 2. Notice that the equilibrium fed funds rate typically lies in an interval $[i_f + \varepsilon, i_f^w + \varepsilon]$. Such an interval is often referred to as a *channel* or *corridor* by central bankers.[50] In Table 1 the corridor gets narrower as i_f increases, and in the limit when $i_f \to i_f^w = 0.0075$ (the last row), the interval collapses to a point: the equilibrium rate can only equal $0.0075 + \varepsilon$, and therefore becomes insensitive to $Q/\bar{k}$. Similarly, the equilibrium rate tends to remain equal to $0.0025 + \varepsilon$ for any value of $Q/\bar{k}$ as $i_f^w \to i_f = 0.0025$ (first row of Table 2).

For the third experiment we set $i_f^w = i_f + w$, where w denotes a number of basis points (per annum), and increase i_f by 25 basis points from 0 to 100 basis points. The implied values of $\bar{\rho}_f$, for $w = 0.0025/360$ are reported in Table 3. The equilibrium fed funds rate is always inside a corridor $[i_f + \varepsilon, i_f + w + \varepsilon]$ (the value of ε is slightly above half a basis point). Thus, these experiments amount to shifting the whole corridor keeping its width, w, constant. As before, the exact position of the equilibrium fed funds rate within this corridor depends on the amount of reserves relative to required reserves, $Q/\bar{k}$. For example, from the last column of Table 3, it is clear that if reserves are very abundant, the equilibrium fed funds rate coincides with the lower limit of the corridor, $i_f + \varepsilon$. As the market becomes more balanced, i.e., as $Q/\bar{k}$ becomes closer to 1, the equilibrium fed funds rate approaches the middle of the corridor. Finally, notice that shifting the whole corridor up by x basis points (keeping the corridor width fixed) increases the

[49] As we explain below, the "0.5" results from the fact that in our baseline calibration, the bargaining power of all banks is equal to one half.

[50] The value of ε is slightly above half a basis point in our calculations. (It is not exactly zero, because in the baseline calibration, banks have a small yet positive, concave intraday payoff from holding reserves.) Many central banks, e.g., the European Central Bank, and the central banks of Australia, Canada, and England, use a *channel* or *corridor system* to implement monetary policy. The system consists of a lending facility that resembles the Discount Window in the United States, from which banks are allowed to borrow freely (typically against acceptable collateral) at an interest rate equal to the target rate plus a fixed number of basis points. There is also a deposit facility that allows banks to earn overnight interest on reserves at a rate equal to the target rate minus a fixed number of basis points. Hence interest rates at the two standing facilities form a *channel* or *corridor* around the target rate.

equilibrium fed funds rate by x basis points.

Figure 5 succinctly summarizes the experiments described above. Each panel corresponds to a particular policy stance defined by a pair of policy rates i_f and i_f^w, and illustrates the equilibrium (value-weighted) daily average fed funds rate, $\bar{\rho}_f$, as a function of the ratio of overall reserves to required reserves, $Q/\bar{k}$. The center left panel, for example, corresponds to the policy used to compute the fourth row of Table 2. If one regards open-market operations as interventions that change the marketwide availability of reserves relative to the reserve requirement, then the curves displayed in Figure 5 show the effect that open market sales or purchases of various sizes would have on the equilibrium fed funds rate, when carried out against the background of different interest-on-reserves or Discount-Window policies, i_f and i_f^w.[51]

8.3.1 Discussion

In order to explain the impact that changes in policy, such as the Discount-Window rate, or the interest rate that the Federal Reserve pays on reserves, have on the equilibrium distribution of fed funds rates negotiated between banks throughout the day, consider the analytical example studied in Section 7.

Assume that $\{U_k\}$ is given by (52)–(54), a specification that captures the essential institutional arrangements currently in place in the United States, set $\bar{k}_0 = 0$, $\bar{k} = 1$, and let $\rho_f(\tau)$ denote the fed funds rate on a bilateral loan at time $T - \tau$ as it is usually calculated by fed funds analysts, i.e., $\rho_f(\tau) \equiv e^{\rho(\tau)(\tau+\Delta)} - 1$. In Appendix A (Lemma 7) we show that

$$\ln[1+\rho_f(\tau)] = (\Delta - \Delta_f^r)r + \ln\Bigg[\beta(\tau)\, i_f^e + [1-\beta(\tau)]\, i_f^s + e^{r\Delta_f^r}\left(e^{r\tau}-1\right)\frac{u_2 - u_1 + c(\tau)(1-\theta)\bar{u}}{r} + e^{-r(\Delta_f - \Delta_f^r)}\Bigg], \tag{55}$$

where

$$\beta(\tau) \equiv 1 - (1-\theta)\left[\int_0^\tau \alpha n_0(z)\, e^{-[\bar{\delta}(z)-rz]}dz + e^{-[\bar{\delta}(\tau)-r\tau]}\right] \tag{56}$$

$$c(\tau) \equiv \frac{\int_0^\tau e^{-r(\tau-z)}\alpha n_0(z)\int_0^z e^{-[\bar{\delta}(z)-\bar{\delta}(x)]}dx dz + \int_0^\tau e^{-[\bar{\delta}(\tau)-\bar{\delta}(z)]}dz}{\left(1-e^{-r\tau}\right)r^{-1}}, \tag{57}$$

[51]Since policy discussion is often organized around the competitive static model of Poole (1968) (see, e.g., Keister et al., 2008), it is interesting to point out that the curves in Figure 5 are reminiscent of those that would be traced out by the equilibrium points that would result from progressively shifting the standard vertical supply of reserves from left to right in the Poole model.

and $i_f^s \equiv i_f^r + e^{r\Delta_f^r} \min(I_f^c, I_f^w - I_f^r)$. The first term on the right side of (55) reflects the fact that the interest on a loan made to another bank is received at time $T + \Delta$, while the interest on reserves held overnight at the Federal Reserve is received at time $T + \Delta_f^r$. At time T, the interest on excess reserves, i_f^e, is the minimum overnight interest rate at which a bank with excess balance is willing to lend, while

$$i_f^s = \min\left\{i_f^r + e^{-r(\Delta_f^c - \Delta_f^r)}(i_f^c + P^c), e^{-r(\Delta_f^w - \Delta_f^r)}(i_f^w + P^w)\right\}$$

is the maximum overnight interest rate that a bank with a deficient balance is willing to pay. If r or the maturity difference $\Delta - \Delta_f^r$ is small, then it is clear from (55) along with the maintained assumptions $i_f^e \leq i_f^r$ and $I_f^r < I_f^w$, that $\rho_f(\tau) \geq i_f^e$, i.e., at no point during the trading session will a bank with two units of reserves lend the second (excess) unit to another bank for an interest rate smaller than the interest it can earn on this second unit from the Fed. The premium that the lender can charge over the interest it can earn on reserves, will depend on the size of the current gain from trade as well as on the bargaining power of the lender.

To explain the effect of policy on the fed funds rate, focus on the case in which $\Delta - \Delta_f^r$ is negligible, so (55) simplifies to

$$\begin{aligned}\rho_f(\tau) &= \beta(\tau)\, i_f^e + [1 - \beta(\tau)]\, i_f^s \\ &\quad + e^{r\Delta}\left(e^{r\tau} - 1\right) \frac{u_2 - u_1 + c(\tau)(1-\theta)\,\bar{u}}{r} + e^{r(\Delta - \Delta_f)} - 1. \qquad (58)\end{aligned}$$

For this case, we have the following characterization of the effects of the policy rates on the equilibrium path of the fed funds rate.

Proposition 12 *Suppose that either $r \approx 0$ or $\Delta - \Delta_f^r \approx 0$. A one percent increase in the overnight interest rate that the Fed pays on excess reserves, i_f^e, causes a $\beta(\tau)$ percent increase in the fed funds rate at time $T - \tau$. A one percent increase in the maximum overnight interest rate that a bank with a deficient balance is willing to pay at the end of the trading session, i_f^s, causes an $1 - \beta(\tau)$ percent increase in the fed funds rate at time $T - \tau$. If $n_2(T) = n_0(T)$, then $\beta(\tau) = \theta$. If $n_2(T) \neq n_0(T)$, then*

$$\begin{aligned}1 - \beta(\tau) &= (1-\theta)\frac{e^{-\alpha[n_2(T)-n_0(T)]\theta\tau}\left\{n_2(T) - n_0(T)\, e^{-\alpha[n_2(T)-n_0(T)](T-\tau)}\right\}}{n_2(T) - n_0(T)\, e^{-\alpha[n_2(T)-n_0(T)]T}} \\ &\quad + \frac{n_0(T)\, e^{-\alpha[n_2(T)-n_0(T)]T}\left\{e^{\alpha[n_2(T)-n_0(T)](1-\theta)\tau} - 1\right\}}{n_2(T) - n_0(T)\, e^{-\alpha[n_2(T)-n_0(T)]T}},\end{aligned}$$

with $\beta(0) = \theta$ and $\beta(\tau) \in [0,1]$ for all τ. Moreover, $0 \leq \beta(\tau) \leq \theta$ and $\beta'(\tau) < 0$ if $n_2(T) < n_0(T)$, and $\theta \leq \beta(\tau) \leq 1$ and $\beta'(\tau) > 0$ if $n_0(T) < n_2(T)$.

Intuitively, $1 - \beta(\tau)$ can be thought of a lender's *effective bargaining power* at time $T - \tau$. It is determined by the lenders' fundamental bargaining power, $1 - \theta$, as well as by their ability to realize gains from trade in the time remaining until the end of the trading session, which depends on the evolution of the endogenous distribution of balances across banks. For example, if $n_0(T) < n_2(T)$, it is relatively difficult for banks with excess balances to find potential borrowers, and $1 - \beta(\tau)$ is smaller than $1 - \theta$ throughout the trading session. In this case the lenders' effective bargaining power, $1 - \beta(\tau)$, increases toward their fundamental bargaining power, $1 - \theta$, as the trading session progresses, reflecting the fact that although borrowers face a favorable distribution of potential trading partners throughout the session, their chances to execute the desired trade diminish as the end of the session draws closer.

The mechanism is most transparent if in addition to $r \approx 0$, we set $u_i \approx 0$ (as is currently the case in the United States) since in this case

$$\rho_f(\tau) = \beta(\tau)\, i_f^e + [1 - \beta(\tau)]\, i_f^s. \tag{59}$$

Recall that $\rho_f(\tau)$ in this example is the rate that a bank with two units of reserves charges a bank with no reserves for a loan of size one. Since required reserves equal one unit, the bank with two units has excess reserves, and a bank with zero units needs to purchase one unit to comply with the reserve requirement. According to (59), the fed funds rate is a time-varying weighted average of the lender's end-of-day return on the second unit of balances, i_f^e, and the borrower's end-of-day reservation rate for the first unit of balances, i_f^s. The weight on the former is $\beta(\tau)$, the borrower's effective bargaining power at time $T-\tau$. Notice that if $\Delta_f^r = \Delta_f^w = \Delta_f^r$, then $i_f^s = \min(i_f^r + i_f^c + P^c, i_f^w + P^w)$. Hence, if $i_f^w + P^w < i_f^r + i_f^c + P^c$, then a bank with zero balance at time T will resort to the Discount Window, and the equilibrium fed funds rate at time $T - \tau$ is $\rho_f(\tau) = \beta(\tau)\, i_f^e + [1 - \beta(\tau)]\,(i_f^w + P^w)$. Conversely, if $i_f^r + i_f^c + P^c < i_f^w + P^w$, then a bank with zero balance at time T will instead choose to bear the deficiency charges, and the equilibrium fed funds rate at time $T - \tau$ is $\rho_f(\tau) = \beta(\tau)\, i_f^e + [1 - \beta(\tau)]\,(i_f^r + i_f^c + P^c)$.

To conclude, we illustrate the importance of basing policy recommendations on a theory that is explicit about the over-the-counter nature of the fed funds market. Let $\rho_f^\infty(\tau)$ denote the fed funds rate (measured as it is usually calculated from Fedwire data) that would prevail in the frictionless economy of Section 7.3; i.e., $1 + \rho_f^\infty(\tau) = e^{\rho^\infty(\tau)(\tau+\Delta)}$. In Appendix A (Lemma

8), we show that $\rho_f^\infty(\tau)$ is independent of τ, so here we denote it ρ_f^∞. For the special case with $\Delta - \Delta_f = u_i = 0$ for all i, and either $r \approx 0$ or $\Delta - \Delta_f^r \approx 0$,

$$\rho_f^\infty = \begin{cases} i_f^s & \text{if } n_2(T) < n_0(T) \\ \theta i_f^e + (1-\theta)\, i_f^s & \text{if } n_2(T) = n_0(T) \\ i_f^e & \text{if } n_0(T) < n_2(T) \end{cases} \tag{60}$$

is the frictionless analogue of the over-the-counter fed funds rate in (59). Generically, the fed funds rate in the frictional market is time-dependent and continuous in the quantity of fed funds in the market (in this case, $Q = n_1(T) + 2n_2(T)$). In contrast, the frictionless rate ρ_f^∞ is independent of τ and discontinuous in Q; ρ_f^∞ jumps from i_f^e up to $\theta i_f^e + (1-\theta)\, i_f^s$ as Q approaches 1 from below, and jumps from i_f^s down to $\theta i_f^e + (1-\theta)\, i_f^s$ as Q approaches 1 from above. In general,

$$\rho_f^\infty - \rho_f(\tau) = \begin{cases} \beta(\tau)\,(i_f^s - i_f^e) & \text{if } n_2(T) < n_0(T) \\ 0 & \text{if } n_2(T) = n_0(T) \\ -[1-\beta(\tau)]\,(i_f^s - i_f^e) & \text{if } n_0(T) < n_2(T). \end{cases}$$

Notice that $\rho_f(\tau) = \rho_f^\infty$ in the non-generic case of a "balanced market", i.e., if $n_2(T) = n_0(T)$ (or equivalently, if $Q = 1$), for in this case the distribution of balances is neutral with respect to borrowers and lenders, and hence their effective bargaining powers, $\beta(\tau)$ and $1 - \beta(\tau)$ coincide with their fundamental bargaining powers, θ and $1-\theta$. But generically, the frictionless approximation overestimates the true frictional fed funds rate if reserves are relatively scarce (i.e., if $Q < 1$ or equivalently, $n_2(T) < n_0(T)$ in this example), and underestimates the true rate if reserves are relatively abundant (i.e., if $Q > 1$). Interestingly, these biases which are nil when the market is perfectly balanced, will also tend to be relatively small if the market is very unbalanced. For example, if $n_2(T)$ is very large relative to $n_0(T)$, then according to Proposition 12, the equilibrium path for $\beta(\tau)$ will be very close to 1 throughout most of the trading session ($\beta(\tau)$ will fall sharply toward θ over a very short interval of time right before the end of the trading session).

9 Conclusion

We have developed a dynamic equilibrium model of trade in the fed funds market that explicitly accounts for the two distinctive features of the over-the-counter structure of the actual market: search for counterparties, and bilateral negotiations. The theory identifies the determinants of the fed funds rate as well as empirical measures of trade volume and trading delays. We have derived theoretical predictions for the equilibrium dynamics of the reserve balances of individual banks, and proposed theory-based measures of the importance of bank-provided intermediation in the process of reallocation of reserves. We have calibrated and simulated a large-scale version of the model to assess the quantitative potential of the theory to capture the salient empirical regularities of the market for federal funds in the United States. We have also used the model as a laboratory to study some topical issues in modern central banking, such as the effectiveness of policies that use the interest rate on reserves or the Discount-Window rate as tools to implement a desired fed funds rate target.

The structure of the model we have studied is strikingly simple: banks randomly contact other banks over time, and bargain the terms of the loans. Given the wide range of theoretical and quantitative insights that the model delivers, we regard its simplicity as a virtue. We recognize, however, that there are several aspects of the real-world market for federal funds that our theory abstracts from, and we think that this opens up several interesting avenues for future work.

Available estimates suggest that the bulk of trade in the fed funds market is direct trade between banks. But there is a segment of the market where trades between banks are intermediated by specialized brokers that are not themselves commercial banks, so it would be interesting to incorporate specialized brokers into the model. The model is perhaps best suited to describe the last 2.5 hours of the typical trading session, when unexpected payment shocks are rare. In order to model trade dynamics throughout the whole day, the theory could be extended to allow for random payment shocks that induce exogenous reallocations of reserve balances among banks. Our model focuses on trading dynamics within a typical day, but standard data sets consist of sequences of trading days, so it would be useful to extend the theory to encompass a sequence of trading sessions much like the representative session we have modeled. This type of extension would also entail a dimension of endogeneity to the beginning-of-day distribution of reserve balances which could also yield interesting insights. The baseline model has banks that only differ in their initial holdings of reserve balances, while several aspects of available data seem driven by more fundamental heterogeneity among banks in terms of their relative bargaining strengths, the rates at which they can contact potential trading partners, and the payoffs from holding reserve balances. While we have already formulated some of these theoretical extensions, there is little doubt that implementing them quantitatively would be a fruitful task. We have assumed random search, which may be a reasonable assumption for settings in which banks are completely uninformed about potential counterparties' balances before a contact takes place. However, in reality some banks may have some information about which counterparties are more likely to be long or short on any given day (e.g., based on the different cash flows associated with different lines of business), so it may be useful to extend the model to incorporate some degree of directedness in search activity. The baseline model abstracts from default risk, which may be an interesting feature to add, especially to study the behavior of interbank markets during times of financial stress such as the period following the bankruptcy of Lehman Brothers in 2008. Other natural extensions involve incorporating private information, e.g., regarding the reserve balances held by each bank, or the likelihood that a given bank may fail to repay a loan at the stipulated time.

A Proofs

Lemma 1 *For any* $(k,k') \in \mathbb{K} \times \mathbb{K}$ *and any* $\tau \in [0,T]$*, consider the following problem:*

$$\max_{b \in \Gamma(k,k'), R \in \mathbb{R}} \left[V_{k-b}(\tau) - V_k(\tau) + e^{-r(\tau+\Delta)}R\right]^{\theta_{kk'}} \left[V_{k'+b}(\tau) - V_{k'}(\tau) - e^{-r(\tau+\Delta)}R\right]^{1-\theta_{kk'}} \tag{61}$$

where $\theta_{kk'} = 1 - \theta_{k'k} \in [0,1]$*, and* $V_k(\tau) : \mathbb{K} \times [0,T] \to \mathbb{R}$ *is bounded. The correspondence*

$$H^*(k,k',\tau;V) = \arg\max_{b \in \Gamma(k,k'), R \in \mathbb{R}} \left\{ \left[V_{k-b}(\tau) - V_k(\tau) + e^{-r(\tau+\Delta)}R\right]^{\theta_{kk'}} \right.$$
$$\left. \left[V_{k'+b}(\tau) - V_{k'}(\tau) - e^{-r(\tau+\Delta)}R\right]^{1-\theta_{kk'}} \right\}$$

is nonempty. Moreover, $(b_{kk'}(\tau), R_{k'k}(\tau)) \in H^*(k,k',\tau;V)$ *if and only if*

$$b_{kk'}(\tau) \in \arg\max_{b \in \Gamma(k,k')} \left[V_{k'+b}(\tau) + V_{k-b}(\tau) - V_{k'}(\tau) - V_k(\tau)\right], \text{ and} \tag{62}$$

$$e^{-r(\tau+\Delta)}R_{k'k}(\tau) = \theta_{kk'}\left[V_{k'+b_{kk'}(\tau)}(\tau) - V_{k'}(\tau)\right] + (1-\theta_{kk'})\left[V_k(\tau) - V_{k-b_{kk'}(\tau)}(\tau)\right]. \tag{63}$$

Proof of Lemma 1. Consider

$$\max_{(b,R) \in \tilde{\mathbf{\Gamma}}(k,k')} \left[V_{k-b}(\tau) - V_k(\tau) + e^{-r(\tau+\Delta)}R\right]^{\theta_{kk'}} \left[V_{k'+b}(\tau) - V_{k'}(\tau) - e^{-r(\tau+\Delta)}R\right]^{1-\theta_{kk'}} \tag{64}$$

where $\tilde{\mathbf{\Gamma}}(k,k') = \{(b,R) \in \Gamma(k,k') \times [-B,B]\}$ for some arbitrary real number $B > 0$. Clearly, this problem has at least one solution. Let (b^*, R^*) denote a solution to (64). If the constraints $-B \leq R \leq B$ are slack at (b^*, R^*), then (b^*, R^*) is also a solution to (61), and (b^*, R^*) must satisfy the following first-order condition

$$e^{-r(\tau+\Delta)}R^* = \theta_{kk'}\left[V_{k'+b^*}(\tau) - V_{k'}(\tau)\right] + (1-\theta_{kk'})\left[V_k(\tau) - V_{k-b^*}(\tau)\right]. \tag{65}$$

Suppose that (b^*, R^*) with R^* given by (65) is a solution to (64) with $-B \leq R^* \leq B$ (given (65), these inequalities can be guaranteed by choosing B large enough), but such that

$$b^* \notin \arg\max_{b \in \Gamma(k,k')} \left[V_{k'+b}(\tau) + V_{k-b}(\tau) - V_{k'}(\tau) - V_k(\tau)\right]. \tag{66}$$

Condition (65) implies

$$V_{k-b^*}(\tau) - V_k(\tau) + e^{-r(\tau+\Delta)}R^* = \theta_{kk'}\left[V_{k'+b^*}(\tau) + V_{k-b^*}(\tau) - V_{k'}(\tau) - V_k(\tau)\right]$$
$$V_{k'+b^*}(\tau) - V_{k'}(\tau) - e^{-r(\tau+\Delta)}R^* = (1-\theta_{kk'})\left[V_{k'+b^*}(\tau) + V_{k-b^*}(\tau) - V_{k'}(\tau) - V_k(\tau)\right],$$

so the value of (64) achieved by (b^*, R^*) is

$$\theta_{kk'}^{\theta_{kk'}} (1-\theta_{kk'})^{1-\theta_{kk'}} \left[V_{k'+b^*}(\tau) + V_{k-b^*}(\tau) - V_{k'}(\tau) - V_k(\tau)\right] \equiv \xi^*.$$

But (66) implies that there exists $b' \in \Gamma(k, k')$ such that

$$\xi^* < \theta_{kk'}^{\theta_{kk'}} (1-\theta_{kk'})^{1-\theta_{kk'}} \left[V_{k'+b'}(\tau) + V_{k-b'}(\tau) - V_{k'}(\tau) - V_k(\tau)\right].$$

Then since B can be chosen large enough so that

$$R' = e^{r(\tau+\Delta)} \left\{\theta_{kk'}\left[V_{k'+b'}(\tau) - V_{k'}(\tau)\right] + (1-\theta_{kk'})\left[V_k(\tau) - V_{k-b'}(\tau)\right]\right\} \in (-B, B),$$

it follows that (b', R') achieves a higher value than (b^*, R^*), so (b^*, R^*) is not a solution to (64); a contradiction. Hence, a solution (b^*, R^*) to (64) with $-B \leq R^* \leq B$ must satisfy (65) and

$$b^* \in \arg\max_{b\in\Gamma(k,k')} \left[V_{k'+b}(\tau) + V_{k-b}(\tau) - V_{k'}(\tau) - V_k(\tau)\right]. \tag{67}$$

Since the right side of (65) is bounded, R^* is finite and B can be chosen large enough such that $R^* \in (-B, B)$, so (61) has at least one solution, and any solution to (61) must satisfy (65) and (67). To conclude, we show that any (b^*, R^*) that satisfies (65) and (67) is a solution to (61). To see this, notice that for all $(b, R) \in \Gamma(k, k') \times \mathbb{R}$,

$$\begin{aligned}
&\left[V_{k-b}(\tau) - V_k(\tau) + e^{-r(\tau+\Delta)}R\right]^{\theta_{kk'}} \left[V_{k'+b}(\tau) - V_{k'}(\tau) - e^{-r(\tau+\Delta)}R\right]^{1-\theta_{kk'}} \\
&\leq \max_{R\in\mathbb{R}} \left[V_{k-b}(\tau) - V_k(\tau) + e^{-r(\tau+\Delta)}R\right]^{\theta_{kk'}} \left[V_{k'+b}(\tau) - V_{k'}(\tau) - e^{-r(\tau+\Delta)}R\right]^{1-\theta_{kk'}} \\
&= \theta_{kk'}^{\theta_{kk'}} (1-\theta_{kk'})^{1-\theta_{kk'}} \left[V_{k'+b}(\tau) + V_{k-b}(\tau) - V_{k'}(\tau) - V_k(\tau)\right] \\
&\leq \theta_{kk'}^{\theta_{kk'}} (1-\theta_{kk'})^{1-\theta_{kk'}} \max_{b\in\Gamma(k,k')} \left[V_{k'+b}(\tau) + V_{k-b}(\tau) - V_{k'}(\tau) - V_k(\tau)\right] = \xi^*. \blacksquare
\end{aligned}$$

Lemma 2 *The function $J_k(x, \tau)$ given in (2) satisfies (1) if and only if $V_k(\tau)$ satisfies (3), given (4) and (5).*

Proof of Lemma 2. Let $\boldsymbol{B}$ denote the space of bounded real-valued functions defined on $\mathbb{K} \times [0, T]$. Let $\boldsymbol{B'}$ denote the space of functions obtained by adding $e^{-r(\tau+\Delta)}x$ for some $x \in \mathbb{R}$, to each element of $\boldsymbol{B}$. That is,

$$\boldsymbol{B'} = \left\{g : \mathbb{S} \to \mathbb{R} \mid g(k, x, \tau) = w(k, \tau) + e^{-r(\tau+\Delta)}x \text{ for some } w \in \boldsymbol{B}\right\},$$

where $\mathbb{S} = \mathbb{K} \times \mathbb{R} \times [0,T]$. Let $\boldsymbol{s} = (k,x)$ and $\boldsymbol{s}' = (k',x')$ denote two elements of $\mathbb{K} \times \mathbb{R}$. For any $g \in \boldsymbol{B'}$ and any $(\boldsymbol{s}, \boldsymbol{s}', \tau) \in \mathbb{K} \times \mathbb{R} \times \mathbb{S}$, let

$$\tilde{H}\left(\boldsymbol{s}, \boldsymbol{s}', \tau; g\right) = \arg \max_{b \in \Gamma(k,k'), R \in \mathbb{R}} \Big\{ \left[g\left(k-b, x+R, \tau\right) - g\left(k,x,\tau\right)\right]^{\theta_{kk'}} \\ \big[g(k'+b, x'-R,\tau) - g(k',x',\tau)\big]^{1-\theta_{kk'}} \Big\},$$

where $\theta_{kk'} = 1 - \theta_{k'k} \in [0,1]$ for any $k,k' \in \mathbb{K}$. Since $g \in \boldsymbol{B'}$, $\tilde{H}\left(\boldsymbol{s}, \boldsymbol{s}', \tau; g\right) = H^*\left(k,k',\tau;w\right)$, where

$$H^*\left(k,k',\tau;w\right) = \arg \max_{b \in \Gamma(k,k'), R \in \mathbb{R}} \left\{ \left[w\left(k-b,\tau\right) - w\left(k,\tau\right) + e^{-r(\tau+\Delta)}R\right]^{\theta_{kk'}} \\ \left[w(k'+b,\tau) - w(k',\tau) - e^{-r(\tau+\Delta)}R\right]^{1-\theta_{kk'}} \right\}$$

for some $w \in \boldsymbol{B}$, as defined in Lemma 1. By Lemma 1, $H^*\left(k,k',\tau;w\right)$ is nonempty, and $(b(k,k',\tau), R(k',k,\tau)) \in H^*\left(k,k',\tau;w\right)$ if and only if

$$b(k,k',\tau) \in \arg \max_{b \in \Gamma(k,k')} \left[w(k'+b,\tau) + w\left(k-b,\tau\right) - w(k',\tau) - w\left(k,\tau\right)\right] \tag{68}$$

and

$$e^{-r(\tau+\Delta)}R(k',k,\tau) = \theta_{kk'}\left\{w\left[k'+b(k,k',\tau),\tau\right] - w(k',\tau)\right\} \\ + \left(1-\theta_{kk'}\right)\left\{w(k,\tau) - w\left[k-b(k,k',\tau),\tau\right]\right\}. \tag{69}$$

The right side of (1) defines a mapping $\mathcal{T}$ on $\boldsymbol{B'}$. That is, for any $g \in \boldsymbol{B'}$ and all $(k,x,\tau) \in \mathbb{S}$,

$$\left(\mathcal{T}g\right)(k,x,\tau) = \mathbb{E}\Bigg[\int_0^{\min(\tau_\alpha,\tau)} e^{-rz}u_k dz + \mathbb{I}_{\{\tau_\alpha > \tau\}}e^{-r\tau}\left(U_k + e^{-r\Delta}x\right) \\ + \mathbb{I}_{\{\tau_\alpha \le \tau\}}e^{-r\tau_\alpha}\int g\left[k - b(k,k',\tau-\tau_\alpha), x + R(k',k,\tau-\tau_\alpha), \tau-\tau_\alpha\right]\mu\left(d\boldsymbol{s}', \tau-\tau_\alpha\right)\Bigg]$$

where $b(k,k',\tau)$ satisfies (68) and $R(k',k,\tau)$ satisfies (69) (for the special case $\theta_{kk'} = 1/2$ for all $k,k' \in \mathbb{K}$), for $w \in \boldsymbol{B}$ defined by $w\left(k,\tau\right) = g\left(k,x,\tau\right) - e^{-r(\tau+\Delta)}x$ for all $(k,\tau) \in \mathbb{K} \times [0,T]$. Substitute $g\left(k,x,\tau\right) = w\left(k,\tau\right) + e^{-r(\tau+\Delta)}x$ on the right side of $\left(\mathcal{T}g\right)(k,x,\tau)$ to obtain

$$\left(\mathcal{T}g\right)(k,x,\tau) = \left(\mathcal{M}w\right)(k,\tau) + e^{-r(\tau+\Delta)}x, \tag{70}$$

where $\mathcal{M}$ is a mapping on $\boldsymbol{B}$ defined by

$$\begin{aligned}(\mathcal{M}w)(k,\tau) = \mathbb{E}\bigg[&\int_0^{\min(\tau_\alpha,\tau)} e^{-rz}u_k dz + \mathbb{I}_{\{\tau_\alpha>\tau\}}e^{-r\tau}U_k \\ &+ \mathbb{I}_{\{\tau_\alpha\le\tau\}}e^{-r\tau_\alpha}\int w\left[k-b(k,k',\tau-\tau_\alpha),\tau-\tau_\alpha\right]\mu\left(d\boldsymbol{s}',\tau-\tau_\alpha\right) \\ &+ \mathbb{I}_{\{\tau_\alpha\le\tau\}}e^{-r\tau_\alpha}\int e^{-r(\tau+\Delta-\tau_\alpha)}R(k',k,\tau-\tau_\alpha)\mu\left(d\boldsymbol{s}',\tau-\tau_\alpha\right)\bigg],\end{aligned} \tag{71}$$

for all $(k,\tau)\in\mathbb{K}\times[0,T]$. Since the right side of (71) is independent of the net credit position x, after recognizing that $\mu\left(\{(k',x)\in\mathbb{K}\times\mathbb{R}:k'=k\},\tau\right)=n_k(\tau)$, (71) can be written as

$$\begin{aligned}(\mathcal{M}w)(k,\tau) = \mathbb{E}&\left[\int_0^{\min(\tau_\alpha,\tau)} e^{-rz}u_k dz + \mathbb{I}_{\{\tau_\alpha>\tau\}}e^{-r\tau}U_k\right] \\ &+ \mathbb{E}\bigg[\mathbb{I}_{\{\tau_\alpha\le\tau\}}e^{-r\tau_\alpha}\sum_{k'\in\mathbb{K}} n_{k'}(\tau-\tau_\alpha)\,w\left[k-b(k,k',\tau-\tau_\alpha),\tau-\tau_\alpha\right] \\ &+ \mathbb{I}_{\{\tau_\alpha\le\tau\}}e^{-r\tau_\alpha}\sum_{k'\in\mathbb{K}} n_{k'}(\tau-\tau_\alpha)\,e^{-r(\tau+\Delta-\tau_\alpha)}R(k',k,\tau-\tau_\alpha)\bigg],\end{aligned} \tag{72}$$

for all $(k,\tau)\in\mathbb{K}\times[0,T]$. From (72), it is clear that $\mathcal{M}$ is the mapping defined by the right side of (3). Since $w\in\boldsymbol{B}$, and $(b(k,k',\tau),R(k',k,\tau))$ satisfy (68) and (69), it follows that $\mathcal{M}:\boldsymbol{B}\to\boldsymbol{B}$, and together with (70), this implies $\mathcal{T}:\boldsymbol{B}'\to\boldsymbol{B}'$. Notice that $g^*=w^*+e^{-r(\tau+\Delta)}x\in\boldsymbol{B}'$ is a fixed point of $\mathcal{T}$ if and only if $w^*\in\boldsymbol{B}$ is a fixed point of $\mathcal{M}$. In the statement of the lemma and in the body of the paper, the fixed points $g^*(k,x,\tau)$ and $w^*(k,\tau)$ are denoted $J_k(x,\tau)$ and $V_k(\tau)$, respectively. ■

Proof of Proposition 1. Start with the mapping (72), and notice that after writing out the expectation explicitly and performing a change of variable, it becomes

$$(\mathcal{M}w)(k,\tau) = v_k(\tau)+\alpha\int_0^\tau\sum_{k'\in\mathbb{K}} n_{k'}(z)\left\{w\left[k-b(k,k',z),z\right]+e^{-r(z+\Delta)}R(k',k,z)\right\}e^{-(r+\alpha)(\tau-z)}dz,$$

for all $(k,\tau)\in\mathbb{K}\times[0,T]$, where

$$v_k(\tau)\equiv\mathbb{E}\left[\int_0^{\min(\tau_\alpha,\tau)} e^{-rz}u_k dz+\mathbb{I}_{\{\tau_\alpha>\tau\}}e^{-r\tau}U_k\right],$$

which can be integrated to obtain the expression in (11). Since $b(k,k',\tau)$ and $R(k',k,\tau)$ satisfy

(68) and (69), the previous expression for the mapping $\mathcal{M}$ can be written as

$$\begin{aligned}(\mathcal{M}w)(k,\tau) &= v_k(\tau) + \alpha\int_0^\tau w(k,z)\,e^{-(r+\alpha)(\tau-z)}dz \\ &+ \alpha\int_0^\tau\Bigg[\sum_{k'\in\mathbb{K}} n_{k'}(z)\,\theta_{kk'}\{w\left[k'+b(k,k',z),z\right] + w\left[k-b(k,k',z),z\right] \\ &- w(k',z) - w(k,z)\}\Bigg]e^{-(r+\alpha)(\tau-z)}dz.\end{aligned}$$

In turn, since

$$\begin{aligned}&w\left[k'+b(k,k',z),z\right] + w\left[k-b(k,k',z),z\right] - w(k',z) - w(k,z) \\ &= \max_{b\in\Gamma(k,k')}\left[w(k'+b,z) + w(k-b,z) - w(k',z) - w(k,z)\right] \\ &= \max_{(i,j)\in\Pi(k,k')}\left[w(j,z) + w(i,z) - w(k',z) - w(k,z)\right],\end{aligned}$$

we have

$$\begin{aligned}(\mathcal{M}w)(k,\tau) &= v_k(\tau) + \alpha\int_0^\tau w(k,z)\,e^{-(r+\alpha)(\tau-z)}dz \\ &+ \alpha\int_0^\tau \sum_{k'\in\mathbb{K}} n_{k'}(z)\,\theta_{kk'} \max_{(i,j)\in\Pi(k,k')}\left[w(i,z)+w(j,z)-w(k,z)-w(k',z)\right]e^{-(r+\alpha)(\tau-z)}dz,\end{aligned}$$

for all $(k,\tau)\in\mathbb{K}\times[0,T]$. With a relabeling, this mapping can be rewritten as

$$\begin{aligned}(\mathcal{M}w)(i,\tau) &= v_i(\tau) + \alpha\int_0^\tau w(i,z)\,e^{-(r+\alpha)(\tau-z)}dz \qquad (73)\\ &+ \alpha\int_0^\tau \sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} n_j(z)\,\theta_{ij}\phi_{ij}^{ks}(z)\left[w(k,z)+w(s,z)-w(i,z)-w(j,z)\right]e^{-(r+\alpha)(\tau-z)}dz,\end{aligned}$$

for all $(i,\tau)\in\mathbb{K}\times[0,T]$, with

$$\phi_{ij}^{ks}(z) = \begin{cases}\tilde{\phi}_{ij}^{ks}(z) & \text{if } (k,s)\in\Omega_{ij}\left[w(\cdot,z)\right] \\ 0 & \text{if } (k,s)\notin\Omega_{ij}\left[w(\cdot,z)\right],\end{cases}$$

for all $i,j,k,s\in\mathbb{K}$ and all $z\in[0,T]$, where $\tilde{\phi}_{ij}^{ks}(z)\geq 0$ and $\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}}\tilde{\phi}_{ij}^{ks}(z)=1$, and

$$\Omega_{ij}\left[w(\cdot,z)\right] \equiv \arg\max_{(k',s')\in\Pi(i,j)}\left[w(k',z)+w(s',z)-w(i,z)-w(j,z)\right].$$

From (73) (with $\theta_{kk'}=1/2$), it is clear that (10) is just $\boldsymbol{V}=\mathcal{M}\boldsymbol{V}$. ■

The following lemma establishes the equivalence between property (DMC) and discrete midpoint concavity.

Lemma 3 *Let g be a real-valued function on $\mathbb{K}$. Then g satisfies*

$$g\left(\left\lceil \tfrac{i+j}{2} \right\rceil\right) + g\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor\right) \geq g(k) + g(s) \tag{74}$$

for any $i, j \in \mathbb{K}$ and all $(k, s) \in \Pi(i, j)$, if and only if it satisfies the discrete midpoint concavity property,

$$g\left(\left\lceil \tfrac{i+j}{2} \right\rceil\right) + g\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor\right) \geq g(i) + g(j) \tag{75}$$

for all $i, j \in \mathbb{K}$.

Proof of Lemma 3. Suppose that g satisfies (74). Since the condition holds for all $(k, s) \in \Pi(i, j)$, and we know that $(i, j) \in \Pi(i, j)$, it holds for the special case $(k, s) = (i, j)$, so g satisfies (75). To show the converse, notice that since (75) holds for all $i, j \in \mathbb{K}$, it also holds for all $i, j \in \mathbb{K}$ such that $(i, j) \in \Pi(k, s)$ for any $k, s \in \mathbb{K}$. But for any such (i, j), we know that $i + j = k + s$, so (75) implies

$$g\left(\left\lceil \tfrac{k+s}{2} \right\rceil\right) + g\left(\left\lfloor \tfrac{k+s}{2} \right\rfloor\right) \geq g(i) + g(j)$$

for any $k, s \in \mathbb{K}$ and all $(i, j) \in \Pi(k, s)$, which is the same as (74) up to a relabeling. ■

The following two lemmas are used in the proof of Proposition 2.

Lemma 4 *For any given path $\boldsymbol{n}(\tau)$, there exists a unique $w^* \in \boldsymbol{B}$ that satisfies $w^* = \mathcal{M}w^*$, and a unique $g^* \in \boldsymbol{B}'$ that satisfies $g^* = \mathcal{T}g^*$, defined by $g^*(k, x, \tau) = w^*(k, \tau) + e^{-r(\tau+\Delta)}x$ for all $(k, x, \tau) \in \mathbb{S}$.*

Proof of Lemma 4. Write the mapping $\mathcal{M}$ defined in the proof of Proposition 1 (with $\theta_{kk'} = 1/2$), as

$$\begin{aligned}(\mathcal{M}w)(i, \tau) &= v_i(\tau) + \alpha \int_0^\tau w(i, z)\, e^{-(r+\alpha)(\tau - z)} dz \\ &\quad + \frac{\alpha}{2} \int_0^\tau \sum_{j \in \mathbb{K}} n_j(z) \max_{(k,s) \in \Pi(i,j)} \left[w(k, z) + w(s, z) - w(i, z) - w(j, z)\right] e^{-(r+\alpha)(\tau - z)} dz,\end{aligned}$$

for all $(i, \tau) \in \mathbb{K} \times [0, T]$. For any $w, w' \in \boldsymbol{B}$, define the metric $D : \boldsymbol{B} \times \boldsymbol{B} \to \mathbb{R}$, by

$$D\left(w, w'\right) = \sup_{(i,\tau) \in \mathbb{K} \times [0,T]} \left[e^{-\beta\tau} \left|w(i, \tau) - w'(i, \tau)\right|\right],$$

where $\beta \in \mathbb{R}$ satisfies

$$\max\{0, 2\alpha - r\} < \beta < \infty. \tag{76}$$

For the case with $\beta = 0$, D reduces to the standard sup metric, d_∞. The metric space $(\boldsymbol{B}, d_\infty)$ is complete, and since $(\boldsymbol{B}, D)$ and $(\boldsymbol{B}, d_\infty)$ are strongly equivalent, it follows that $(\boldsymbol{B}, D)$ is also a complete metric space (see Ok, 2007, p. 136 and 167). For any $w, w' \in \boldsymbol{B}$, and any $(i,\tau) \in \mathbb{K} \times [0,T]$,

$$\begin{aligned}
&e^{-\beta\tau} \left|(\mathcal{M}w)(i,\tau) - (\mathcal{M}w')(i,\tau)\right| = \\
&\quad = e^{-\beta\tau} \Bigg| \alpha \int_0^\tau w(i,z)\, e^{-(r+\alpha)(\tau-z)} dz - \alpha \int_0^\tau w'(i,z)\, e^{-(r+\alpha)(\tau-z)} dz \\
&\quad + \frac{\alpha}{2} \int_0^\tau \sum_{j\in\mathbb{K}} n_j(z) \max_{(k,s)\in\Pi(i,j)} \left[w(k,z) + w(s,z) - w(i,z) - w(j,z)\right] e^{-(r+\alpha)(\tau-z)} dz \\
&\quad - \frac{\alpha}{2} \int_0^\tau \sum_{j\in\mathbb{K}} n_j(z) \max_{(k,s)\in\Pi(i,j)} \left[w'(k,z) + w'(s,z) - w'(i,z) - w'(j,z)\right] e^{-(r+\alpha)(\tau-z)} dz \Bigg| \\
&\quad \le \alpha e^{-\beta\tau} \int_0^\tau \left| w(i,z) - w'(i,z) \right| e^{-(r+\alpha)(\tau-z)} dz \\
&\quad + \frac{\alpha}{2} e^{-\beta\tau} \int_0^\tau \sum_{j\in\mathbb{K}} n_j(z) \Bigg| \max_{(k,s)\in\Pi(i,j)} \left[w(k,z) + w(s,z) - w(i,z) - w(j,z)\right] \\
&\quad - \max_{(k,s)\in\Pi(i,j)} \left[w'(k,z) + w'(s,z) - w'(i,z) - w'(j,z)\right] \Bigg| e^{-(r+\alpha)(\tau-z)} dz.
\end{aligned}$$

Use $(k^*_{ij}(z), s^*_{ij}(z))$ to denote a solution to the maximization on the right side of $\mathcal{M}w$, that is,

$$\left(k^*_{ij}(z), s^*_{ij}(z)\right) \in \max_{(k,s)\in\Pi(i,j)} \left[w(k,z) + w(s,z) - w(i,z) - w(j,z)\right].$$

A solution exists because $w \in \boldsymbol{B}$, and $\Pi(i,j)$ is a finite set for all $(i,j) \in \mathbb{K} \times \mathbb{K}$. Then

$$\begin{aligned}
&e^{-\beta\tau} \left|(\mathcal{M}w)(i,\tau) - (\mathcal{M}w')(i,\tau)\right| \le \\
&\le \alpha \int_0^\tau e^{-\beta z} \left| w(i,z) - w'(i,z) \right| e^{-(r+\alpha+\beta)(\tau-z)} dz \\
&+ \frac{\alpha}{2} \int_0^\tau \sum_{j\in\mathbb{K}} n_j(z) \Big\{ e^{-\beta z} \left| w\left(k^*_{ij}(z), z\right) - w'\left(k^*_{ij}(z), z\right) \right| + e^{-\beta z} \left| w\left(s^*_{ij}(z), z\right) - w'\left(s^*_{ij}(z), z\right) \right| \\
&+ e^{-\beta z} \left| w'(i,z) - w(i,z) \right| + e^{-\beta z} \left| w'(j,z) - w(j,z) \right| \Big\} e^{-(r+\alpha+\beta)(\tau-z)} dz \\
&\le \frac{3\alpha}{r+\alpha+\beta} \left[1 - e^{-(r+\alpha+\beta)\tau}\right] D(w,w') \\
&\le \frac{3\alpha}{r+\alpha+\beta} D(w,w').
\end{aligned}$$

Since this last inequality holds for all $(i,\tau) \in \mathbb{K}\times [0,T]$, and w and w' are arbitrary,

$$D\left(\mathcal{M}w, \mathcal{M}w'\right) \leq \frac{3\alpha}{r+\alpha+\beta} D\left(w, w'\right), \qquad \text{for all } w, w' \in \boldsymbol{B}. \tag{77}$$

Notice that (76) implies $\frac{3\alpha}{r+\alpha+\beta} \in (0,1)$, so $\mathcal{M}$ is a contraction mapping on the complete metric space $(\boldsymbol{B}, D)$. By the Contraction Mapping Theorem (Theorem 3.2 in Stokey and Lucas, 1989), for any given path $\boldsymbol{n}(\tau)$, there exists a unique $w^* \in \boldsymbol{B}$ that satisfies $w^* = \mathcal{M}w^*$, and therefore, by (70), there exists a unique $g^* \in \boldsymbol{B}'$ that satisfies $g^* = \mathcal{T}g^*$, and it is defined by $g^*(k,x,\tau) = w^*(k,\tau) + e^{-r(\tau+\Delta)}x$ for all $(k,x,\tau) \in \mathbb{S}$. ■

Lemma 5 *Let $i,j,q \in \mathbb{K}$, and $(k,s) \in \Pi(i,j)$.*

(i) *If either $i+j$ or $s+q$ is even, then*

$$\left(\left\lceil \tfrac{k+q}{2}\right\rceil, \left\lfloor \tfrac{s+q}{2}\right\rfloor\right) \in \Pi\left(\left\lceil \tfrac{i+j}{2}\right\rceil, q\right) \quad \textit{and} \quad \left(\left\lfloor \tfrac{k+q}{2}\right\rfloor, \left\lceil \tfrac{s+q}{2}\right\rceil\right) \in \Pi\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor, q\right).$$

(ii) *If $i+j$ and $s+q$ are odd, then*

$$\left(\left\lfloor \tfrac{k+q}{2}\right\rfloor, \left\lceil \tfrac{s+q}{2}\right\rceil\right) \in \Pi\left(\left\lceil \tfrac{i+j}{2}\right\rceil, q\right) \quad \textit{and} \quad \left(\left\lceil \tfrac{k+q}{2}\right\rceil, \left\lfloor \tfrac{s+q}{2}\right\rfloor\right) \in \Pi\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor, q\right).$$

Proof of Lemma 5. Notice that for any $i,j,q \in \mathbb{K}$,

$$\Pi(i,j) = \{(i+j-y, y) \in \mathbb{K}\times\mathbb{K} : y \in \{0,1,\ldots,i+j\}\},$$

so

$$\Pi\left(\left\lceil \tfrac{i+j}{2}\right\rceil, q\right) = \left\{\left(\left\lceil \tfrac{i+j}{2}\right\rceil + q - y, y\right) \in \mathbb{K}\times\mathbb{K} : y \in \left\{0,1,\ldots,\left\lceil \tfrac{i+j}{2}\right\rceil + q\right\}\right\} \tag{78}$$

$$\Pi\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor, q\right) = \left\{\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor + q - y, y\right) \in \mathbb{K}\times\mathbb{K} : y \in \left\{0,1,\ldots,\left\lfloor \tfrac{i+j}{2}\right\rfloor + q\right\}\right\}. \tag{79}$$

For any $i,j,q \in \mathbb{K}$, define

$$\tilde{\Pi}(i,j,q) = \left\{\left(\left\lceil \tfrac{k+q}{2}\right\rceil, \left\lfloor \tfrac{s+q}{2}\right\rfloor\right) \in \mathbb{K}\times\mathbb{K} : (k,s) \in \Pi(i,j)\right\}$$
$$\hat{\Pi}(i,j,q) = \left\{\left(\left\lfloor \tfrac{k+q}{2}\right\rfloor, \left\lceil \tfrac{s+q}{2}\right\rceil\right) \in \mathbb{K}\times\mathbb{K} : (k,s) \in \Pi(i,j)\right\},$$

and recall that $(k,s) \in \Pi(i,j)$ implies $k+s = i+j$.

(i) Assume that either $i+j$ or $s+q$ is even. We first show that given any $i,j,q \in \mathbb{K}$, $(k,s) \in \Pi(i,j)$ implies $\left(\left\lceil \frac{k+q}{2} \right\rceil, \left\lfloor \frac{s+q}{2} \right\rfloor\right) \in \Pi\left(\left\lceil \frac{i+j}{2} \right\rceil, q\right)$. Notice that if either $i+j$ or $s+q$ is even, then

$$\left\lceil \frac{k+q}{2} \right\rceil + \left\lfloor \frac{s+q}{2} \right\rfloor = \left\lceil \frac{i+j}{2} \right\rceil + q. \tag{80}$$

With (80),

$$\begin{aligned}
\tilde{\Pi}(i,j,q) &= \left\{\left(\left\lceil \tfrac{k+q}{2} \right\rceil, \left\lceil \tfrac{i+j}{2} \right\rceil + q - \left\lceil \tfrac{k+q}{2} \right\rceil\right) \in \mathbb{K}\times\mathbb{K} : (k, i+j-k) \in \Pi(i,j)\right\} \\
&= \left\{\left(y, \left\lceil \tfrac{i+j}{2} \right\rceil + q - y\right) \in \mathbb{K}\times\mathbb{K} : y \in \left\{\left\lceil \tfrac{q}{2} \right\rceil, \left\lceil \tfrac{q+1}{2} \right\rceil, \ldots, \left\lceil \tfrac{q+i+j}{2} \right\rceil\right\}\right\} \\
&\equiv \tilde{\Pi}^e\left(\left\lceil \tfrac{i+j}{2} \right\rceil, q\right).
\end{aligned} \tag{81}$$

By construction, given any $i,j,q \in \mathbb{K}$, $\left(\left\lceil \frac{k+q}{2} \right\rceil, \left\lfloor \frac{s+q}{2} \right\rfloor\right) \in \tilde{\Pi}^e\left(\left\lceil \frac{i+j}{2} \right\rceil, q\right)$ for all $(k,s) \in \Pi(i,j)$. Since $0 \leq \left\lceil \frac{q}{2} \right\rceil$, and $\left\lceil \frac{q+i+j}{2} \right\rceil \leq \left\lceil \frac{i+j}{2} \right\rceil + q$, it follows from (78) and (81) that $\tilde{\Pi}^e\left(\left\lceil \frac{i+j}{2} \right\rceil, q\right) \subseteq \Pi\left(\left\lceil \frac{i+j}{2} \right\rceil, q\right)$ for all $i,j,q \in \mathbb{K}$, which implies $\left(\left\lceil \frac{k+q}{2} \right\rceil, \left\lfloor \frac{s+q}{2} \right\rfloor\right) \in \Pi\left(\left\lceil \frac{i+j}{2} \right\rceil, q\right)$ for all $(k,s) \in \Pi(i,j)$, and any $i,j,q \in \mathbb{K}$.

Next, we show that given any $i,j,q \in \mathbb{K}$, $(k,s) \in \Pi(i,j)$ implies $\left(\left\lfloor \frac{k+q}{2} \right\rfloor, \left\lceil \frac{s+q}{2} \right\rceil\right) \in \Pi\left(\left\lfloor \frac{i+j}{2} \right\rfloor, q\right)$. Notice that if either $i+j$ or $s+q$ is even, then

$$\left\lfloor \frac{k+q}{2} \right\rfloor + \left\lceil \frac{s+q}{2} \right\rceil = \left\lfloor \frac{i+j}{2} \right\rfloor + q. \tag{82}$$

With (82),

$$\begin{aligned}
\hat{\Pi}(i,j,q) &= \left\{\left(\left\lfloor \tfrac{k+q}{2} \right\rfloor, \left\lfloor \tfrac{i+j}{2} \right\rfloor + q - \left\lfloor \tfrac{k+q}{2} \right\rfloor\right) \in \mathbb{K}\times\mathbb{K} : (k, i+j-k) \in \Pi(i,j)\right\} \\
&= \left\{\left(y, \left\lfloor \tfrac{i+j}{2} \right\rfloor + q - y\right) \in \mathbb{K}\times\mathbb{K} : y \in \left\{\left\lfloor \tfrac{q}{2} \right\rfloor, \left\lfloor \tfrac{q+1}{2} \right\rfloor, \ldots, \left\lfloor \tfrac{q+i+j}{2} \right\rfloor\right\}\right\} \\
&\equiv \hat{\Pi}^e\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, q\right).
\end{aligned} \tag{83}$$

By construction, given any $i,j,q \in \mathbb{K}$, $\left(\left\lfloor \frac{k+q}{2} \right\rfloor, \left\lceil \frac{s+q}{2} \right\rceil\right) \in \hat{\Pi}^e\left(\left\lfloor \frac{i+j}{2} \right\rfloor, q\right)$ for all $(k,s) \in \Pi(i,j)$. Since $0 \leq \left\lfloor \frac{q}{2} \right\rfloor$, and $\left\lfloor \frac{q+i+j}{2} \right\rfloor \leq \left\lfloor \frac{i+j}{2} \right\rfloor + q$, it follows from (79) and (83) that $\hat{\Pi}^e\left(\left\lfloor \frac{i+j}{2} \right\rfloor, q\right) \subseteq \Pi\left(\left\lfloor \frac{i+j}{2} \right\rfloor, q\right)$ for all $i,j,q \in \mathbb{K}$, which implies $\left(\left\lfloor \frac{k+q}{2} \right\rfloor, \left\lceil \frac{s+q}{2} \right\rceil\right) \in \Pi\left(\left\lfloor \frac{i+j}{2} \right\rfloor, q\right)$ for all $(k,s) \in \Pi(i,j)$, and any $i,j,q \in \mathbb{K}$.

(ii) Suppose that $i+j$ and $s+q$ are odd. We first show that given any $i,j,q \in \mathbb{K}$, $(k,s) \in \Pi(i,j)$ implies $\left(\left\lfloor \frac{k+q}{2} \right\rfloor, \left\lceil \frac{s+q}{2} \right\rceil\right) \in \Pi\left(\left\lceil \frac{i+j}{2} \right\rceil, q\right)$. Notice that if $i+j$ and $s+q$ are odd, then

$$\left\lfloor \frac{k+q}{2} \right\rfloor + \left\lceil \frac{s+q}{2} \right\rceil = \left\lceil \frac{i+j}{2} \right\rceil + q. \tag{84}$$

With (84),

$$\begin{aligned}\hat{\Pi}(i,j,q) &= \left\{\left(\left\lceil \tfrac{i+j}{2}\right\rceil + q - \left\lceil \tfrac{s+q}{2}\right\rceil, \left\lceil \tfrac{s+q}{2}\right\rceil\right) \in \mathbb{K}\times\mathbb{K} : (k,s) \in \Pi(i,j)\right\} \\ &= \left\{\left(\left\lceil \tfrac{i+j}{2}\right\rceil + q - y, y\right) \in \mathbb{K}\times\mathbb{K} : y \in \left\{\left\lceil \tfrac{q}{2}\right\rceil, \left\lceil \tfrac{q+1}{2}\right\rceil, \ldots, \left\lceil \tfrac{q+i+j}{2}\right\rceil\right\}\right\} \\ &\equiv \hat{\Pi}^{o}\left(\left\lceil \tfrac{i+j}{2}\right\rceil, q\right). \end{aligned} \tag{85}$$

By construction, given any $i,j,q \in \mathbb{K}$, $\left(\left\lfloor \frac{k+q}{2}\right\rfloor, \left\lceil \frac{s+q}{2}\right\rceil\right) \in \hat{\Pi}^{o}\left(\left\lceil \frac{i+j}{2}\right\rceil, q\right)$ for all $(k,s) \in \Pi(i,j)$. Since $0 \leq \left\lceil \frac{q}{2}\right\rceil$, and $\left\lceil \frac{q+i+j}{2}\right\rceil \leq \left\lceil \frac{i+j}{2}\right\rceil + q$, it follows from (78) and (85) that $\hat{\Pi}^{o}\left(\left\lceil \frac{i+j}{2}\right\rceil, q\right) \subseteq \Pi\left(\left\lceil \frac{i+j}{2}\right\rceil, q\right)$ for all $i,j,q \in \mathbb{K}$, which implies $\left(\left\lfloor \frac{k+q}{2}\right\rfloor, \left\lceil \frac{s+q}{2}\right\rceil\right) \in \Pi\left(\left\lceil \frac{i+j}{2}\right\rceil, q\right)$ for all $(k,s) \in \Pi(i,j)$, and any $i,j,q \in \mathbb{K}$.

Finally, we show that given any $i,j,q \in \mathbb{K}$, $(k,s) \in \Pi(i,j)$ implies $\left(\left\lceil \frac{k+q}{2}\right\rceil, \left\lfloor \frac{s+q}{2}\right\rfloor\right) \in \Pi\left(\left\lfloor \frac{i+j}{2}\right\rfloor, q\right)$. Notice that if $i+j$ and $s+q$ are odd, then

$$\left\lceil \frac{k+q}{2}\right\rceil + \left\lfloor \frac{s+q}{2}\right\rfloor = \left\lfloor \frac{i+j}{2}\right\rfloor + q. \tag{86}$$

With (86),

$$\begin{aligned}\tilde{\Pi}(i,j,q) &= \left\{\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor + q - \left\lfloor \tfrac{s+q}{2}\right\rfloor, \left\lfloor \tfrac{s+q}{2}\right\rfloor\right) \in \mathbb{K}\times\mathbb{K} : (k,s) \in \Pi(i,j)\right\} \\ &= \left\{\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor + q - y, y\right) \in \mathbb{K}\times\mathbb{K} : y \in \left\{\left\lfloor \tfrac{q}{2}\right\rfloor, \left\lfloor \tfrac{q+1}{2}\right\rfloor, \ldots, \left\lfloor \tfrac{q+i+j}{2}\right\rfloor\right\}\right\} \\ &\equiv \tilde{\Pi}^{o}\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor, q\right). \end{aligned} \tag{87}$$

By construction, given any $i,j,q \in \mathbb{K}$, $\left(\left\lceil \frac{k+q}{2}\right\rceil, \left\lfloor \frac{s+q}{2}\right\rfloor\right) \in \tilde{\Pi}^{o}\left(\left\lfloor \frac{i+j}{2}\right\rfloor, q\right)$ for all $(k,s) \in \Pi(i,j)$. Since $0 \leq \left\lfloor \frac{q}{2}\right\rfloor$, and $\left\lfloor \frac{q+i+j}{2}\right\rfloor \leq \left\lfloor \frac{i+j}{2}\right\rfloor + q$, it follows from (79) and (87) that $\tilde{\Pi}^{o}\left(\left\lfloor \frac{i+j}{2}\right\rfloor, q\right) \subseteq \left(\left\lfloor \frac{i+j}{2}\right\rfloor, q\right)$ for all $i,j,q \in \mathbb{K}$, which implies $\left(\left\lceil \frac{k+q}{2}\right\rceil, \left\lfloor \frac{s+q}{2}\right\rfloor\right) \in \Pi\left(\left\lfloor \frac{i+j}{2}\right\rfloor, q\right)$ for all $(k,s) \in \Pi(i,j)$, and any $i,j,q \in \mathbb{K}$. ■

Proof of Proposition 2. Consider the metric space $(\boldsymbol{B}, D)$ used in the proof of Lemma 4. A function $w \in \boldsymbol{B}$ satisfies the *bilateral-trade asset-holding Equalization Property* (EP) if for all $(i,j,\tau) \in \mathbb{K}\times\mathbb{K}\times[0,T]$,

$$\begin{aligned}&\max_{(k,s)\in\Pi(i,j)} \left[w(k,\tau) + w(s,\tau) - w(i,\tau) - w(j,\tau)\right] \\ &\qquad = w\left(\left\lceil \tfrac{i+j}{2}\right\rceil, \tau\right) + w\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor, \tau\right) - w(i,\tau) - w(j,\tau). \end{aligned} \tag{EP}$$

A function $w \in \boldsymbol{B}$ satisfies the *bilateral-trade asset-holding Strict Equalization Property* (SEP) if for all $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times [0, T]$,

$$\arg \max_{(k,s) \in \Pi(i,j)} \left[w\left(k, \tau\right) + w\left(s, \tau\right) - w\left(i, \tau\right) - w\left(j, \tau\right)\right] = \Omega_{ij}^{*}, \tag{SEP}$$

where Ω_{ij}^{*} is defined in (15). Let

$$\begin{aligned}
\boldsymbol{B}'' &= \{w \in \boldsymbol{B} : w \text{ satisfies (EP)}\} \\
\boldsymbol{B}''' &= \{w \in \boldsymbol{B} : w \text{ satisfies (SEP)}\}.
\end{aligned}$$

Clearly, $\boldsymbol{B}''' \subseteq \boldsymbol{B}'' \subseteq \boldsymbol{B}$.

We first establish that $\boldsymbol{B}''$ is a closed subset of $\boldsymbol{B}$. Let $\{w_n\}_{n=0}^{\infty}$ be a sequence of functions in $\boldsymbol{B}''$, with $\lim_{n\to\infty} w_n = \bar{w}$. If $\bar{w} \notin \boldsymbol{B}''$, then there exists some $(k, s) \in \Pi(i, j)$ and $\varsigma \in \mathbb{R}$ such that

$$0 < \varsigma = \bar{w}\left(k, \tau\right) + \bar{w}\left(s, \tau\right) - \left[\bar{w}\left(\left\lceil \tfrac{i+j}{2} \right\rceil, \tau\right) + \bar{w}\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, \tau\right)\right],$$

for some $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times [0, T]$. This implies

$$\begin{aligned}
w_n\left(k, \tau\right) + w_n\left(s, \tau\right) = {} & w_n\left(\left\lceil \tfrac{i+j}{2} \right\rceil, \tau\right) + w_n\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, \tau\right) + \varsigma \\
& - \left\{\bar{w}\left(k, \tau\right) + \bar{w}\left(s, \tau\right) - \left[w_n\left(k, \tau\right) + w_n\left(s, \tau\right)\right]\right\} \\
& + \bar{w}\left(\left\lceil \tfrac{i+j}{2} \right\rceil, \tau\right) + \bar{w}\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, \tau\right) - \left[w_n\left(\left\lceil \tfrac{i+j}{2} \right\rceil, \tau\right) + w_n\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, \tau\right)\right].
\end{aligned}$$

For this particular $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times [0, T]$, for all n large enough we can ensure that

$$\left|\bar{w}\left(k, \tau\right) + \bar{w}\left(s, \tau\right) - \left[w_n\left(k, \tau\right) + w_n\left(s, \tau\right)\right]\right| < \frac{\varsigma}{4}$$

and

$$\left|\bar{w}\left(\left\lceil \tfrac{i+j}{2} \right\rceil, \tau\right) + \bar{w}\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, \tau\right) - \left[w_n\left(\left\lceil \tfrac{i+j}{2} \right\rceil, \tau\right) + w_n\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, \tau\right)\right]\right| < \frac{\varsigma}{4},$$

but then

$$0 < \varsigma/2 < w_n\left(k, \tau\right) + w_n\left(s, \tau\right) - \left[w_n\left(\left\lceil \tfrac{i+j}{2} \right\rceil, \tau\right) + w_n\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, \tau\right)\right],$$

which contradicts the fact that $w_n \in \boldsymbol{B}''$. Thus, we conclude that $\bar{w} \in \boldsymbol{B}''$, so $\boldsymbol{B}''$ is closed.

The second step is to show that the mapping $\mathcal{M}$ defined in (71) preserves property (EP), i.e., that $\mathcal{M}\left(\boldsymbol{B}''\right) \subseteq \boldsymbol{B}''$. That is, we wish to show that for any $w \in \boldsymbol{B}''$, $w' = \mathcal{M}w \in \boldsymbol{B}''$, or equivalently, that

$$w\left(\left\lceil \tfrac{i+j}{2} \right\rceil, \tau\right) + w\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, \tau\right) \geq w\left(k, \tau\right) + w\left(s, \tau\right) \text{ for all } (k, s) \in \Pi\left(i, j\right),$$

for any $(i,j,\tau) \in \mathbb{K}\times\mathbb{K}\times[0,T]$, implies that

$$w'\left(\left\lceil \tfrac{i+j}{2}\right\rceil,\tau\right)+w'\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor,\tau\right)-w'(k,\tau)-w'(s,\tau)\geq 0 \text{ for all } (k,s)\in\Pi(i,j), \tag{88}$$

for any $(i,j,\tau) \in \mathbb{K}\times\mathbb{K}\times[0,T]$. Since $w \in \boldsymbol{B}''$, using (73) (with $\theta_{kk'} = 1/2$ for all $k,k' \in \mathbb{K}$),

$$\begin{aligned}(\mathcal{M}w)(i,\tau) &= v_i(\tau)+\alpha\int_0^\tau w(i,z)\,e^{-(r+\alpha)(\tau-z)}dz \\ &\quad+\frac{\alpha}{2}\int_0^\tau\sum_{q\in\mathbb{K}} n_q(z)\left[w\left(\left\lceil \tfrac{i+q}{2}\right\rceil,z\right)+w\left(\left\lfloor \tfrac{i+q}{2}\right\rfloor,z\right)-w(i,z)-w(q,z)\right]e^{-(r+\alpha)(\tau-z)}dz,\end{aligned}$$

for all $(i,\tau)\in\mathbb{K}\times[0,T]$. For any $(i,j,\tau) \in \mathbb{K}\times\mathbb{K}\times[0,T]$ and $(k,s)\in\Pi(i,j)$, let $G(i,j,k,s,\tau)$ denote the left side of inequality (88). Then,

$$\begin{aligned}G(i,j,k,s,\tau) &= v_{\left\lceil \frac{i+j}{2}\right\rceil}(\tau)+v_{\left\lfloor \frac{i+j}{2}\right\rfloor}(\tau)-v_k(\tau)-v_s(\tau) \\ &\quad+\alpha\int_0^\tau\left[w\left(\left\lceil \tfrac{i+j}{2}\right\rceil,z\right)+w\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor,z\right)-w(k,z)-w(s,z)\right]e^{-(r+\alpha)(\tau-z)}dz \\ &\quad+\frac{\alpha}{2}\int_0^\tau\sum_{q\in\mathbb{K}} n_q(z)\left[w\left(\left\lceil \frac{\left\lceil \frac{i+j}{2}\right\rceil+q}{2}\right\rceil,z\right)+w\left(\left\lfloor \frac{\left\lceil \frac{i+j}{2}\right\rceil+q}{2}\right\rfloor,z\right)\right. \\ &\qquad\qquad\left.-w\left(\left\lceil \tfrac{i+j}{2}\right\rceil,z\right)-w(q,z)\right]e^{-(r+\alpha)(\tau-z)}dz \\ &\quad+\frac{\alpha}{2}\int_0^\tau\sum_{q\in\mathbb{K}} n_q(z)\left[w\left(\left\lceil \frac{\left\lfloor \frac{i+j}{2}\right\rfloor+q}{2}\right\rceil,z\right)+w\left(\left\lfloor \frac{\left\lfloor \frac{i+j}{2}\right\rfloor+q}{2}\right\rfloor,z\right)\right. \\ &\qquad\qquad\left.-w\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor,z\right)-w(q,z)\right]e^{-(r+\alpha)(\tau-z)}dz \\ &\quad-\frac{\alpha}{2}\int_0^\tau\sum_{q\in\mathbb{K}} n_q(z)\left[w\left(\left\lceil \tfrac{k+q}{2}\right\rceil,z\right)+w\left(\left\lfloor \tfrac{k+q}{2}\right\rfloor,z\right)-w(k,z)-w(q,z)\right]e^{-(r+\alpha)(\tau-z)}dz \\ &\quad-\frac{\alpha}{2}\int_0^\tau\sum_{q\in\mathbb{K}} n_q(z)\left[w\left(\left\lceil \tfrac{s+q}{2}\right\rceil,z\right)+w\left(\left\lfloor \tfrac{s+q}{2}\right\rfloor,z\right)-w(s,z)-w(q,z)\right]e^{-(r+\alpha)(\tau-z)}dz.\end{aligned}$$

With (11) and after deleting redundant terms, this expression can be rearranged to yield

$$
\begin{aligned}
G(i,j,k,s,\tau) = {} & \frac{1-e^{-(r+\alpha)\tau}}{r+\alpha}\left(u_{\lceil\frac{i+j}{2}\rceil}+u_{\lfloor\frac{i+j}{2}\rfloor}-u_k-u_s\right) \\
& + e^{-(r+\alpha)\tau}\left[U_{\lceil\frac{i+j}{2}\rceil}+U_{\lfloor\frac{i+j}{2}\rfloor}-U_k-U_s\right] \\
& + \frac{\alpha}{2}\int_0^{\tau}\left[w\left(\left\lceil\tfrac{i+j}{2}\right\rceil,z\right)+w\left(\left\lfloor\tfrac{i+j}{2}\right\rfloor,z\right)-w(k,z)-w(s,z)\right]e^{-(r+\alpha)(\tau-z)}dz \\
& + \frac{\alpha}{2}\int_0^{\tau}\sum_{q\in\mathbb{K}}n_q(z)\Bigg\{w\left(\left\lceil\frac{\left\lfloor\frac{i+j}{2}\right\rfloor+q}{2}\right\rceil,z\right)+w\left(\left\lfloor\frac{\left\lceil\frac{i+j}{2}\right\rceil+q}{2}\right\rfloor,z\right) \\
& \qquad - w\left(\left\lceil\tfrac{k+q}{2}\right\rceil,z\right)-w\left(\left\lfloor\tfrac{s+q}{2}\right\rfloor,z\right) \\
& \qquad - w\left(\left\lfloor\tfrac{k+q}{2}\right\rfloor,z\right)-w\left(\left\lceil\tfrac{s+q}{2}\right\rceil,z\right) \\
& \qquad + w\left(\left\lceil\frac{\left\lceil\frac{i+j}{2}\right\rceil+q}{2}\right\rceil,z\right)+w\left(\left\lfloor\frac{\left\lfloor\frac{i+j}{2}\right\rfloor+q}{2}\right\rfloor,z\right)\Bigg\}e^{-(r+\alpha)(\tau-z)}dz.
\end{aligned}
$$

What needs to be shown is that $w\in\boldsymbol{B}''$ implies that for any $(i,j,\tau)\in\mathbb{K}\times\mathbb{K}\times[0,T]$, $G(i,j,k,s,\tau)\geq 0$ for all $(k,s)\in\Pi(i,j)$. The fact that $w\in\boldsymbol{B}''$ immediately implies that the first integral in the last expression is nonnegative. By Lemma 5, $w\in\boldsymbol{B}''$ also implies that the second integral in the last expression is nonnegative. Together with Assumption A, these observations imply

$$
\begin{aligned}
0 < {} & \frac{1-e^{-(r+\alpha)\tau}}{r+\alpha}\left(u_{\lceil\frac{i+j}{2}\rceil}+u_{\lfloor\frac{i+j}{2}\rfloor}-u_k-u_s\right)+e^{-(r+\alpha)\tau}\left[U_{\lceil\frac{i+j}{2}\rceil}+U_{\lfloor\frac{i+j}{2}\rfloor}-U_k-U_s\right] \\
& \leq G(i,j,k,s,\tau),
\end{aligned} \tag{89}
$$

so we conclude that $\mathcal{M}(\boldsymbol{B}'')\subseteq\boldsymbol{B}'''\subseteq\boldsymbol{B}''$.

The third step is to show that (14) is the equilibrium distribution of trading probabilities. From Lemma 4, we know that $\mathcal{M}$ is a contraction mapping on the complete metric space $(\boldsymbol{B},D)$, so it has a unique fixed point $w^*(k,\tau)\equiv V_k(\tau)\in\boldsymbol{B}$. In addition, we have now established that $\boldsymbol{B}''$ is a closed subset of $\boldsymbol{B}$, and that $\mathcal{M}(\boldsymbol{B}'')\subseteq\boldsymbol{B}'''\subseteq\boldsymbol{B}''$. Therefore, by Corollary 1 in Stokey and Lucas (1989, p. 52) we conclude that $V_k(\tau)\in\boldsymbol{B}'''$. This implies that the set $\Omega_{ij}[\boldsymbol{V}(\tau)]$ defined in (13) reduces to Ω^*_{ij} for all $(i,j,\tau)\in\mathbb{K}\times\mathbb{K}\times[0,T]$, and consequently, that (12) reduces to (14) for all $(i,j,\tau)\in\mathbb{K}\times\mathbb{K}\times[0,T]$. This establishes part (*ii*) in the statement of the proposition.

We can now show that the paths $\boldsymbol{n}(\tau)$ and $\boldsymbol{V}(\tau)$ are uniquely determined. Since (by Lemma 4) the fixed point $V_k(\tau)\in\boldsymbol{B}'''$ is unique given any path for the distribution of reserve balances, $\boldsymbol{n}(\tau)$, all that has to be shown is that given the initial condition $\{n_k(T)\}_{k\in\mathbb{K}}$, and given that the path $\boldsymbol{\phi}(\tau)$ satisfies (14), the system of first-order ordinary differential equations, $\dot{\boldsymbol{n}}(\tau) = f[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)]$, has a unique solution. But since f is continuously differentiable, this follows from Propositions 6.3 and 7.6 in Amann (1990). This establishes part (i) in the statement of the proposition.

By Proposition 1, the equilibrium value function, $\boldsymbol{V}$, satisfies (10). Notice that (10) implies (17). Differentiate both sides of (10) with respect to τ, and rearrange terms to obtain

$$\dot{V}_i(\tau)+rV_i(\tau) = \dot{v}_i(\tau)+(r+\alpha)v_i(\tau)+\frac{\alpha}{2}\sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} n_j(\tau)\phi_{ij}^{ks}(\tau)\left[V_k(\tau)+V_s(\tau)-V_i(\tau)-V_j(\tau)\right],$$

which together with the fact that $\dot{v}_i(\tau) = u_i - (r+\alpha)v_i(\tau)$ implies (16). This establishes part (iii) in the statement of the proposition.

Suppose that at time $T-\tau$, a bank with balance j extends a loan of size b to a bank with balance i. Then (5) implies that the present discounted value of the repayment from the latter to the former is

$$\frac{1}{2}\left[V_{i+b}(\tau)-V_i(\tau)\right]+\frac{1}{2}\left[V_j(\tau)-V_{j-b}(\tau)\right],$$

which reduces to the right side of (18) if the loan size is $b = j-s = k-i$, as specified by part (iv) in the statement of the proposition. ■

Corollary 1 *Assume that $\{U_k\}_{k\in\mathbb{K}}$ satisfies the discrete midpoint concavity property and $\{u_k\}_{k\in\mathbb{K}}$ satisfies the discrete midpoint strict concavity property. An equilibrium exists, and the equilibrium paths for the distribution of reserve balances, $\boldsymbol{n}(\tau)$, and maximum attainable payoffs, $\boldsymbol{V}(\tau)$, are uniquely determined, and identical to those in Proposition 2. The equilibrium distribution of trading probabilities is*

$$\phi_{ij}^{ks}(\tau) = \begin{cases} \tilde{\phi}_{ij}^{ks}(\tau) & \text{if } (k,s)\in\Omega_{ij}^{*}(\tau) \\ 0 & \text{if } (k,s)\notin\Omega_{ij}^{*}(\tau) \end{cases} \tag{90}$$

for all $i,j,k,s\in\mathbb{K}$ and $\tau\in[0,T]$, with $\tilde{\phi}_{ij}^{ks}(\tau)\geq 0$ and $\sum_{(k,s)\in\Omega_{ij}^{*}(\tau)}\tilde{\phi}_{ij}^{ks}(\tau) = 1$, and where $\Omega_{ij}^{*}(\tau) = \Omega_{ij}^{*}$, with Ω_{ij}^{*} given by (15) for all $\tau\in(0,T]$, and $\Omega_{ij}^{*}(0) = \Omega_{ij}^{*}\cup\Omega_{ij}^{0}$, where

$$\Omega_{ij}^{0} = \left\{(k,s)\in\Pi(i,j) : U_k+U_s = U_{\lceil\frac{i+j}{2}\rceil}+U_{\lfloor\frac{i+j}{2}\rfloor}\right\}.$$

Proof of Corollary 1. The proof proceeds exactly as the proof of Proposition 2 up to (89). Notice that under Assumption A, (89) holds for all $\tau \in [0, T]$. Instead, under the assumption that $\{U_k\}_{k\in\mathbb{K}}$ satisfies *discrete midpoint concavity* and $\{u_k\}_{k\in\mathbb{K}}$ satisfies *discrete midpoint strict concavity*, the inequality in (89) holds as a strict inequality for all $\tau \in (0, T]$, but only as a weak inequality for $\tau = 0$. As before, the unique fixed point $V_k(\tau) \in \boldsymbol{B}''$, but now $V_k(\tau) \notin \boldsymbol{B}'''$, since $V_k(\tau)$ satisfies (SEP) for all $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times (0, T]$, rather than for all $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times [0, T]$. However, it is clear from (89) that in this case $\mathcal{M}V_k(\tau) = V_k(\tau) \in \boldsymbol{B}_0'''$, where $\boldsymbol{B}_0'''$ is the subset of elements of $\boldsymbol{B}$ that satisfy (SEP) for all $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times (0, T]$. This implies that the set $\Omega_{ij}[\boldsymbol{V}(\tau)]$ defined in (13) now reduces to the set $\Omega^*_{ij}(\tau)$ defined in the statement of the corollary for all $\tau \in [0, T]$, and consequently, that (12) reduces to (90) for all $\tau \in [0, T]$. Notice that despite the potential multiplicity of optimal post-trade portfolios in bilateral meetings at $\tau = 0$ (which is the only difference between this case and the one treated in Proposition 2), as can be seen from (73) and (90), the mapping $\mathcal{M}$ is unaffected by this multiplicity, and hence so is its fixed point. Therefore, (by Lemma 2) the fixed point $V_k(\tau) \in \boldsymbol{B}_0'''$ is unique given any path for $\boldsymbol{n}(\tau)$. Finally, if we cast (8) in integral equation form,

$$n_k(\tau) = n_k(T) - \alpha \int_\tau^T \sum_{i\in\mathbb{K}} \sum_{j\in\mathbb{K}} \sum_{s\in\mathbb{K}} n_i(z) \left[n_k(z)\, \phi^{sj}_{ki}(z) - n_j(z)\, \phi^{ks}_{ij}(z) \right] dz \tag{91}$$

for all $k \in \mathbb{K}$, then it becomes clear that for all $k \in \mathbb{K}$ and all $\tau \in [0, T]$, $n_k(\tau)$ is independent of $\phi^{ks}_{ij}(0)$ (changing the integral at a single point leaves the right side of (91) unaffected). Therefore, by the same arguments used in the final step of the proof of Proposition 2, there exists a unique $\boldsymbol{n}(\tau)$ that solves the system (91), and it is the same solution that obtains under Assumption A. ■

Proof of Proposition 3. The planner's current-value Hamiltonian can be written as

$$L = \sum_{k\in\mathbb{K}} m_k(t)\, u_k + \alpha \sum_{i\in\mathbb{K}} \sum_{j\in\mathbb{K}} \sum_{k\in\mathbb{K}} \sum_{s\in\mathbb{K}} m_i(t)\, m_j(t)\, \chi^{ks}_{ij}(t) \left[\mu_k(t) - \mu_i(t)\right],$$

where $\boldsymbol{\mu}(t) = \{\mu_k(t)\}_{k\in\mathbb{K}}$ is the vector of co-states associated with the law of motion for the distribution of banks across reserve balances. In an optimum, the co-states and the controls must satisfy $\frac{\partial L}{\partial m_i(t)} = r\mu_i(t) - \dot{\mu}_i(t)$, and

$$\chi_{ij}^{ks}(t) \begin{cases} = 1 & \text{if } \left.\frac{\partial L}{\partial \chi_{ij}^{ks}(t)}\right|_{\chi_{ji}^{sk}(t)=\chi_{ij}^{ks}(t)} > 0 \\ \in [0,1] & \text{if } \left.\frac{\partial L}{\partial \chi_{ij}^{ks}(t)}\right|_{\chi_{ji}^{sk}(t)=\chi_{ij}^{ks}(t)} = 0 \\ 0 & \text{if } \left.\frac{\partial L}{\partial \chi_{ij}^{ks}(t)}\right|_{\chi_{ji}^{sk}(t)=\chi_{ij}^{ks}(t)} < 0. \end{cases}$$

Notice that

$$\left.\frac{\partial L}{\partial \chi_{ij}^{ks}(t)}\right|_{\chi_{ji}^{sk}(t)=\chi_{ij}^{ks}(t)} = \alpha m_i(t)\, m_j(t)\left[\mu_k(t) + \mu_s(t) - \mu_i(t) - \mu_j(t)\right],$$

and that given $\chi_{ji}^{sk}(t) = \chi_{ij}^{ks}(t)$,

$$\frac{\partial L}{\partial m_i} = u_i + \alpha \sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} m_j(t)\, \chi_{ij}^{ks}(t)\left[\mu_k(t) + \mu_s(t) - \mu_i(t) - \mu_j(t)\right].$$

Thus the necessary conditions for optimality are:

$$\chi_{ij}^{ks}(t) = \begin{cases} \tilde{\chi}_{ij}^{ks}(t) & \text{if } (k,s) \in \Omega_{ij}[\boldsymbol{\mu}(t)] \\ 0 & \text{if } (k,s) \notin \Omega_{ij}[\boldsymbol{\mu}(t)], \end{cases} \tag{92}$$

for all $i,j,k,s \in \mathbb{K}$ and all $t \in [0,T]$, where $\tilde{\chi}_{ij}^{ks}(t) \geq 0$ and $\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} \tilde{\chi}_{ij}^{ks}(t) = 1$, the Euler equations,

$$r\mu_i(t) - \dot{\mu}_i(t) = u_i + \alpha \sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} m_j(t)\, \chi_{ij}^{ks}(t)\left[\mu_k(t) + \mu_s(t) - \mu_i(t) - \mu_j(t)\right] \tag{93}$$

for all $i \in \mathbb{K}$, with the path for $\boldsymbol{m}(t)$ given by (19), and

$$\mu_i(T) = U_i \quad \text{for all } i \in \mathbb{K}. \tag{94}$$

In summary, the necessary conditions are (19), (92), (93), and (94). Next, we use the fact that $\tau \equiv T - t$ to define $m_k(t) = m_k(T-\tau) \equiv n_k(\tau)$, $\chi_{ij}^{ks}(t) = \chi_{ij}^{ks}(T-\tau) \equiv \psi_{ij}^{ks}(\tau)$, and $\mu_i(t) = \mu_i(T-\tau) \equiv \lambda_i(\tau)$. With these new variables, (93) leads to (20), (19) leads to $\dot{\boldsymbol{n}}(\tau) = f[\boldsymbol{n}(\tau), \boldsymbol{\psi}(\tau)]$, (94) leads to (21), and (92) leads to (22). ■

Proof of Proposition 4. The function $\boldsymbol{\lambda} \equiv [\lambda(\tau)]_{\tau\in[0,T]}$ satisfies (20) and (21) if and only if it satisfies

$$\lambda_i(\tau) = v_i(\tau) + \alpha \int_0^\tau \lambda_i(z)\, e^{-(r+\alpha)(\tau-z)} dz$$
$$+ \alpha \int_0^\tau \sum_{j\in\mathbb{K}} \sum_{k\in\mathbb{K}} \sum_{s\in\mathbb{K}} n_j(z)\, \psi_{ij}^{ks}(z) \left[\lambda_k(z) + \lambda_s(z) - \lambda_i(z) - \lambda_j(z)\right] e^{-(r+\alpha)(\tau-z)} dz.$$

The right side of this functional equation defines a mapping $\mathcal{P} : \boldsymbol{B} \to \boldsymbol{B}$, that is for any $w \in \boldsymbol{B}$,

$$(\mathcal{P}w)(i,\tau) = v_i(\tau) + \alpha \int_0^\tau w(i,z)\, e^{-(r+\alpha)(\tau-z)} dz$$
$$+ \alpha \int_0^\tau \sum_{j\in\mathbb{K}} \sum_{k\in\mathbb{K}} \sum_{s\in\mathbb{K}} n_j(z)\, \psi_{ij}^{ks}(z) \left[w(k,z) + w(s,z) - w(i,z) - w(j,z)\right] e^{-(r+\alpha)(\tau-z)} dz,$$

for all $(i,\tau) \in \mathbb{K} \times [0,T]$. Hence a function $\boldsymbol{\lambda}$ satisfies (20) and (21) if and only if it satisfies $\boldsymbol{\lambda} = \mathcal{P}\boldsymbol{\lambda}$. Rewrite the mapping $\mathcal{P}$ as

$$(\mathcal{P}w)(i,\tau) = v_i(\tau) + \alpha \int_0^\tau w(i,z)\, e^{-(r+\alpha)(\tau-z)} dz \tag{95}$$
$$+ \alpha \int_0^\tau \sum_{j\in\mathbb{K}} n_j(z) \max_{(k,s)\in\Pi(i,j)} \left[w(k,z) + w(s,z) - w(i,z) - w(j,z)\right] e^{-(r+\alpha)(\tau-z)} dz,$$

and for any $w, w' \in \boldsymbol{B}$, define the metric $D^* : \boldsymbol{B} \times \boldsymbol{B} \to \mathbb{R}$ by

$$D^*(w,w') = \sup_{(i,\tau)\in\mathbb{K}\times[0,T]} \left[e^{-\kappa\tau} \left|w(i,\tau) - w'(i,\tau)\right|\right],$$

where $\kappa \in \mathbb{R}$ satisfies

$$\max\{0, 5\alpha - r\} < \kappa < \infty. \tag{96}$$

The metric space $(\boldsymbol{B}, D^*)$ is complete (by the same argument used to argue that $(\boldsymbol{B}, D)$ is complete, in the proof of Lemma 4). For any $w, w' \in \boldsymbol{B}$, and any $(i,\tau) \in \mathbb{K} \times [0,T]$, the same steps that led to (77), now lead to

$$D^*(\mathcal{P}w, \mathcal{P}w') \leq \frac{5\alpha}{r+\alpha+\kappa} D^*(w,w'), \quad \text{for all } w, w' \in \boldsymbol{B}.$$

Notice that (96) implies $\frac{5\alpha}{r+\alpha+\kappa} \in (0,1)$, so $\mathcal{P}$ is a contraction mapping on the complete metric space $(\boldsymbol{B}, D^*)$. By the Contraction Mapping Theorem (Theorem 3.2 in Stokey and Lucas, 1989), for any given path $\boldsymbol{n}(\tau)$, there exists a unique $\boldsymbol{\lambda} \in \boldsymbol{B}$ that satisfies $\boldsymbol{\lambda} = \mathcal{P}\boldsymbol{\lambda}$.

Consider the sets $\boldsymbol{B}''$ and $\boldsymbol{B}'''$ defined in the proof of Proposition 2. By following the same steps as in the first part of that proof, it can be shown that $\boldsymbol{B}''$ is closed under D^*. Next we show that the mapping $\mathcal{P}$ defined in (95) preserves property (EP), i.e., that $\mathcal{P}(\boldsymbol{B}'') \subseteq \boldsymbol{B}''$. That is, we wish to show that for any $w \in \boldsymbol{B}''$, $w' = \mathcal{P}w \in \boldsymbol{B}''$, or equivalently, that

$$w\left(\left\lceil \tfrac{i+j}{2} \right\rceil, \tau\right) + w\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, \tau\right) \geq w(k,\tau) + w(s,\tau) \text{ for all } (k,s) \in \Pi(i,j),$$

for any $(i,j,\tau) \in \mathbb{K} \times \mathbb{K} \times [0,T]$, implies that

$$w'\left(\left\lceil \tfrac{i+j}{2} \right\rceil, \tau\right) + w'\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, \tau\right) - w'(k,\tau) - w'(s,\tau) \geq 0 \text{ for all } (k,s) \in \Pi(i,j), \tag{97}$$

for any $(i,j,\tau) \in \mathbb{K} \times \mathbb{K} \times [0,T]$. Since $w \in \boldsymbol{B}''$,

$$\begin{aligned}(\mathcal{P}w)(i,\tau) &= v_i(\tau) + \alpha \int_0^\tau w(i,z)\, e^{-(r+\alpha)(\tau-z)} dz \\ &\quad + \alpha \int_0^\tau \sum_{q\in\mathbb{K}} n_q(z) \left[w\left(\left\lceil \tfrac{i+q}{2} \right\rceil, z\right) + w\left(\left\lfloor \tfrac{i+q}{2} \right\rfloor, z\right) - w(i,z) - w(q,z) \right] e^{-(r+\alpha)(\tau-z)} dz,\end{aligned}$$

for any $(i,\tau) \in \mathbb{K} \times [0,T]$. For any $(i,j,\tau) \in \mathbb{K}\times\mathbb{K}\times[0,T]$ and $(k,s) \in \Pi(i,j)$, let $G'(i,j,k,s,\tau)$ denote the left side of inequality (97). Then,

$$\begin{aligned}G'(i,j,k,s,\tau) &= \frac{1-e^{-(r+\alpha)\tau}}{r+\alpha} \left(u_{\lceil \frac{i+j}{2} \rceil} + u_{\lfloor \frac{i+j}{2} \rfloor} - u_k - u_s \right) \\ &\quad + e^{-(r+\alpha)\tau} \left[U_{\lceil \frac{i+j}{2} \rceil} + U_{\lfloor \frac{i+j}{2} \rfloor} - U_k - U_s \right] \\ &\quad + \alpha \int_0^\tau \sum_{q\in\mathbb{K}} n_q(z) \Bigg\{ w\left(\left\lceil \frac{\left\lceil \frac{i+j}{2} \right\rceil + q}{2} \right\rceil, z \right) + w\left(\left\lfloor \frac{\left\lceil \frac{i+j}{2} \right\rceil + q}{2} \right\rfloor, z \right) \\ &\qquad - w\left(\left\lceil \tfrac{k+q}{2} \right\rceil, z\right) - w\left(\left\lfloor \tfrac{s+q}{2} \right\rfloor, z\right) \\ &\qquad - w\left(\left\lfloor \tfrac{k+q}{2} \right\rfloor, z\right) - w\left(\left\lceil \tfrac{s+q}{2} \right\rceil, z\right) \\ &\qquad + w\left(\left\lceil \frac{\left\lfloor \frac{i+j}{2} \right\rfloor + q}{2} \right\rceil, z \right) + w\left(\left\lfloor \frac{\left\lfloor \frac{i+j}{2} \right\rfloor + q}{2} \right\rfloor, z \right) \Bigg\} e^{-(r+\alpha)(\tau-z)} dz.\end{aligned}$$

What needs to be shown is that $w \in \boldsymbol{B}''$ implies that for any $(i,j,\tau) \in \mathbb{K} \times \mathbb{K} \times [0,T]$, $G'(i,j,k,s,\tau) \geq 0$ for all $(k,s) \in \Pi(i,j)$. By Lemma 5, $w \in \boldsymbol{B}''$ implies that the integral in the last expression is nonnegative. Together with Assumption A, this implies

$$\begin{aligned}0 &< \frac{1-e^{-(r+\alpha)\tau}}{r+\alpha} \left(u_{\lceil \frac{i+j}{2} \rceil} + u_{\lfloor \frac{i+j}{2} \rfloor} - u_k - u_s \right) + e^{-(r+\alpha)\tau} \left[U_{\lceil \frac{i+j}{2} \rceil} + U_{\lfloor \frac{i+j}{2} \rfloor} - U_k - U_s \right] \\ &\leq G'(i,j,k,s,\tau),\end{aligned}$$

so $\mathcal{M}(\boldsymbol{B}'') \subseteq \boldsymbol{B}''' \subseteq \boldsymbol{B}''$.

At this point, we have shown that $\mathcal{P}$ is a contraction mapping on the complete metric space $(\boldsymbol{B}, D^*)$, so it has a unique fixed point $\boldsymbol{\lambda} \in \boldsymbol{B}$. We have also established that $\boldsymbol{B}''$ is a closed subset of $\boldsymbol{B}$, and that $\mathcal{M}(\boldsymbol{B}'') \subseteq \boldsymbol{B}''' \subseteq \boldsymbol{B}''$. Therefore, by Corollary 1 in Stokey and Lucas (1989, p. 52), $\boldsymbol{\lambda} = \mathcal{P}\boldsymbol{\lambda} \in \boldsymbol{B}'''$, that is, the unique fixed point $\boldsymbol{\lambda}$ satisfies (SEP). This implies that the set $\Omega_{ij}[\boldsymbol{\lambda}(\tau)]$ in (22) reduces to Ω^*_{ij} (as defined in (15)) for all $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times [0, T]$, and consequently, that (22) reduces to (23). This establishes part (i) in the statement of the proposition.

Given the initial condition $\{n_k(T)\}_{k \in \mathbb{K}}$, and given that the path $\boldsymbol{\psi}(\tau)$ satisfies (23), the system of first-order ordinary differential equations, $\dot{\boldsymbol{n}}(\tau) = f[\boldsymbol{n}(\tau), \boldsymbol{\psi}(\tau)]$ is identical to the one in part (iii) of Proposition 2, and therefore also has a unique solution. Given the resulting path $\boldsymbol{n}(\tau)$, according to Proposition 3, the path for the vector of co-states must satisfy the necessary condition $\boldsymbol{\lambda} = \mathcal{P}\boldsymbol{\lambda}$, or equivalently, (20) and (21), which establishes part (ii) in the statement of the proposition. ■

Proof of Proposition 6. The (i, j) element of the transition matrix $\boldsymbol{\Pi}^{(N)}(\boldsymbol{t}^{(N)})$ as defined in (26), denoted $\pi^{(N)}_{ij}(\boldsymbol{t}^{(N)})$, is the probability that a bank with balance i at time t_0 has balance j at time t, conditional on a realization of the number of trading opportunities, $N \in \{0, 1, 2, \ldots\}$, and a realization of the corresponding trading times, $\boldsymbol{t}^{(N)} \in \mathbb{T}^{(N)}$. Let $n_{[t_0,t]}$ denote the random number of trading opportunities that a bank encounters during the time interval $[t_0, t]$. Since trading opportunities follow a Poisson process with intensity α,

$$\Pr\left(n_{[t_0,t]} = N\right) = \frac{[\alpha(t - t_0)]^N e^{-\alpha(t-t_0)}}{N!}. \tag{98}$$

Let $h(\boldsymbol{t}^{(N)} | n_{[t_0,t]} = N)$ denote the probability density of $\boldsymbol{t}^{(N)} \in \mathbb{T}^{(N)}$ conditional on N trading opportunities in $[t_0, t]$, and notice that $h(\boldsymbol{t}^{(N)} | n_{[t_0,t]} = N) = h'((T_1, \ldots, T_N) | n_{[t_0,t]} = N)$, where $h'((T_1, \ldots, T_N) | n_{[t_0,t]} = N)$ is the conditional probability density for the N interarrival times, $T_n \equiv t_n - t_{n-1}$, for $n = 1, \ldots, N$. Then, by the definition of conditional density,

$$\begin{aligned} h'(T_1, \ldots, T_N | n_{[t_0,t]} = N) &= \frac{\Pr\left(n_{[t_0,t]} = N \mid T_1, \ldots, T_N\right) \prod_{n=1}^{N} \left(\alpha e^{-\alpha(T_n - T_{n-1})}\right)}{\Pr\left(n_{[t_0,t]} = N\right)} \\ &= \frac{\Pr\left(T_{N+1} > t - t_N\right) \alpha^N e^{-\alpha(t_N - t_0)}}{\Pr\left(n_{[t_0,t]} = N\right)} \\ &= \frac{N!}{(t - t_0)^N}. \end{aligned} \tag{99}$$

Notice that the volume of $[t_0, t]^N$ is $(t - t_0)^N$, but the volume of $\mathbb{T}^{(N)}$ is $(t - t_0)^N / N!$, since for all possible draws of N-vectors from $[t_0, t]^N$, the ascending ordering $\boldsymbol{t}^{(N)} = (t_1, t_2, ..., t_N)$ is only one of $N!$ possible orderings. Thus by (99), the conditional probability distribution for the trading times $\boldsymbol{t}^{(N)}$ given $n_{[t_0,t]} = N$, is uniform on $\mathbb{T}^{(N)}$. For a bank holding any balance in $\mathbb{K}$ at time t_0, we can now use (98) and (99) to write the unconditional transition probabilities to any balance at time t, as

$$P(t|t_0) = \sum_{N=0}^{\infty} \frac{[\alpha(t - t_0)]^N e^{-\alpha(t-t_0)}}{N!} \int_{\mathbb{T}^{(N)}} \boldsymbol{\Pi}^{(N)}(\boldsymbol{t}^{(N)}) \frac{N!}{(t - t_0)^N} d\boldsymbol{t}^{(N)},$$

which simplifies to (27). ■

Proof of Proposition 7. Given an initial balance $a(t_0) = k_0 \in \mathbb{K}$, and given the realization of trading times $\boldsymbol{t}^{(N)} \in [t_0, t]^N$, the probability distribution over the post-trade balances at these trading times, i.e., over vectors $(a(t_1), \ldots, a(t_N)) = \boldsymbol{k}^{(N)} \in \mathbb{K}^N$, is given by (25). Hence,

$$\mathbb{E}\left[O^j(k_0, \boldsymbol{k}^{(N)}) \mid k_0, \boldsymbol{t}^{(N)}\right] = \sum_{\boldsymbol{k}^{(N)} \in \mathbb{K}^N} \left(\prod_{n=1}^{N} \pi_{k_{n-1}k_n}(t_n)\right) O^j(k_0, \boldsymbol{k}^{(N)}) \equiv \tilde{O}^j(k_0, \boldsymbol{t}^{(N)})$$

is the expected cumulative volume of funds purchased (for $j = p$, or sold, for $j = s$) during $[t_0, t]$ by banks that hold balance k_0 at t_0 and have N trading opportunities, at times $\boldsymbol{t}^{(N)} = (t_1, \ldots, t_N)$. By (98) and (99), the expected cumulative volume of funds purchased (for $j = p$, or sold, for $j = s$) during $[t_0, t]$ by banks that hold balance k_0 at t_0 is

$$\mathbb{E}\left[\tilde{O}^j(k_0, \boldsymbol{t}^{(N)}) \mid k_0\right] = \sum_{N=0}^{\infty} \frac{[\alpha(t - t_0)]^N e^{-\alpha(t-t_0)}}{N!} \int_{\mathbb{T}^{(N)}} \tilde{O}^j(k_0, \boldsymbol{t}^{(N)}) \frac{N!}{(t - t_0)^N} d\boldsymbol{t}^{(N)}.$$

Since the density of banks with balance k_0 at time t_0 is $m_{k_0}(t_0)$,

$$\mathbb{E}\left[\mathbb{E}\left[\tilde{O}^j(k_0, \boldsymbol{t}^{(N)}) \mid k_0\right]\right] = \sum_{k_0 \in \mathbb{K}} m_{k_0}(t_0) \sum_{N=0}^{\infty} \frac{[\alpha(t - t_0)]^N e^{-\alpha(t-t_0)}}{N!} \int_{\mathbb{T}^{(N)}} \tilde{O}^j(k_0, \boldsymbol{t}^{(N)}) \frac{N!}{(t - t_0)^N} d\boldsymbol{t}^{(N)}$$

is the expected cumulative volume of funds purchased (for $j = p$, or sold, for $j = s$) by all banks during $[t_0, t]$, which after simplification reduces to $\bar{O}^j(t|t_0)$ in (30). An identical calculation but replacing $\bar{O}^j(k_0, \boldsymbol{k}^{(N)})$ with $X(k_0, \boldsymbol{k}^{(N)})$ leads to (32). Finally, from (28) and (29) it is easy to check that $I(k_0, \boldsymbol{k}^{(N)}) = \frac{1}{2} X(k_0, \boldsymbol{k}^{(N)})$ for all $(k_0, \boldsymbol{k}^{(N)}) \in \mathbb{K}^{N+1}$, which implies (31). ■

Proof of Proposition 8. The right side of (40) can be integrated to obtain the closed-form expression for $S(\tau)$, where

$$\xi(\tau) = [n_2(T) - n_0(T)] \int_0^\tau \frac{e^{\{r+\theta\alpha[n_2(T)-n_0(T)]\}z} n_0(T)}{n_2(T)\, e^{\alpha[n_2(T)-n_0(T)]T} - e^{\alpha[n_2(T)-n_0(T)]z} n_0(T)} dz$$

can be integrated to yield the expression reported in the statement of the proposition. Conditions (36), (37) and (38) imply

$$\dot{V}_1(\tau) - \dot{V}_0(\tau) + r[V_1(\tau) - V_0(\tau)] = u_1 - u_0 - \theta\alpha n_2(\tau) S(\tau),$$

a differential equation in $V_1(\tau) - V_0(\tau)$, with boundary condition $V_1(0) - V_0(0) = U_1 - U_0$. The solution to this differential equation is

$$V_1(\tau) - V_0(\tau) = e^{-r\tau}(U_1 - U_0) + \int_0^\tau [u_1 - u_0 - \theta\alpha n_2(z) S(z)] e^{-r(\tau - z)} dz. \tag{100}$$

With (35) and the closed-form expression for $S(\tau)$, the integral on the right side of (100) can be calculated explicitly to yield

$$\left(1 - e^{-r\tau}\right)\frac{u_1 - u_0}{r} - e^{-r\tau}\frac{e^{\alpha[n_2(T)-n_0(T)]T} n_2(T)}{n_0(T)} \theta\zeta[\tau, \bar{u}, S(0)],$$

with

$$\zeta[\tau, \bar{u}, S(0)] = \sum_{k=1}^{\infty} \frac{\left[\frac{n_2(T)}{n_0(T)}\right]^{k-1}}{\frac{r}{\alpha[n_0(T)-n_2(T)]} + k - \theta} \frac{\frac{e^{\{r+\alpha k[n_0(T)-n_2(T)]\}\tau} - 1}{r+\alpha k[n_0(T)-n_2(T)]} - \frac{e^{\alpha\theta[n_0(T)-n_2(T)]\tau} - 1}{\alpha\theta[n_0(T)-n_2(T)]}}{e^{\alpha(k-1)[n_0(T)-n_2(T)]T}} \bar{u}$$
$$+ \frac{\left[e^{\alpha\theta[n_0(T)-n_2(T)]\tau} - 1\right] n_0(T)}{\theta\left[n_0(T) - e^{-\alpha[n_0(T)-n_2(T)]T} n_2(T)\right]} S(0)$$

if $n_2(T) < n_0(T)$,

$$\zeta[\tau, \bar{u}, S(0)] = e^{r\left[\frac{1}{\alpha n_0(T)} + T\right]} \sum_{k=0}^{\infty} \frac{(-r)^k}{k!k} \left\{ \left[\frac{1}{\alpha n_0(T)} + T\right]^k \tau - \frac{\left\{\left[\frac{1}{\alpha n_0(T)} + T\right]^{k+1} - \left[\frac{1}{\alpha n_0(T)} + T - \tau\right]^{k+1}\right\}}{k+1} \right\} \bar{u}$$
$$+ \frac{\tau}{\frac{1}{\alpha n_0(T)} + T} S(0)$$

if $n_2(T) = n_0(T)$, and

$$\zeta[\tau, \bar{u}, S(0)] = \sum_{k=0}^{\infty} \frac{\left[\frac{n_0(T)}{n_2(T)}\right]^{k+1}}{\frac{r}{\alpha[n_2(T)-n_0(T)]} + \theta + k} \frac{\frac{e^{\{r+\alpha k[n_2(T)-n_0(T)]\}\tau} - 1}{r+\alpha k[n_2(T)-n_0(T)]} - \frac{1 - e^{-\alpha\theta[n_2(T)-n_0(T)]\tau}}{\alpha\theta[n_2(T)-n_0(T)]}}{e^{\alpha(k+1)[n_2(T)-n_0(T)]T}} \bar{u}$$
$$+ \frac{\left[1 - e^{-\alpha\theta[n_2(T)-n_0(T)]\tau}\right] n_0(T)}{\theta\left[e^{\alpha[n_2(T)-n_0(T)]T} n_2(T) - n_0(T)\right]} S(0)$$

if $n_0(T) < n_2(T)$. ■

Proof of Proposition 9. From (40), since $\bar{u} \geq 0$ and $S(0) > 0$, we have $S(\tau) > 0$ for all $\tau \in [0, T]$.

(i) Differentiate (40) to obtain

$$\frac{\partial S(\tau)}{\partial r} = -\left[\left(\int_0^\tau (\tau - z)\, e^{-\left[\bar{\delta}(\tau)-\bar{\delta}(z)\right]} dz\right)\bar{u} + \tau e^{-\bar{\delta}(\tau)} S(0)\right],$$

which is clearly negative for $\tau > 0$.

(ii) Differentiate (40) to obtain

$$\frac{\partial S(\tau)}{\partial \theta} = -\alpha\left\{\bar{u}\int_0^\tau (\tau - z)\, e^{-\left[\bar{\delta}(\tau)-\bar{\delta}(z)\right]} dz + \tau e^{-\bar{\delta}(\tau)} S(0)\right\}\left[n_2(T) - n_0(T)\right],$$

which has the sign of $n_0(T) - n_2(T)$.

(iii) Differentiate (40) to obtain $\frac{\partial S(\tau)}{\partial U_0} = \frac{\partial S(\tau)}{\partial U_2} = -\frac{1}{2}\frac{\partial S(\tau)}{\partial U_1} = -\frac{S(\tau)}{S(0)} < 0$. ■

Proof of Proposition 10. For $\bar{u} = 0$, $R(\tau)$ is given by (41), but with $S(\tau)$ given by

$$S(\tau) = e^{-\int_0^\tau \{r+\alpha[\theta n_2(s)+(1-\theta)n_0(s)]\}ds} S(0),$$

and with $V_1(\tau) - V_0(\tau)$ given by

$$V_1(\tau) - V_0(\tau) = e^{-r\tau}(U_1 - U_0) + \left(1 - e^{-r\tau}\right)\frac{u_1 - u_0}{r} - \frac{\left[e^{-r\tau} - e^{-\{r+\alpha\theta[n_2(T)-n_0(T)]\}\tau}\right]n_2(T)}{n_2(T) - e^{-\alpha[n_2(T)-n_0(T)]T} n_0(T)} S(0)$$

for the case $n_2(T) \neq n_0(T)$, and

$$V_1(\tau) - V_0(\tau) = e^{-r\tau}(U_1 - U_0) + \left(1 - e^{-r\tau}\right)\frac{u_1 - u_0}{r} - \frac{\tau e^{-r\tau}}{\frac{1}{\alpha n_0(T)} + T}\theta S(0)$$

for the case $n_2(T) = n_0(T)$. From (42),

$$\frac{\partial \rho(\tau)}{\partial x} = \frac{1}{\tau + \Delta}\frac{1}{R(\tau)}\frac{\partial R(\tau)}{\partial x},$$

for $x = \theta$, r, U_0.

(i) Differentiate (41) to obtain

$$\frac{\partial R(\tau)}{\partial r} = R(\tau)\Delta - e^{r(\tau+\Delta)}\frac{u_1 - u_0}{r^2}\left(1 - r\tau - e^{-r\tau}\right) > 0,$$

since $1 - r\tau - e^{-r\tau} \leq 0$. Thus, $\frac{\partial \rho(\tau)}{\partial r} > 0$.

(ii) For any $\tau > 0$, differentiate (41) to obtain

$$\frac{\partial R(\tau)}{\partial \theta} = -e^{r(\tau+\Delta)}\left[1 + \frac{\left\{\theta e^{\alpha[n_2(T)-n_0(T)]\tau}n_0(T)+(1-\theta)e^{\alpha[n_2(T)-n_0(T)]T}n_2(T)\right\}[n_2(T)-n_0(T)]\alpha\tau}{e^{\alpha[n_2(T)-n_0(T)]T}n_2(T)-e^{\alpha[n_2(T)-n_0(T)]\tau}n_0(T)}\right]S(\tau) < 0$$

for the case $n_2(T) \neq n_0(T)$, and

$$\frac{\partial R(\tau)}{\partial \theta} = -e^{r(\tau+\Delta)}\left[1 + \frac{\alpha\tau n_0(T)}{1+\alpha(T-\tau)n_0(T)}\right]S(\tau) < 0$$

for the case $n_2(T) = n_0(T)$. Hence $\frac{\partial \rho(\tau)}{\partial \theta} < 0$.

(iii) Differentiate (41) to obtain

$$\frac{\partial R(\tau)}{\partial U_0} = -e^{r(\tau+\Delta)}\frac{(1-\theta)e^{\alpha[n_2(T)-n_0(T)]T}n_2(T)-\left[1-\theta e^{\alpha(1-\theta)[n_2(T)-n_0(T)]\tau}\right]e^{\alpha\theta[n_2(T)-n_0(T)]\tau}n_0(T)}{e^{\alpha[n_2(T)-n_0(T)]T}n_2(T)-e^{\alpha[n_2(T)-n_0(T)]\tau}n_0(T)}\frac{S(\tau)}{S(0)}$$

for the case $n_2(T) \neq n_0(T)$, and

$$\frac{\partial R(\tau)}{\partial U_0} = -e^{r(\tau+\Delta)}\frac{(1-\theta)[1+\alpha T n_0(T)]}{1+\alpha(T-\tau)n_0(T)}\frac{S(\tau)}{S(0)}$$

for the case $n_2(T) = n_0(T)$. It can be verified that $\frac{\partial R(\tau)}{\partial U_0} < 0$ in both cases, so we conclude that $\frac{\partial \rho(\tau)}{\partial U_0} < 0$. ∎

Proof of Proposition 11. The expression for $S^{\infty}(\tau)$ is obtained by letting $\alpha \to \infty$ in the analytical expression for $S(\tau)$ reported in Proposition 8. To obtain $\rho^{\infty}(\tau)$, proceed as follows. Use (43), together with (35) and the expression for $S(\tau)$ reported in Proposition 8 to obtain

$$\begin{aligned} V_0(\tau) &= \left(1-e^{-r\tau}\right)\frac{u_0}{r} + e^{-r\tau}U_0 \\ &+ e^{-r\tau}\frac{n_2(T)}{n_0(T)}\sum_{k=1}^{\infty}\left[\frac{n_2(T)}{n_0(T)}\right]^{k-1}\frac{\theta}{\frac{r}{\alpha[n_0(T)-n_2(T)]}+(k-\theta)}\frac{e^{r\tau}e^{-\alpha[n_0(T)-n_2(T)]k(T-\tau)}-e^{-\alpha[n_0(T)-n_2(T)]kT}}{r+\alpha[n_0(T)-n_2(T)]k}\bar{u} \\ &- e^{-r\tau}\frac{n_2(T)}{n_0(T)}\sum_{k=1}^{\infty}\left[\frac{n_2(T)}{n_0(T)}\right]^{k-1}\frac{e^{-\alpha[n_0(T)-n_2(T)](kT-\theta\tau)}-e^{-\alpha[n_0(T)-n_2(T)]kT}}{r+\alpha[n_0(T)-n_2(T)](k-\theta)}\bar{u} \\ &+ e^{-r\tau}n_2(T)\frac{1-e^{-\alpha[n_0(T)-n_2(T)]\theta\tau}}{n_0(T)e^{\alpha[n_0(T)-n_2(T)](T-\theta\tau)}-\frac{n_2(T)}{e^{\alpha[n_0(T)-n_2(T)]\theta\tau}}}S(0) \end{aligned}$$

if $n_2(T) < n_0(T)$,

$$
\begin{aligned}
V_0(\tau) &= \left(1 - e^{-r\tau}\right)\frac{u_0}{r} + e^{-r\tau}U_0 \\
&+ e^{-r\tau}\frac{n_2(T)}{n_0(T)}\sum_{k=0}^{\infty}\left[\frac{n_0(T)}{n_2(T)}\right]^{k+1}\frac{\theta}{\frac{r}{\alpha[n_2(T)-n_0(T)]}+(\theta+k)}\frac{e^{r\tau}e^{-\alpha[n_2(T)-n_0(T)]k(T-\tau)}-e^{-\alpha[n_2(T)-n_0(T)]kT}}{r+\alpha[n_2(T)-n_0(T)]k}\bar{u} \\
&- e^{-r\tau}\frac{n_2(T)}{n_0(T)}\sum_{k=0}^{\infty}\left[\frac{n_0(T)}{n_2(T)}\right]^{k+1}\frac{e^{-\alpha[n_2(T)-n_0(T)]kT}-e^{-\alpha[n_2(T)-n_0(T)](kT+\theta\tau)}}{r+\alpha[n_2(T)-n_0(T)](\theta+k)}\bar{u} \\
&+ e^{-r\tau}n_2(T)\frac{1-e^{-\alpha[n_2(T)-n_0(T)]\theta\tau}}{n_2(T)-\frac{n_0(T)}{e^{\alpha[n_2(T)-n_0(T)]T}}}S(0)
\end{aligned}
$$

if $n_0(T) < n_2(T)$, and

$$
\begin{aligned}
V_0(\tau) &= \left(1 - e^{-r\tau}\right)\frac{u_0}{r} + e^{-r\tau}U_0 \\
&+ e^{r\left[T-\tau+\frac{1}{\alpha n_0(T)}\right]}\theta\bar{u}\sum_{k=0}^{\infty}\frac{(-r)^k}{kk!}\tau\left(T+\frac{1}{\alpha n_0(T)}\right)^k \\
&+ e^{r\left[T-\tau+\frac{1}{\alpha n_0(T)}\right]}\theta\bar{u}\sum_{k=0}^{\infty}\frac{(-r)^k}{kk!}\frac{1}{k+1}\left[\left(T-\tau+\frac{1}{\alpha n_0(T)}\right)^{k+1}-\left(T+\frac{1}{\alpha n_0(T)}\right)^{k+1}\right] \\
&+ e^{-r\tau}\theta\frac{\alpha n_0(T)}{1+\alpha n_0(T)T}\tau S(0)
\end{aligned}
$$

if $n_2(T) = n_0(T)$. Then let $\alpha \to \infty$ to arrive at (50), for $i = 0$ (the derivation is similar for $i = 2$). Next, recall that $e^{-r(\tau+\Delta)}R(\tau) = V_1(\tau) - V_0(\tau) - \theta S(\tau)$, so

$$
\lim_{\alpha\to\infty}\left[e^{-r(\tau+\Delta)}R(\tau)\right] = \left(1-e^{-r\tau}\right)\frac{u_1}{r} + e^{-r\tau}U_1 - \lim_{\alpha\to\infty}V_0(\tau) - \theta S_\infty(\tau).
$$

Substitute (50) and $S^\infty(\tau)$ to arrive at

$$
\frac{\lim_{\alpha\to\infty}R(\tau)}{e^{r(\tau+\Delta)}} = \begin{cases} \left(1-e^{-r\tau}\right)\frac{u_1-u_0}{r} + e^{-r\tau}\left(U_1-U_0\right) & \text{if } n_2(T) < n_0(T) \\ \left(1-e^{-r\tau}\right)\frac{u_1-u_0-\theta\bar{u}}{r} + e^{-r\tau}\left[U_1-U_0-\theta S(0)\right] & \text{if } n_2(T) = n_0(T) \\ \left(1-e^{-r\tau}\right)\frac{u_2-u_1}{r} + e^{-r\tau}\left(U_2-U_1\right) & \text{if } n_0(T) < n_2(T). \end{cases} \qquad (101)
$$

Since $\rho(\tau) = \frac{\ln R(\tau)}{\tau+\Delta}$, we have

$$
\rho^\infty(\tau) = \frac{\ln\left[\lim_{\alpha\to\infty}R(\tau)\right]}{\tau+\Delta},
$$

which given (101), yields the expression in the statement of the proposition. ■

Lemma 6 *Assume $I_f^r < I_f^o$ and $I_f^r < I_f^w$. Consider a bank with end-of-day balance k that has the option to borrow from the Discount Window right after the end of the trading session, and let k^w denote the bank's balance after having borrowed from the Window.*

(i) *If $\bar{k} \leq k - \bar{k}_0$, then $k^w = k - \bar{k}_0$ and the bank's maximum terminal payoff is given by (52) with $F_k = F^e(k)$.*

(ii) *If $0 \leq k - \bar{k}_0 < \bar{k}$, then*

$$k^w \begin{cases} = \bar{k} & \text{if } I_f^w < I_f^c + I_f^r \\ \in \left[k - \bar{k}_0, \bar{k}\right] & \text{if } I_f^w = I_f^c + I_f^r \\ = k - \bar{k}_0 & \text{if } I_f^c + I_f^r < I_f^w, \end{cases} \tag{102}$$

and the bank's maximum terminal payoff is given by (52) with

$$F_k = \begin{cases} F_{\bar{k}}^w(k) & \text{if } I_f^w \leq I_f^r + I_f^c \\ F^c(k) & \text{if } I_f^r + I_f^c < I_f^w. \end{cases} \tag{103}$$

(iii) *If $k - \bar{k}_0 < 0$, then*

$$k^w \begin{cases} = \bar{k} & \text{if } I_f^w < I_f^c + I_f^r \\ \in \left[0, \bar{k}\right] & \text{if } I_f^w = I_f^c + I_f^r \\ = 0 & \text{if } I_f^c + I_f^r < I_f^w < I_f^c + I_f^o \\ \in \left[k - \bar{k}_0, 0\right] & \text{if } I_f^w = I_f^c + I_f^o \\ = k - \bar{k}_0 & \text{if } I_f^c + I_f^o < I_f^w \end{cases}$$

and the bank's maximum terminal payoff is given by (52) with

$$F_k = \begin{cases} F_{\bar{k}}^w(k) & \text{if } I_f^w \leq I_f^c + I_f^r \\ F_0^w(k) & \text{if } I_f^c + I_f^r < I_f^w \leq I_f^c + I_f^o \\ F^o(k) & \text{if } I_f^c + I_f^o < I_f^w. \end{cases}$$

Proof of Lemma 6. A bank that ends the trading session with balance k chooses k^w by solving

$$\begin{aligned} F_k = \max_{k - \bar{k}_0 \leq k^w} & \left\{ I_f^r \max\left[0, \min\left(k^w, \bar{k}\right)\right] + I_f^e \max\left(k^w - \bar{k}, 0\right) \right. \\ & \left. - I_f^c \max\left(\bar{k} - k^w, 0\right) - I_f^o \max\left(-k^w, 0\right) - I_f^w \left[k^w - (k - \bar{k}_0)\right] \right\}. \end{aligned} \tag{104}$$

(i) Given $\bar{k} \leq k - \bar{k}_0$, (104) reduces to

$$F_k = \max_{k - \bar{k}_0 \leq k^w} \left\{ (I_f^r - I_f^e)\bar{k} + I_f^w (k - \bar{k}_0) + (I_f^e - I_f^w) k^w \right\}.$$

The assumption $I_f^e < I_f^w$ implies the solution is $k^w = k - \bar{k}_0$, and substituting it into the objective function yields $F_k = F^e(k)$.

(ii) Given $0 \leq k - \bar{k}_0 < \bar{k}$ and part *(i)*, (104) reduces to

$$F_k = \max_{k-\bar{k}_0 \leq k^w \leq \bar{k}} \left\{ I_f^w (k - \bar{k}_0) - I_f^c \bar{k} + (I_f^c + I_f^r - I_f^w) k^w \right\},$$

which implies that the solution is given by (102), and substituting this solution into the objective function yields (103).

(iii) Given $k - \bar{k}_0 < 0$ and part *(i)*, (104) becomes

$$\begin{aligned} F_k &= I_f^w (k - \bar{k}_0) + \max_{k-\bar{k}_0 \leq k^w \leq \bar{k}} \left\{ I_f^r \max(0, k^w) - I_f^c \max\left(\bar{k} - k^w, 0\right) - I_f^o \max(-k^w, 0) - I_f^w k^w \right\} \\ &= I_f^w (k - \bar{k}_0) - I_f^c \bar{k} + \max \left\{ \max_{k-\bar{k}_0 \leq k^w \leq 0} (I_f^c + I_f^o - I_f^w) k^w, \max_{0 \leq k^w \leq \bar{k}} (I_f^c + I_f^r - I_f^w) k^w \right\}. \end{aligned}$$

To find the maximizer(s) consider the five possible rankings for the values of I_f^w, $I_f^c + I_f^r$, and $I_f^c + I_f^o$. (The maintained assumptions $I_f^e \leq I_f^r$, $I_f^r < I_f^w$, and $I_f^r < I_f^o$, imply $I_f^e < I_f^c + I_f^r < I_f^c + I_f^o$.) Suppose that $I_f^w < I_f^c + I_f^r$, then the maximizer is $k^w = \bar{k}$, and $F_k = F_{\bar{k}}^w(k)$. Suppose that $I_f^w = I_f^c + I_f^r$, then the set of maximizers is $[0, \bar{k}]$, and $F_k = F_{\bar{k}}^w(k) = F_0^w(k)$. Suppose that $I_f^c + I_f^r < I_f^w < I_f^c + I_f^o$, then the maximizer is $k^w = 0$, and $F_k = F_0^w(k)$. Suppose that $I_f^w = I_f^c + I_f^o$, then the set of maximizers is $\left[k - \bar{k}_0, 0\right]$, and $F_k = F_0^w(k) = F^o(k)$. Finally, suppose that $I_f^c + I_f^o < I_f^w$, then the maximizer is $k^w = k - \bar{k}_0$, and $F_k = F^o(k)$. ■

Lemma 7 *Consider the model of Section 7. Assume that* $\{U_k\}$ *is given by (52) with* $\bar{k}_0 = 0$ *and* $\bar{k} = 1$, *and define* $\rho_f(\tau) = e^{\rho(\tau)(\tau+\Delta)} - 1$. *Then* $\ln[1 + \rho_f(\tau)]$ *is as in (55).*

Proof of Lemma 7. Combine (40), (44) and (45) to obtain

$$\begin{aligned} V_2(\tau) - V_1(\tau) + (1-\theta) S(\tau) &= e^{-r\tau} \{U_2 - U_1 + [1 - \beta(\tau)] S(0)\} \\ &\quad + (1 - e^{-r\tau}) \frac{u_2 - u_1 + c(\tau)(1-\theta)\bar{u}}{r}, \end{aligned}$$

where $\beta(\tau)$ and $c(\tau)$ are given by (56) and (57), respectively. Then (42) implies

$$\rho(\tau)(\tau + \Delta) = \Delta r + \ln \left[U_2 - U_1 + [1 - \beta(\tau)] S(0) + (e^{r\tau} - 1) \frac{u_2 - u_1 + c(\tau)(1-\theta)\bar{u}}{r} \right].$$

From (52) with $\bar{k}_0 = 0$ and $\bar{k} = 1$, we have $U_2 = 2e^{-r\Delta_f} + e^{-r\Delta_f^r}(i_f^r + i_f^e)$, $U_1 = e^{-r\Delta_f} + e^{-r\Delta_f^r} i_f^r$, and $U_0 = -\min(I_f^c, I_f^w - I_f^r)$, so $U_2 - U_1 = e^{-r\Delta_f} + e^{-r\Delta_f^r} i_f^e$, and $S(0) = I_f^r - I_f^e + \min(I_f^c, I_f^w - I_f^r)$.

(The maintained assumptions $i_f^e \leq i_f^r$, $e^{-r\Delta_f^c}(i_f^c + P^c) \equiv I_f^c > 0$, and $e^{-r\Delta_f^r} i_f^r \equiv I_f^r < e^{-r\Delta_f^w}(i_f^w + P^w) \equiv I_f^w$ imply $S(0) > 0$.) Thus

$$\begin{aligned}\rho(\tau)(\tau+\Delta) = (\Delta - \Delta_f^r)r + \ln\Bigg[& i_f^e + e^{-r(\Delta_f - \Delta_f^r)} \\ & + [1-\beta(\tau)]\left[i_f^r - i_f^e + e^{r\Delta_f^r}\min\left(I_f^c, I_f^w - I_f^r\right)\right] \\ & + e^{r\Delta_f^r}(e^{r\tau}-1)\frac{u_2 - u_1 + c(\tau)(1-\theta)\bar{u}}{r}\Bigg]. \end{aligned} \tag{105}$$

Rearrange and substitute $\rho(\tau) = [1/(\tau+\Delta)]\ln[1+\rho_f(\tau)]$ in (105) to arrive at (55). ■

Proof of Proposition 12. Substitute the definition of $\bar{\delta}(\tau)$, (33) and (35) in (56), and integrate to arrive at the expression for $1-\beta(\tau)$ reported in the statement of the proposition. Differentiate to obtain

$$-\beta'(\tau) = \theta(1-\theta)\frac{\alpha[n_0(T) - n_2(T)]e^{\alpha[n_0(T)-n_2(T)]\theta\tau}}{n_0(T)e^{\alpha[n_0(T)-n_2(T)]T} - n_2(T)}\left[n_0(T)e^{\alpha[n_0(T)-n_2(T)](T-\tau)} - n_2(T)\right].$$

Clearly, $\beta'(\tau)$ has the same sign as $n_2(T) - n_0(T)$. Since $\beta(0) = \theta$, it follows that $\beta(\tau) \leq \theta$ if $n_2(T) < n_0(T)$, and that $\theta \leq \beta(\tau)$ if $n_0(T) < n_2(T)$. To conclude, verify that $0 \leq \beta(T)$ if $n_2(T) < n_0(T)$, and that $\beta(T) \leq 1$ if $n_0(T) < n_2(T)$, which respectively imply that $0 \leq \beta(\tau)$ if $n_2(T) < n_0(T)$, and that $\beta(\tau) \leq 1$ if $n_0(T) < n_2(T)$. Notice that

$$\begin{aligned}1-\beta(T) &= \frac{e^{-\alpha[n_2(T)-n_0(T)]\theta T}\left\{(1-\theta)[n_2(T)-n_0(T)] + n_0(T)\left[1-e^{-\alpha[n_2(T)-n_0(T)](1-\theta)T}\right]\right\}}{n_2(T) - n_0(T)e^{-\alpha[n_2(T)-n_0(T)]T}} \\ &= 1 - \frac{\left[e^{\alpha[n_0(T)-n_2(T)]\theta T} - 1\right]n_2(T) + \theta e^{\alpha[n_0(T)-n_2(T)]\theta T}[n_0(T)-n_2(T)]}{n_0(T)e^{\alpha[n_0(T)-n_2(T)]T} - n_2(T)},\end{aligned}$$

so it is immediate from the first expression, that $0 \leq 1-\beta(T)$ if $n_0(T) < n_2(T)$ (with equality only if $\theta = 1$), and from the second expression, that $1-\beta(T) \leq 1$ if $n_2(T) < n_0(T)$ (with equality only if $\theta = 0$). ■

Lemma 8 *Consider the model of Section 7. Assume that $\{U_k\}$ is given by (52), with $\bar{k}_0 = 0$ and $\bar{k} = 1$, and define $\rho_f^\infty(\tau) = e^{\rho^\infty(\tau)(\tau+\Delta)} - 1$. Then $\rho_f^\infty(\tau)$ is independent of τ. If in addition, $\Delta - \Delta_f^r \approx 0$ or $r \approx 0$, and $\Delta - \Delta_f = u_i = 0$, then $\rho_f^\infty(\tau)$ is given by (60).*

Proof of Lemma 8. Start with (51), use (52) to substitute $\{U_k\}$, and replace the theoretical rate $\rho^\infty(\tau)$ with its empirical counterpart, $\rho_f^\infty(\tau) = e^{\rho^\infty(\tau)(\tau+\Delta)} - 1$. This yields

$$\ln\left[1+\rho_f^\infty(\tau)\right] = (\Delta - \Delta_f^r)r + \ln\left[i_f^s + e^{r\Delta_f^r}(e^{r\tau}-1)\frac{u_1 - u_0}{r} + e^{r(\Delta_f^r - \Delta_f)}\right]$$

if $n_2(T) < n_0(T)$,

$$\ln\left[1+\rho_f^{\infty}(\tau)\right] = (\Delta - \Delta_f^r)r + \ln\left[\theta i_f^e + (1-\theta)\, i_f^s + e^{r\Delta_f^r}\left(e^{r\tau}-1\right)\frac{u_2 - u_1 + (1-\theta)\,\bar{u}}{r} + e^{r(\Delta_f^r - \Delta_f)}\right]$$

if $n_2(T) = n_0(T)$, and

$$\ln\left[1+\rho_f^{\infty}(\tau)\right] = (\Delta - \Delta_f^r)r + \ln\left[i_f^e + e^{r\Delta_f^r}\left(e^{r\tau}-1\right)\frac{u_2 - u_1}{r} + e^{r(\Delta_f^r - \Delta_f)}\right]$$

if $n_0(T) < n_2(T)$. Set $\Delta - \Delta_f^r \approx 0$ or $r \approx 0$, and $\Delta - \Delta_f = u_i = 0$ to obtain (60). ■

B Extensions

In this section we develop several extensions of the theory to allow for ex-ante heterogeneity in bank types. Each extension is motivated by a particular aspect of the fed funds market that our baseline model has abstracted from. First, according to practitioners, some banks (e.g., large banks) consistently exhibit a stronger bargaining position when trading against other (e.g., small) banks. Our first extension allows banks to differ in their bargaining power parameter. Second, empirical studies of the fed funds market have emphasized that a few banks trade with much higher intensity than others, and are consistently more likely to act as borrowers and as lenders during the same trading session.[52] Our second extension allows for banks to differ in the rate at which they contact and are contacted by potential trading partners. Third, in practice, in any given trading session institutions may value end-of-day reserve balances differently. For example, some banks may have balance sheets that call for larger balances to meet their reserve requirements. Policy considerations can also induce differences among fed funds participants, as the Federal Reserve remunerates the reserve balances of some participants, e.g., depository institutions, but not others, e.g., Government Sponsored Enterprises (GSEs). Our third extension allows for heterogeneity in the fed funds participants' payoffs from holding end-of-day balances.

For each extension, we describe the evolution of the distribution of balances and the value function, and the determination of the trading decisions, i.e., all the ingredients needed to define

[52] See Bech and Atalay (2008). The intensity of a bank's trading activity in the fed funds market is also correlated with the interest rates that the bank charges when it lends, and the rates that it pays when it borrows. Ashcraft and Duffie (2007) find that rates tend to be higher on loans that involve lenders who are more active in the federal funds market relative to the borrower. They also document that rates tend to be lower on loans that involve borrowers who are more active relative to the lender.

equilibrium. In each case, we give the relevant variables a superscript that identifies the bank's type. The set of types, $\mathbb{Y}$, is finite and η^y denotes the fraction of banks of type $y \in \mathbb{Y}$, i.e., $\eta^y \in [0,1]$ and $\sum_{y\in\mathbb{Y}} \eta^y = 1$. The measure of banks of type y with balance k at time $T-\tau$, is denoted $n_k^y(\tau)$, so $\sum_{k\in\mathbb{K}} n_k^y(\tau) = \eta^y$. In a meeting at time $T-\tau$ between a bank of type x with i balances and a bank of type y with j balances, $\phi_{ij,xy}^{ks}(\tau)$ is used to denote the probability that the former and the latter hold k and s balances after the meeting, respectively. In this section, $\boldsymbol{n}(\tau) = \left\{n_k^y(\tau)\right\}_{y\in\mathbb{Y},k\in\mathbb{K}}$ and $\boldsymbol{V}(\tau) = \left\{V_k^y(\tau)\right\}_{y\in\mathbb{Y},k\in\mathbb{K}}$ denote the distribution of balances and the value function, respectively, at time $T-\tau$. The distribution of trading probabilities at time $T-\tau$, $\boldsymbol{\phi}(\tau) = \{\{\phi_{ij,xy}^{ks}(\tau)\}_{x,y\in\mathbb{Y}}\}_{i,j,k,s\in\mathbb{K}}$, satisfies $\phi_{ij,xy}^{ks}(\tau) \in [0,1]$ with $\sum\limits_{k\in\mathbb{K}}\sum\limits_{s\in\mathbb{K}} \phi_{ij,xy}^{ks}(\tau) = 1$, and is feasible if $\phi_{ij,xy}^{ks}(\tau) = 0$ if $(k,s) \notin \Pi(i,j)$ for all $i,j,k,s \in \mathbb{K}$ and all $x,y \in \mathbb{Y}$.

B.1 Heterogeneous bargaining powers

Let $\theta_{xy} \in [0,1]$ be the bargaining power of a bank type $x \in \mathbb{Y}$ in negotiations with a bank of type $y \in \mathbb{Y}$, where $\theta_{xy} + \theta_{yx} = 1$.[53] Given any feasible path for the distribution of trading probabilities, $\boldsymbol{\phi}(\tau)$, the distribution of balances evolves according to

$$\dot{n}_k^x(\tau) = f^x[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)] \quad \text{for all } k \in \mathbb{K} \text{ and } x \in \mathbb{Y}, \tag{106}$$

where

$$\begin{aligned} f^x[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)] \equiv \; & \alpha n_k^x(\tau) \sum_{y\in\mathbb{Y}}\sum_{i\in\mathbb{K}}\sum_{j\in\mathbb{K}}\sum_{s\in\mathbb{K}} n_i^y(\tau)\, \phi_{ki,xy}^{sj}(\tau) \\ & - \alpha \sum_{y\in\mathbb{Y}}\sum_{i\in\mathbb{K}}\sum_{j\in\mathbb{K}}\sum_{s\in\mathbb{K}} n_i^x(\tau)\, n_j^y(\tau)\, \phi_{ij,xy}^{ks}(\tau). \end{aligned} \tag{107}$$

The value function satisfies

$$rV_i^x(\tau) + \dot{V}_i^x(\tau) = u_i + \alpha \sum_{y\in\mathbb{Y}}\sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} n_j^y(\tau)\, \phi_{ij,xy}^{ks}(\tau)\, \theta_{xy} \left[V_k^x(\tau) + V_s^y(\tau) - V_i^x(\tau) - V_j^y(\tau)\right] \tag{108}$$

for all $(x,i,\tau) \in \mathbb{Y} \times \mathbb{K} \times [0,T]$, with

$$V_i^x(0) = U_i \quad \text{for all } x \in \mathbb{Y} \text{ and all } i \in \mathbb{K}. \tag{109}$$

[53] For example, a natural specification would be $\mathbb{Y} = \{1,\ldots,N\}$ with $\theta_{xy} \le \theta_{yx}$ if $x \le y$. In this case, a higher type corresponds to a stronger bargaining power.

The path for $\boldsymbol{\phi}(\tau)$ is given by

$$\phi_{ij,xy}^{ks}(\tau) = \begin{cases} \tilde{\phi}_{ij,xy}^{ks}(\tau) & \text{if } (k,s) \in \Omega_{ij,xy}[\boldsymbol{V}(\tau)] \\ 0 & \text{if } (k,s) \notin \Omega_{ij,xy}[\boldsymbol{V}(\tau)], \end{cases} \tag{110}$$

for all $x, y \in \mathbb{Y}$, all $i, j, k, s \in \mathbb{K}$, and all $\tau \in [0, T]$, where $\tilde{\phi}_{ij,xy}^{ks}(\tau) \geq 0$ and $\sum_{k\in\mathbb{K}} \sum_{s\in\mathbb{K}} \tilde{\phi}_{ij,xy}^{ks}(\tau) = 1$, with

$$\Omega_{ij,xy}[\boldsymbol{V}(\tau)] \equiv \arg \max_{(k',s')\in\Pi(i,j)} \left[V_{k'}^{x}(\tau) + V_{s'}^{y}(\tau) - V_{i}^{x}(\tau) - V_{j}^{y}(\tau)\right]. \tag{111}$$

If at time $T-\tau$, a bank of type y with balance j extends a loan of size $j-s=k-i$ to a bank of type x with balance i, the present value of the equilibrium repayment from the latter to the former is

$$e^{-r(\tau+\Delta)} R_{ij,x,y}^{ks}(\tau) = \frac{1}{2}\left[V_{k}^{x}(\tau) - V_{i}^{x}(\tau)\right] + \frac{1}{2}\left[V_{j}^{y}(\tau) - V_{s}^{y}(\tau)\right]. \tag{112}$$

B.2 Heterogeneous contact rates

Let α^x be the contact rate of a bank of type $x \in \mathbb{Y}$. Notice that from the perspective of any bank, the probability of finding a trading partner of type $y \in \mathbb{Y}$ with balance $j \in \mathbb{K}$ at time $T-\tau$, conditional on having contacted a random partner, is $\bar{\eta}^y n_j^y(\tau)$, where

$$\bar{\eta}^y \equiv \frac{\alpha^y \eta^y}{\sum_{x\in\mathbb{Y}} \alpha^x \eta^x}.$$

Hence the rate at which a bank of type x contacts a bank of type y who holds balance j at time $T-\tau$, is $\alpha^x \bar{\eta}^y n_j^y(\tau)$, and $\alpha^x \bar{\eta}^y n_j^y(\tau) n_i^x(\tau)$ is the measure of banks of type x who hold balance i, that meet a bank of type y who holds balance j. Therefore, given any feasible path for the distribution of trading probabilities, $\boldsymbol{\phi}(\tau)$, the distribution of balances evolves according to

$$\dot{n}_k^x(\tau) = f^x[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)] \quad \text{for all } k \in \mathbb{K} \text{ and } x \in \mathbb{Y}, \tag{113}$$

where

$$\begin{aligned} f^x[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)] \equiv{}& \alpha^x n_k^x(\tau) \sum_{y\in\mathbb{Y}} \sum_{i\in\mathbb{K}} \sum_{j\in\mathbb{K}} \sum_{s\in\mathbb{K}} \bar{\eta}^y n_i^y(\tau) \phi_{ki,xy}^{sj}(\tau) \\ &- \alpha^x \sum_{y\in\mathbb{Y}} \sum_{i\in\mathbb{K}} \sum_{j\in\mathbb{K}} \sum_{s\in\mathbb{K}} \bar{\eta}^y n_i^x(\tau) n_j^y(\tau) \phi_{ij,xy}^{ks}(\tau). \end{aligned} \tag{114}$$

The value function satisfies

$$rV_i^x(\tau) + \dot{V}_i^x(\tau) = u_i + \frac{\alpha^x}{2} \sum_{y\in\mathbb{Y}} \sum_{j\in\mathbb{K}} \sum_{k\in\mathbb{K}} \sum_{s\in\mathbb{K}} \bar{\eta}^y n_j^y(\tau) \phi_{ij,xy}^{ks}(\tau) \left[V_k^x(\tau) + V_s^y(\tau) - V_i^x(\tau) - V_j^y(\tau)\right]$$

for all $(x, i, \tau) \in \mathbb{Y} \times \mathbb{K} \times [0, T]$, subject to (109). Given $\boldsymbol{V}(\tau)$, the path for $\boldsymbol{\phi}(\tau)$ is as in (110), and the repayment as in (112).

B.3 Payoff heterogeneity

Let $U_k^y \in \mathbb{R}$ be the payoff to a bank of type $y \in \mathbb{Y}$ from holding a balance $k \in \mathbb{K}$ at the end of the trading session. Given any feasible path for the distribution of trading probabilities, $\boldsymbol{\phi}(\tau)$, the distribution of balances evolves according to (106) and (107). The value function satisfies (108), but with terminal condition

$$V_i^x(0) = U_i^x \quad \text{for all } x \in \mathbb{Y} \text{ and all } i \in \mathbb{K},$$

and $\theta_{xy} = 1/2$ for all $x, y \in \mathbb{Y}$. Given $\boldsymbol{V}(\tau)$, the path for $\boldsymbol{\phi}(\tau)$ is as in (110), and the repayment as in (112).

On October 9, 2008, the Federal Reserve began to pay interest on the required reserve balances and on the excess balances held by depository institutions, but not on the balances held by non-depository institutions.[54] This means that some large lenders in the federal funds market which are non-depository institutions, such as the GSEs, do not receive interest on their reserve balances.[55] It has been argued (see Bech and Klee, 2011) that such institutions may have an incentive to lend at rates below the rate that banks receive on reserve balances, which might have contributed to an increase of their market share and to the effective federal funds rate (a daily volume-weighted average of brokered transactions rates) being lower than the rate of interest banks earn on reserve balances. In our extended model, this feature of GSEs, and its implication for the determination of the distribution of fed funds rates, can be handled by regarding GSEs as a particular type, $y_0 \in \mathbb{Y}$, with $U_k^{y_0} = 0$ for all $k \in \mathbb{K}$.

C Trade dynamics

For the numerical exercises in this section we set $\mathbb{K} = \{0, 1, 2\}$, $\bar{k} = 1$, and consider two initial distributions of funds, $\{n_k(T)\}_{k=0}^2 = \{0.6, 0.1, 0.3\}$, and $\{n_k(T)\}_{k=0}^2 = \{0.3, 0.1, 0.6\}$. All other parameter values are as in the baseline calibration described in Section 8.3.

[54]The Financial Services Regulatory Relief Act of 2006 gives the Federal Reserve authority to pay interest on reserve balances only to depository institutions, including banks, savings associations, saving banks and credit unions, trust companies, and U.S. agencies and branches of foreign banks.

[55]Fannie Mae and Freddie Mac are large lenders of fed funds because their business model involves using the fed funds market as a short-term investment for incoming mortgage payments, before passing the funds on to investors in the form of principal and/or interest payments. Similarly, the Federal Home Loan Banks use the fed funds market to keep their funds readily available to meet unexpected borrowing demands from members.

C.1 Bargaining power

Figure 6 (with actual time, $t = T - \tau$, on the horizontal axis) shows the time paths for the trade surplus, the opportunity cost to a lender from giving up the second unit of reserves, and the fed funds rate, for different values of the borrower's bargaining power, $\theta = 0.1$, $\theta = 0.5$ (the baseline), and $\theta = 0.9$. The top row of panels corresponds to the case in which the initial number of lenders is smaller than the initial number of borrowers, i.e., $n_2(T) = 0.3 < n_0(T) = 0.6$. Notice that in this case, reserve balances are relatively scarce since $Q = 0.7 < 1 = \bar{k}$. First consider the left panel on the top row. Since $S(0) = 2U_1 - U_2 - U_0$, the trade surplus at the end of the session is the same for all values of θ. For all $t < T$, however, the time-path for the trade surplus is shifted upward as the borrower's bargaining power, θ, increases. The reason is that while for each τ, an increase in θ increases the borrower's outside option, $V_0(\tau)$, and decreases the lender's outside option, $V_2(\tau)$, the fact that $n_2(\tau) < n_0(\tau)$ for all τ, implies that the decrease in the lender's outside option is larger than the increase in the borrower's outside option, so the resulting trade surplus is larger at each point in time along the trading session. The middle panel shows that as θ increases, the path for the value of a lender is shifted down for all $\tau \in (0, T]$. (Agents with one unit of balances do not trade in this example, so the path for $V_1(\tau)$ is effectively exogenous.) The right panel confirms that the path for the fed funds rate is shifted down as the bargaining power of the borrower increases, as was to be expected from (42) and the effect of θ on $V_2(\tau) - V_1(\tau)$ illustrated in the middle panel. Naturally, the borrower pays less for the loan if his bargaining power is higher.

The panels on the bottom row correspond to the case in which reserve balances are abundant; since the initial number of borrowers is smaller than the initial number of lenders, i.e., $n_0(T) = 0.3 < n_2(T) = 0.6$, we have $\bar{k} = 1 < 1.3 = Q$. In this case an increase in θ still increases $V_0(\tau)$ and decreases $V_2(\tau)$ for each $\tau \in (0, T]$, but the fact that $n_0(\tau) < n_2(\tau)$ for all τ implies that the decrease in the lender's outside option is smaller than the increase in the borrower's outside option, so the resulting trade surplus is now smaller at each point during the trading session. As in the top panel, the path for the value of a lender is shifted down for all $\tau \in (0, T]$ as θ increases, but notice that the size of this effect is smaller for smaller $n_0(\tau)$ (because in this case the lender meets borrowers very infrequently, which makes his expected gain from trade small to begin with). Again, the right panel confirms that the path for the fed funds rate is shifted down as the bargaining power of the borrower increases.[56]

[56]In order to understand the intraday dynamics of the fed funds rate, it is useful to compare the right panel

C.2 Discount-Window lending rate

Figure 7 (with $t = T - \tau$, on the horizontal axis) shows the time paths for the trade surplus, the opportunity cost to a lender from giving up the second unit of reserves, and the fed funds rate, for different values of the Discount Window policy rate as it is reported by the Federal Reserve, i.e., $i_f^w \equiv e^{i^w \Delta_f} - 1$. The different values considered are $i_f^w = 0.005$, $i_f^w = 0.0075$ (the baseline), and $i_f^w = 0.01$. The panels on the top row correspond to the case in which reserve balances are scarce (the initial number of lenders is smaller than the initial number of borrowers, i.e., $n_2(T) = 0.3 < n_0(T) = 0.6$), while the bottom row corresponds to the case with $n_0(T) = 0.3 < n_2(T) = 0.6$, in which reserve balances are abundant. The left panels on the top and bottom rows show that making it more costly to borrow from the Discount Window shifts up the path of the surplus, an effect driven by the fact that the first-order implication of a larger i_f^w is to reduce the borrower's outside option, $V_0(\tau)$, making it more valuable for borrowers to trade and avoid having to resort to the Window. Naturally, this effect also causes the paths for the interest rate to shift up. The middle panels show that an increase in i_f^w leads to an increase in the value of lenders for all $t \in [0, T)$.

C.3 Trading delays

Figure 8 (with $t = T - \tau$, on the horizontal axis) shows the time paths for the trade surplus, the opportunity cost to a lender from giving up the second unit of reserves, and the fed funds rate, for different values of the contact rate, $\alpha = 25$, $\alpha = 50$ (the baseline), and $\alpha = 100$. The panels on the top row correspond to the case in which reserve balances are scarce (the initial number of lenders is smaller than the initial number of borrowers, i.e., $n_2(T) = 0.3 < n_0(T) = 0.6$), while the bottom row corresponds to the case with abundant reserve balances, $n_0(T) = 0.3 < n_2(T) = 0.6$. The middle panel on the top row shows that traders on the short side of the market benefit from increases in the contact rate. In contrast, the middle panel on the bottom row shows that in this example, increases in α decrease the expected payoffs of the agents who are on the long side of the market. This is explained by the fact that, from the standpoint

on the top row with the right panel on the bottom row. In general, the fed funds rate tends to increase over time (i.e., as the end of the trading session approaches) when there are more lenders than borrowers, but it can decrease over time when there are more borrowers than lenders, provided θ is not too small. In both scenarios $S(\tau)$ is increasing over time, which tends to make $\rho(\tau)$ increasing over time (see (42)). But when the number of borrowers is large relative to the number of lenders, the difference $V_2(\tau) - V_1(\tau)$ is large and decreases steeply over time, and this effect can (e.g., for θ large enough) dominate the dynamics of the fed funds rate, resulting in an equilibrium fed funds rate that decreases over time.

of the agents on the long side, a faster contact rate has the undesirable effect of taking scarce potential trading partners off the market, which can adversely affect the effective rate at which they are able to trade.[57]

For all $t < T$ the time-path for the trade surplus is shifted downward as α increases. In the parametrization illustrated in the top row, an increase in α increases $V_2(\tau)$ for all $\tau \in (0, T]$ and decreases $V_0(\tau)$ for all $\tau \in (0, T]$. However, the former outweights the latter since $n_2(\tau)$ is small relative to $n_0(\tau)$ for all τ. In the parametrization illustrated in the bottom row, an increase in α increases $V_0(\tau)$ for all $\tau \in (0, T]$ and decreases $V_2(\tau)$ for all $\tau \in (0, T]$ and the former effect outweights the latter since $n_0(\tau)$ is small relative to $n_2(\tau)$ for all τ.

Together, the dynamics of $V_2(\tau) - V_1(\tau)$ and $S(\tau)$ account for the pattern of interest rates displayed in the right panels of the top and bottom rows. In each case, the right panel shows that traders on the short side of the market benefit from increases in the contact rate. Specifically, when lenders are on the short side, increases in the contact rate take scarce lenders off the market which makes borrowers willing to pay higher rates for the loans. Similarly, when borrowers are on the short side, a faster contact rate takes scarce borrowers off the market making lenders more willing to accept lower rates for the loans.

D Policy evaluation (2007 counterfactual)

In this section we use the model calibrated to mimic the salient features of a typical day in 2007 to conduct the policy experiments conducted in Section 8.3. Table 4 reports the equilibrium values of $\bar{\rho}$ that result from varying i_f from 0 to 6 percent in 1 percent increments (as before, each column corresponds to a different value of $Q/\bar{k}$). All other parameter values are as in Section 8.1. Table 5 reports the equilibrium values of $\bar{\rho}$ that result from varying i_f^w from 575 basis points to 700 basis points in 25 basis point increments, while keeping all other parameter values as in Section 8.1.

[57]In general, the effect of changes in α on equilibrium payoffs can be subtle. For example, in some of our numerical simulations we have found that, if $n_2(T) < n_0(T)$, then $V_0(\tau)$ can be nonmonotonic in α: increasing in α for small values of α, but decreasing in α for large values. If $n_2(T) < n_0(T)$, however, $V_2(\tau)$ is typically increasing in α. We have found the converse to be the case for $n_0(T) < n_2(T)$, i.e., $V_0(\tau)$ is increasing in α, while increases in α from relatively small values tend to shift $V_2(\tau)$ up, while increases in α at large values tend to shift $V_2(\tau)$ down.

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i_f	$Q/\bar{k}=0.1$	$Q/\bar{k}=0.5$	$Q/\bar{k}=1$	$Q/\bar{k}=5$	$Q/\bar{k}=10$	$Q/\bar{k}=15$	$Q/\bar{k}=30$
0	0.0039	0.0038	0.0036	0.0027	0.0017	0.0009	0.0002
0.0025	0.0051	0.0051	0.0050	0.0043	0.0036	0.0032	0.0026
0.0050	0.0064	0.0064	0.0063	0.0060	0.0056	0.0054	0.0051
0.0075	0.0077	0.0077	0.0077	0.0076	0.0076	0.0076	0.0076

Table 1: Effects of i_f and $Q/\bar{k}$ on the fed funds rate

i_f^w	$Q/\bar{k}=0.1$	$Q/\bar{k}=0.5$	$Q/\bar{k}=1$	$Q/\bar{k}=5$	$Q/\bar{k}=10$	$Q/\bar{k}=15$	$Q/\bar{k}=30$
0.0025	0.0027	0.0027	0.0027	0.0026	0.0026	0.0026	0.0026
0.0050	0.0039	0.0039	0.0038	0.0035	0.0031	0.0029	0.0026
0.0075	0.0051	0.0051	0.0050	0.0043	0.0036	0.0032	0.0026
0.0100	0.0064	0.0063	0.0061	0.0052	0.0042	0.0034	0.0027
0.0125	0.0076	0.0074	0.0073	0.0060	0.0047	0.0037	0.0027
0.0150	0.0088	0.0086	0.0084	0.0068	0.0052	0.0040	0.0027

Table 2: Effects of i_f^w and $Q/\bar{k}$ on the fed funds rate

(i_f, i_f^w)	$Q/\bar{k} = 0.1$	$Q/\bar{k} = 0.5$	$Q/\bar{k} = 1$	$Q/\bar{k} = 5$	$Q/\bar{k} = 10$	$Q/\bar{k} = 15$	$Q/\bar{k} = 30$
$(0.0000, 0.0025)$	0.0014	0.0014	0.0013	0.0010	0.0006	0.0004	0.0001
$(0.0025, 0.0050)$	0.0039	0.0039	0.0038	0.0035	0.0031	0.0029	0.0026
$(0.0050, 0.0075)$	0.0064	0.0064	0.0063	0.0060	0.0056	0.0054	0.0051
$(0.0075, 0.0100)$	0.0089	0.0089	0.0088	0.0085	0.0081	0.0079	0.0076
$(0.0100, 0.0125)$	0.0114	0.0114	0.0113	0.0110	0.0106	0.0104	0.0101

Table 3: Determination of the fed funds rate under a 25-basis-point corridor system

i_f	$Q/\bar{k} = 0.01$	$Q/\bar{k} = 0.25$	$Q/\bar{k} = 0.5$	$Q/\bar{k} = 1$	$Q/\bar{k} = 5$	$Q/\bar{k} = 15$	$Q/\bar{k} = 30$
0	0.0861	0.0799	0.0722	0.0544	0.0031	0.0002	0.0001
0.01	0.0886	0.0830	0.0760	0.0597	0.0128	0.0101	0.0101
0.02	0.0912	0.0860	0.0797	0.0650	0.0226	0.0201	0.0201
0.03	0.0937	0.0891	0.0834	0.0703	0.0323	0.0301	0.0301
0.04	0.0962	0.0922	0.0872	0.0756	0.0420	0.0401	0.0401
0.05	0.0988	0.0952	0.0909	0.0808	0.0518	0.0501	0.0501
0.06	0.1013	0.0983	0.0946	0.0861	0.0615	0.0601	0.0601

Table 4: Effects of i_f and $Q/\bar{k}$ on the fed funds rate (2007 counterfactual)

i_f^w	$Q/\bar{k} = 0.01$	$Q/\bar{k} = 0.25$	$Q/\bar{k} = 0.5$	$Q/\bar{k} = 1$	$Q/\bar{k} = 5$	$Q/\bar{k} = 15$	$Q/\bar{k} = 30$
0.0575	0.0824	0.0764	0.0691	0.0521	0.0029	0.0002	0.0001
0.0600	0.0842	0.0782	0.0706	0.0532	0.0030	0.0002	0.0001
0.0625	0.0861	0.0799	0.0722	0.0544	0.0031	0.0002	0.0001
0.0650	0.0880	0.0816	0.0738	0.0556	0.0031	0.0002	0.0001
0.0675	0.0899	0.0834	0.0753	0.0568	0.0032	0.0002	0.0001
0.0700	0.0917	0.0851	0.0769	0.0580	0.0033	0.0002	0.0001

Table 5: Effects of i_f^w and $Q/\bar{k}$ on the fed funds rate (2007 counterfactual)

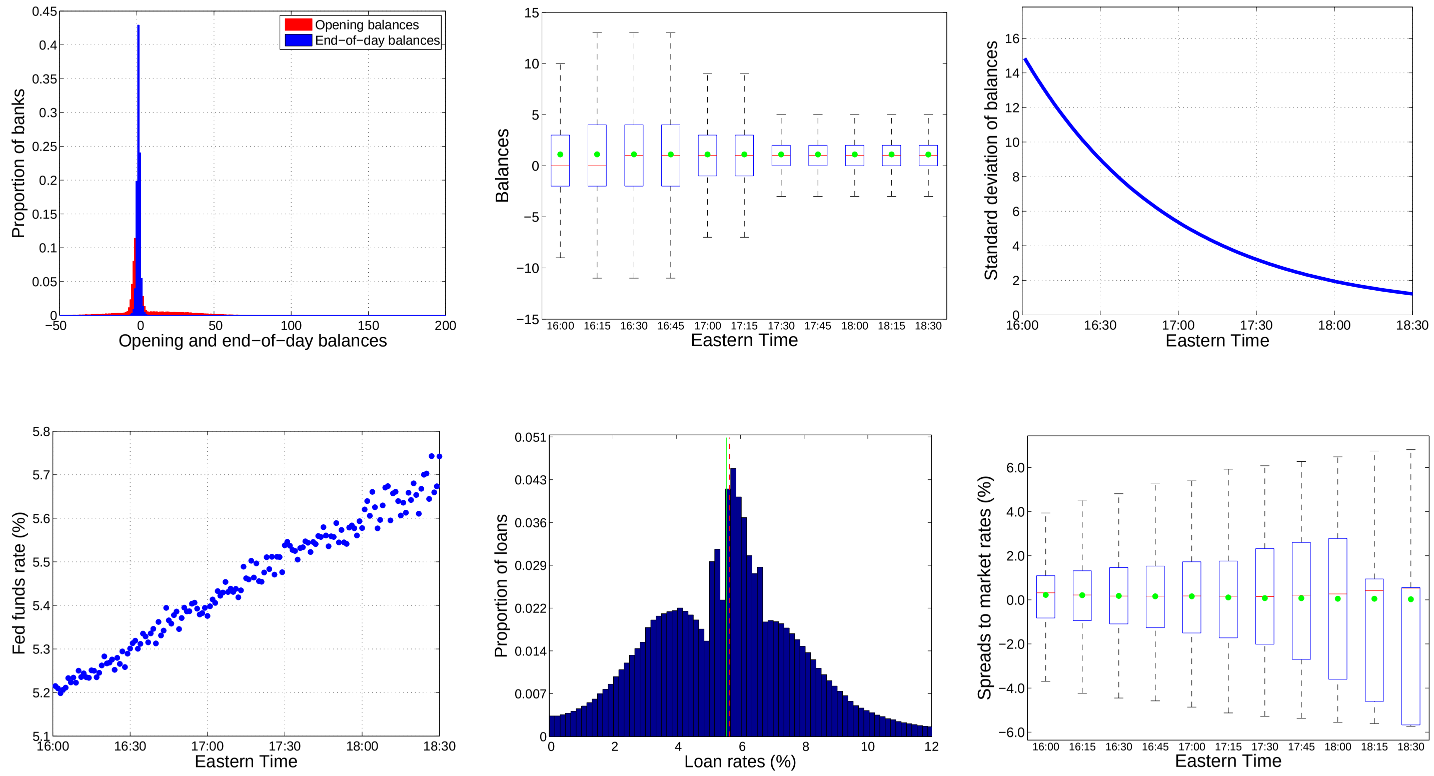

Figure 1: Model calibrated to 2007. Distribution of opening and end-of-day balances (top left). Box plot of balances every 15 minutes during a trading session (outliers not shown) (top center). Standard deviation of the distribution of balances during a trading session (top right). Fed funds rate (value-weighted) at each minute of a trading session (bottom left). Histogram of fed funds rates on all daily transactions (bottom center). Box plot every 15 minutes of the spread between rates on loans traded at time t and the value-weighted average of the rates on all transactions traded at t (bottom right). In a histogram, the solid (green) vertical line indicates the mean, and the dashed (red) vertical line represents the median. In a box plot, the (green) solid dot indicates the mean and the (red) horizontal line represents the median.

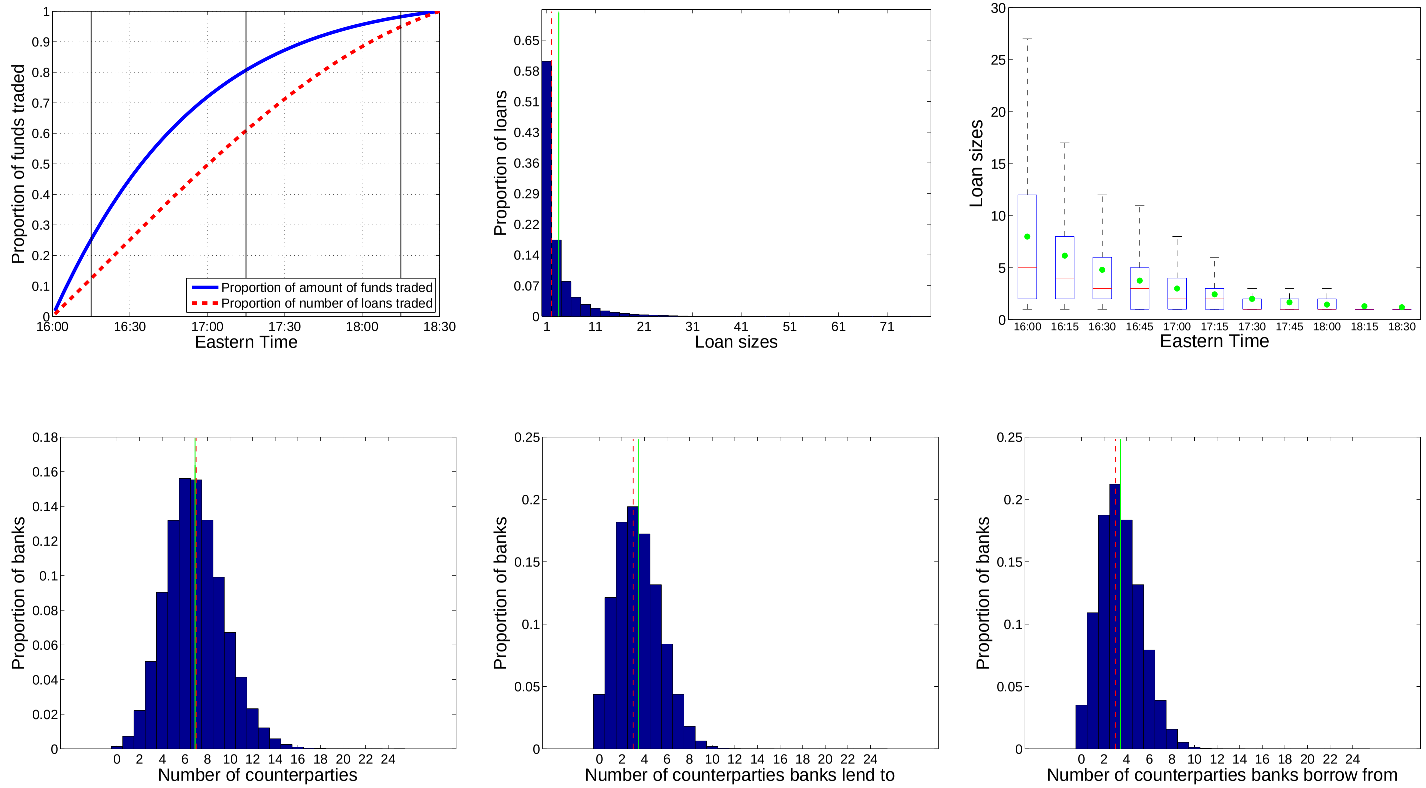

Figure 2: Model calibrated to 2007. Amount of funds and number of loans traded by time of day (top left). Histogram of daily loan sizes (top center). Box plot of loan sizes every 15 minutes during a trading session (top right). Histogram of the number of counterparties banks trade with during a trading session (bottom left). Histogram of the number of counterparties banks lend to (borrowers) during a trading session (bottom center). Histogram of the number of counterparties banks borrow from (lenders) during a trading session (bottom right). In a histogram, the solid (green) vertical line indicates the mean, and the dashed (red) vertical line represents the median. In a box plot, the (green) solid dot indicates the mean and the (red) horizontal line represents the median.

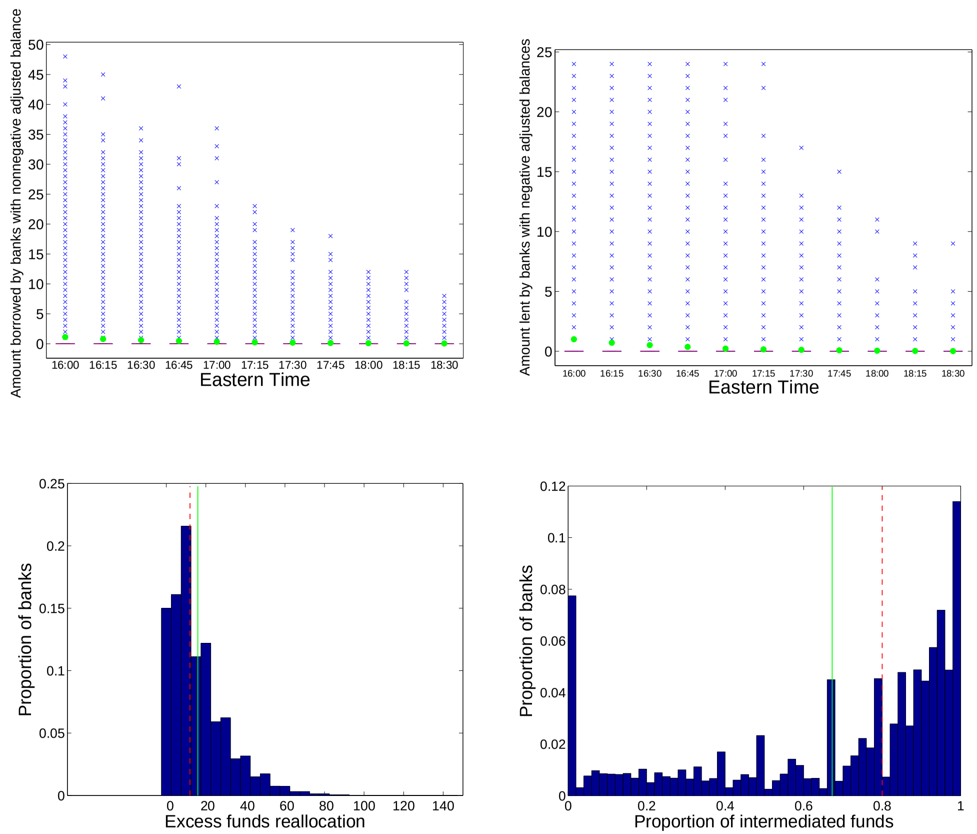

Figure 3: Model calibrated to 2007. Box plot of the amount borrowed by banks with large non-negative pre-trade balances (left). Histogram of the excess funds reallocation during a trading session (center). Histogram of the proportion of intermediated funds during a trading session (right). In a histogram, the solid (green) vertical line indicates the mean, and the dashed (red) vertical line represents the median. In a box plot, the (green) solid dot indicates the mean and the (red) horizontal line represents the median. Outliers are indicated by (blue) crosses.

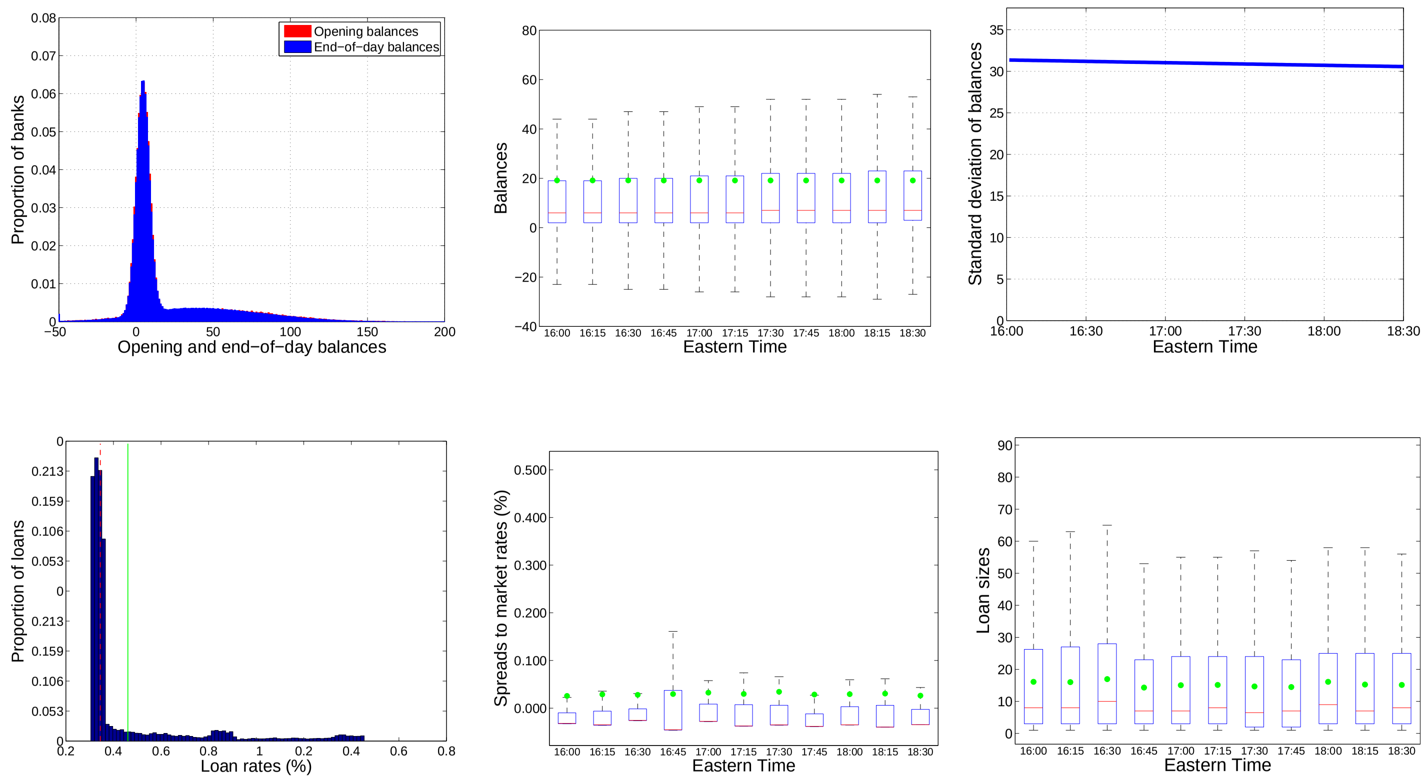

Figure 4: Model calibrated to 2011. Distribution of opening and end-of-day balances (top left). Box plot of balances every 15 minutes during a trading session (outliers not shown) (top center). Standard deviation of the distribution of balances during a trading session (top right). Histogram of fed funds rates on all daily transactions (bottom left). Box plot every 15 minutes of the spread between rates on loans traded at time t and the value-weighted average of the rates on all transactions traded at t (bottom center). Histogram of daily loan sizes (bottom right). In a histogram, the solid (green) vertical line indicates the mean, and the dashed (red) vertical line represents the median. In a box plot, the (green) solid dot indicates the mean and the (red) horizontal line represents the median.

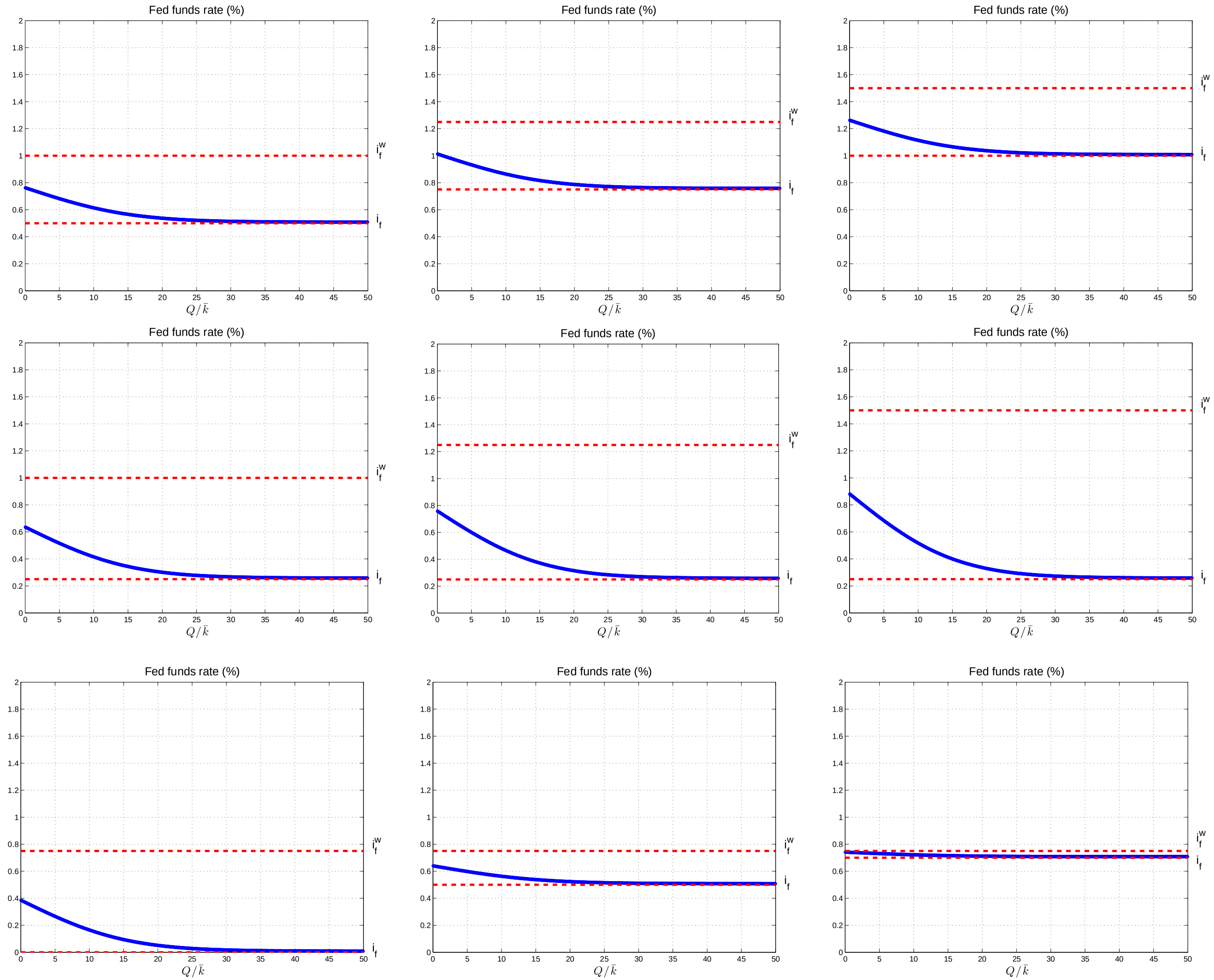

Figure 5: Equilibrium fed funds rate ($\bar{\rho}$) as a function of the consolidated level of reserves (relative to required reserves) in the banking sector, for different policies (i_f, i_f^w)

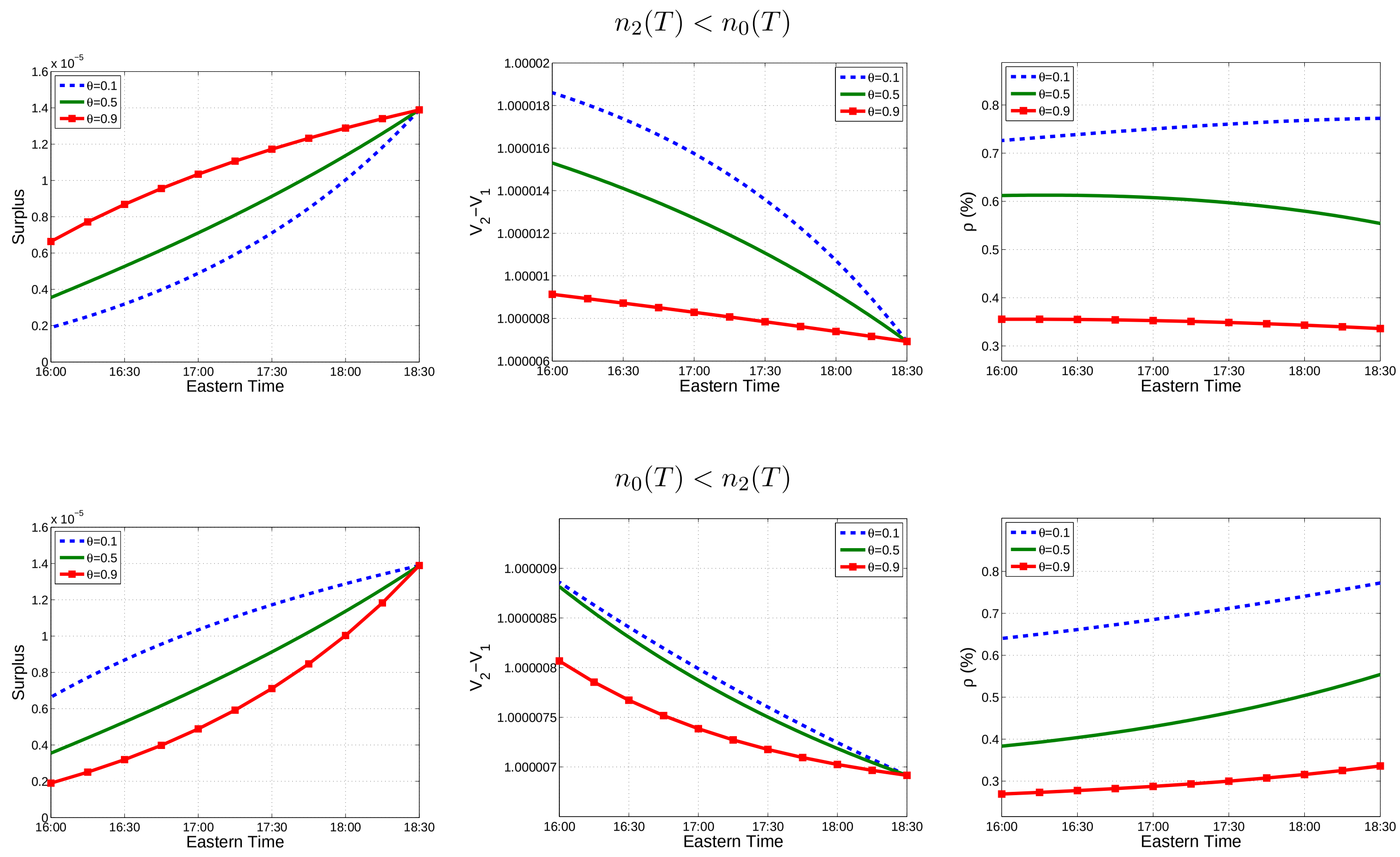

Figure 6: Surplus (left), $V_2(t) - V_1(t)$ (center), and fed funds rate (right) for different values of the bargaining power: $\theta = 0.1$, $\theta = 0.5$ (baseline), and $\theta = 0.9$, when $\{n_k(T)\}_{k=0}^{2} = \{0.6, 0.1, 0.3\}$ (top row), and when $\{n_k(T)\}_{k=0}^{2} = \{0.3, 0.1, 0.6\}$ (bottom row). Other parameter values: $\alpha = 50$, $T = 2.5/24$, $\Delta = 22/24$, $\Delta_f^r = \Delta_f^w = \Delta_f^c = \Delta_f = 2.5/24$, $\Delta_f^d = 0$, $r = 0.0001/365$, $i_+^d = 10^{-7}/360$, $i_f^r = i_f^e = 0.0025/360$, $i_f^w = 0.0075$, $i_f^c = 0.0175$, and $P^w = P^c = 0$.

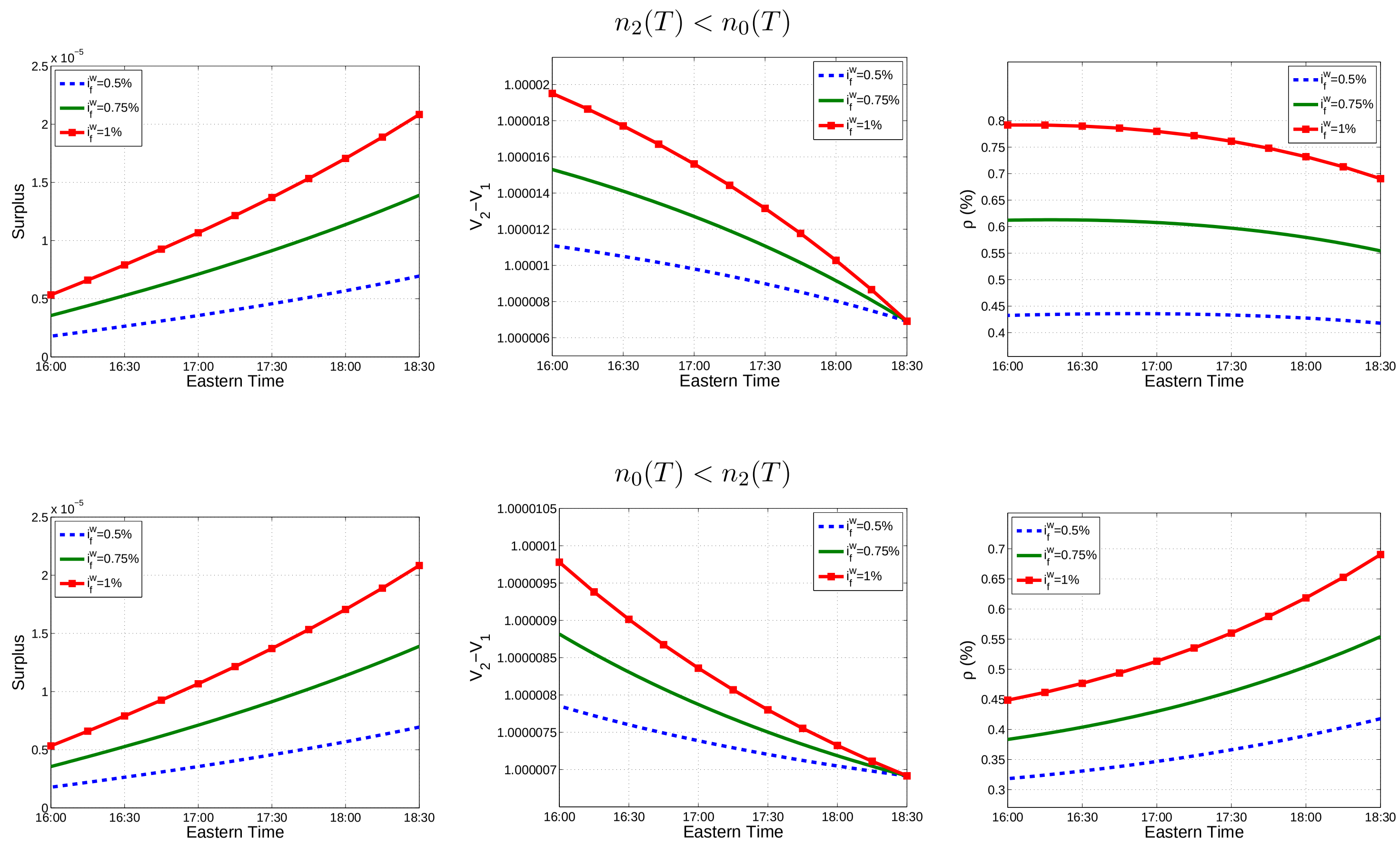

Figure 7: Surplus (left), $V_2(t) - V_1(t)$ (center), and fed funds rate (right) for different values of the Discount Window rate: $i_f^w = 0.005$, $i_f^w = 0.0075$ (baseline), and $i_f^w = 0.01$, when $\{n_k(T)\}_{k=0}^2 = \{0.6, 0.1, 0.3\}$ (top row), and when $\{n_k(T)\}_{k=0}^2 = \{0.3, 0.1, 0.6\}$ (bottom row). Other parameter values: $\alpha = 50$, $\theta = 1/2$, $T = 2.5/24$, $\Delta = 22/24$, $\Delta_f^r = \Delta_f^w = \Delta_f^c = \Delta_f = 2.5/24$, $\Delta_f^d = 0$, $r = 0.0001/365$, $i_+^d = 10^{-7}/360$, $i_f^r = i_f^e = 0.0025/360$, $i_f^c = 0.0175$, and $P^w = P^c = 0$.

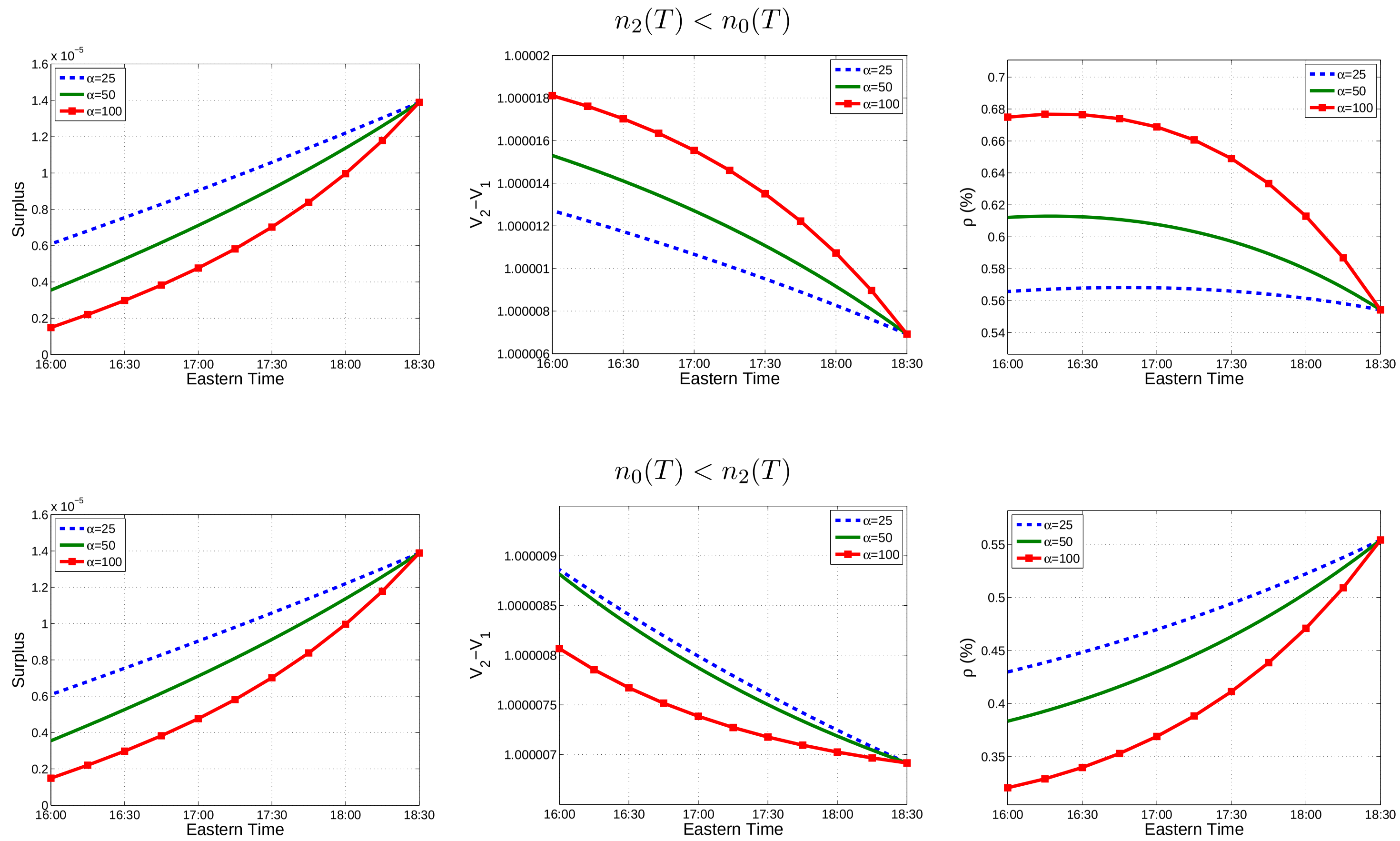

Figure 8: Surplus (left), $V_2(t) - V_1(t)$ (center), and fed funds rate (right) for different values of the frequency of meetings: $\alpha = 25$, $\alpha = 50$ (baseline), and $\alpha = 100$, when $\{n_k(T)\}_{k=0}^{2} = \{0.6, 0.1, 0.3\}$ (top row), and when $\{n_k(T)\}_{k=0}^{2} = \{0.3, 0.1, 0.6\}$ (bottom row). Other parameter values: $\theta = 1/2$, $T = 2.5/24$, $\Delta = 22/24$, $\Delta_f^r = \Delta_f^w = \Delta_f^c = \Delta_f = 2.5/24$, $\Delta_f^d = 0$, $r = 0.0001/365$, $i_+^d = 10^{-7}/360$, $i_f^r = i_f^e = 0.0025/360$, $i_f^w = 0.0075$, $i_f^c = 0.0175$, and $P^w = P^c = 0$.