Some simple Bitcoin Economics

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Linda Schilling[1] and Harald Uhlig[2]

[1]École Polytechnique - CREST
Department of Economics
lin.schilling@gmail.com

[2]University of Chicago
Department of Economics
huhlig@uchicago.edu

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Outline

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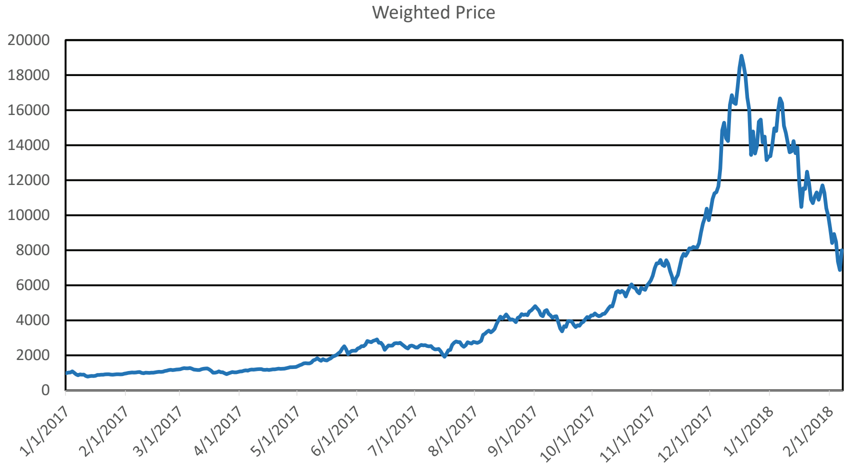
Questions

- Bitcoin, cybercurrencies: increasingly hard to ignore.
- Satoshi Nakamoto, "Bitcoin: A Peer-to-Peer Electronic Cash System," www.bitcoin.org.
- Increasing number of cybercurrencies. Regulatory concerns.
- Blockchain technology. (Not a topic today)
- Literature: growing. Increasingly: serious academics. See paper.
- Imagine a world, where Bitcoin (or cybercurrencies) are important.
- Key questions:
    - How do Bitcoin prices evolve?
    - What are the consequences for monetary policy?

Bitcoin Price, 2011-09-13 to 2018-02-07

Data: quandl.com

Bitcoin Price, 2017-01-01 to 2018-02-07



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This paper

Approach: a simple model, with money as a medium of exchange.

- A novel, yet simple endowment economy: two types of agents keep trading.
- Two types of money: Bitcoins and Dollars.
- A central bank keeps real value of Dollars constant...
- ... while Bitcoin production is private and decentralized.

Results:

- "Fundamental condition": a version of Kareken-Wallace (1981)
- "Speculative condition".
- Under some conditions: no speculation.
- Under some conditions: Bitcoin price converges.
- Implications for monetary policy: two scenarios.
- Construction of equilibria.

Literature

Bitcoin Pricing

- Athey et al
- Garratt and Wallace (2017)
- Huberman, Leshno, Moallemi (2017)

Currency Competition

- Kareken and Wallace (1981)

(Monetary) Theory

- Bewley (1977)
- Townsend (1980)
- Kyotaki and Wright (1989)
- Lagos and Wright (2005)

Outline

The model

- $t = 0, 1, 2, \ldots$. Randomness: θ_t, at beg. of per.. History: θ^t.
- Two types of money: Bitcoins B_t and Dollars D_t (aggregates).
- Assume: Central Bank keeps Dollar price constant, $P_t \equiv 1$.
- Goods (= Dollar) price of Bitcoins: $Q_t = Q(\theta^t)$.
- Two types of infinitely lived agents: green and red.
- Green agent j in even periods t:
  - receives lump sum Dollar transfer ("tax", if < 0) from Central Bank.
  - purchases goods from red agents, with Bitcoins or Dollars.
  - enjoys consumption $c_{t,j}$, utility $\beta^t u(c_{t,j})$.
- Green agents in odd periods t:
  - mines new Bitcoins $A_{t,j} = f(e_{t,j}; B_t)$ at effort $e_{t,j} \geq 0$, disutil. $-\beta^t e_{t,j}$.
  - receives goods endowment $y_{t,j}$. Not storable.
  - can sell goods to red agents, against Bitcoins or Dollars.
- Red agents: flip even and odd periods.
- Assume: whoever consumes first has all the money.

Optimization problem of green agents: (drop "*j*")

$$\text{Maximize} \;\; U = E\left[\sum_{t=0}^{\infty} \beta^t \left(\xi_{t,g} u(c_t) - e_t\right)\right]$$

where $\xi_{t,g} = 1$ in even periods, $\xi_{t,g} = 0$ in odd periods, s.t.

$$
\begin{aligned}
\text{in even periods } t: \quad 0 \leq b_t &\leq Q_t B_{t,g} && (1)\\
0 \leq P_t d_t &\leq D_{t,g} && (2)\\
0 \leq c_t &= b_t + d_t && (3)\\
0 \leq B_{t+1,g} &= B_{t,g} - b_t / Q_t && (4)\\
0 \leq D_{t+1,g} &= D_{t,g} - P_t d_t && (5)\\
\text{in odd periods } t: \quad A_t &= f(e_t; B_t), \text{ with } e_t \geq 0 && (6)\\
y_t &= x_t + z_t, \text{ with } x_t \geq 0,\, z_t \geq 0 && (7)\\
0 \leq B_{t+1,g} &= A_t + B_{t,g} + x_t / Q_t && (8)\\
0 \leq D_{t+1,g} &= D_{t,g} + P_t z_t + \tau_{t+1} && (9)
\end{aligned}
$$

Monetary Policy and Market clearing

- The **Central Bank** achieves $P_t \equiv 1$, per suitable transfers τ_t.
- **Markets clear**:

$$\text{Bitcoin market:} \quad B_t = B_{t,r} + B_{t,g} \tag{10}$$

$$\text{Dollar market:} \quad D_t = D_{t,r} + D_{t,g} \tag{11}$$

$$\text{Bitcoin denom. cons. market:} \quad b_t = x_t \tag{12}$$

$$\text{Dollar denom. cons. market:} \quad d_t = z_t \tag{13}$$

Equilibrium

An equilibrium is a stochastic sequence
$(A_t, [B_t, B_{t,g}, B_{t,r}], [D_t, D_{t,g}, D_{t,r}], \tau_t, (P_t, z_t, d_t), (Q_t, x_t, b_t), e_t)_{t\geq 0}$

- Given prices, choices maximize utility for green and red agents.
  Budget constraints — Evolution money stock
  - $0 \leq b_{t,j} \leq B_{t,j} Q_t$ — $B_{t+1,j} = B_{t,j} - b_{t,j}/Q_t \geq 0$
  - $0 \leq P_t d_{t,j} \leq D_{t,j}$ — $D_{t+1,j} = D_{t,j} - P_t d_{t,j} \geq 0$
  - $B_{t+1,j} = B_{t,j} + x_{t,j}/Q_t + A_{t,j}(e_{t,j})$
  - $B_{t+1,j} = B_{t,j} + x_{t,j}/Q_t + A_{t,j}(e_{t,j})$
- Markets clear (for goods, Bitcoin, Dollars):
  - $y_t = \int_0^2 c_{t,j}\, dj$
  - $\int_0^2 z_{t,j}\, dj = \int_0^2 d_{t,j}\, dj$ — $y_t = x_{t,j} + z_{t,j}$
  - $\int_0^2 x_{t,j}\, dj = \int_0^2 b_{t,j}\, dj$ — $c_{t,j} = b_{t,j} + d_{t,j}$
  - $D_t = D_{t,g} + D_{t,r}$
  - $B_t = B_{t,g} + B_{t,r}$
- Dollar monetary policy: $P_t = 1$

Outline

Consolidate:

$$
\begin{aligned}
B_{t+1} &= B_t + f(e_t; B_t) \\
D_t &= D_{t-1} + \tau_t \\
c_t &= y_t
\end{aligned}
$$

Avoid speculation with Dollars

Assumption A.

Assume throughout: for all t,

$$u'(y_t) - \beta^2 \mathbb{E}_t[u'(y_{t+2})] > 0 \tag{14}$$

Proposition

(All Dollars are spent:) *Agents will always spend all Dollars. Thus, $D_t = D_{t,g}$ and $D_{t,r} = 0$ in even periods and $D_t = D_{t,r}$ and $D_{t,g} = 0$ in odd periods.*

This is a consequence of assumption 14 and $P_t \equiv 1$.

Proposition

(Dollar Injections:) *In equilibrium,*

$$D_t = z_t \quad \text{and} \quad \tau_t = z_t - z_{t-1}$$

Bitcoin Production

Proposition

(Bitcoin Production Condition:) *Suppose that Dollar sales are nonzero, $z_t > 0$ in period t. Then*

$$1 \geq \beta \mathbb{E}_t \left[u'(c_{t+1}) \frac{\partial f(e_t; B_t)}{\partial e_t} Q_{t+1} \right] \quad (15)$$

This inequality is an equality, if there is positive production $A_t > 0$ of Bitcoins and associated positive effort $e_t > 0$ at time t as well as positive spending of Bitcoins $b_{t+1} > 0$ in $t + 1$.

The Fundamental Condition

The following is a version of Kareken-Wallace (1981).

Proposition

(Fundamental Condition:)
Suppose that sales happen both in the Bitcoin-denom. cons. market as well as the Dollar-denom. cons. market at time t *as well as at time* $t+1$*, i.e. suppose that* $x_t > 0$, $z_t > 0$, $x_{t+1} > 0$ *and* $z_{t+1} > 0$*. Then*

$$\mathbb{E}_t \left[u'(c_{t+1}) \right] = \mathbb{E}_t \left[u'(c_{t+1}) \frac{Q_{t+1}}{Q_t} \right] \tag{16}$$

In particular, if consumption and production is constant at $t+1$*,* $c_{t+1} = y_{t+1} \equiv \bar{y}$*, then*

$$Q_t = \mathbb{E}_t \left[Q_{t+1} \right] \tag{17}$$

i.e., the price of a Bitcoin in Dollar is a martingale.

The Speculative Condition

Proposition

(Speculative Condition:)

Suppose that $B_t > 0$, $Q_t > 0$, $z_t > 0$ and that $b_t < Q_t B_t$. Then,

$$u'(c_t) \leq \beta^2 \mathbb{E}_t \left[u'(c_{t+2}) \frac{Q_{t+2}}{Q_t} \right] \tag{18}$$

where this equation furthermore holds with equality, if $x_t > 0$ and $x_{t+2} > 0$.

Seller Participation Condition

Proposition

(Seller Participation Condition:)
Suppose that $B_t > 0$, $Q_t > 0$, $z_t > 0$. *Then*

$$\mathbb{E}_t\left[u'(c_{t+1})\right] \geq \mathbb{E}_t\left[u'(c_{t+1})\frac{Q_{t+1}}{Q_t}\right] \tag{19}$$

The Sharpened No-Speculation Assumption

Assumption A.

For all t,

$$u'(y_t) - \beta \mathbb{E}_t[u'(y_{t+1})] > 0 \tag{20}$$

This is a slightly sharper version of assumption 1, which only required

$$u'(y_t) - \beta^2 \mathbb{E}_t[u'(y_{t+2})] > 0$$

The No-Bitcoin-Speculation Theorem

Theorem

(No-Bitcoin-Speculation Theorem.) *Suppose that* $B_t > 0$ *and* $Q_t > 0$ *for all* t*. Impose assumption 2. Then in every period, all Bitcoins are spent.*

Proof.

$$\begin{aligned}
\beta^2 \mathbb{E}_t[u'(c_{t+2})Q_{t+2}] &= \beta^2 \mathbb{E}_t[\mathbb{E}_{t+1}[u'(c_{t+2})Q_{t+2}]] && \text{(law of iter. expect.)} \\
&\leq \beta^2 \mathbb{E}_t[\mathbb{E}_{t+1}[u'(c_{t+2})] \cdot Q_{t+1}] && \text{(equ. (19) at } t+1\text{)} \\
&< \beta \mathbb{E}_t[u'(c_{t+1})Q_{t+1}] && \text{(ass. 2 at } t+1\text{)} \\
&\leq \beta \mathbb{E}_t[u'(c_{t+1})]Q_t && \text{(equ. (19) at } t\text{)} \\
&< u'(c_t)Q_t && \text{(ass. 2 at } t\text{)}
\end{aligned}$$

Thus, the specul. cond. (18) cannot hold in t. Hence $b_t = Q_t B_t$. □

A (very high) bound for Bitcoin Prices

Corollary

(Bitcoin price bound) *Suppose that* $B_t > 0$ *and* $Q_t > 0$ *for all* t*. The Bitcoin price is bounded by*

$$0 \leq Q_t \leq \bar{Q}$$

where

$$\bar{Q} = \frac{\bar{y}}{B_0} \tag{21}$$

Bitcoin Correlation-Pricing

Rewrite (16) as

$$Q_t = \frac{\text{cov}_t(u'(c_{t+1}), Q_{t+1})}{\mathbb{E}_t[u'(c_{t+1})]} + \mathbb{E}_t[Q_{t+1}] \tag{22}$$

Corollary

(Bitcoin Correlation Pricing Formula:)

Suppose that $B_t > 0$ and $Q_t > 0$ for all t. Impose assumption 2. In equilibrium,

$$Q_t = \kappa_t \cdot corr_t(u'(c_{t+1}), Q_{t+1}) + \mathbb{E}_t[Q_{t+1}] \tag{23}$$

where

$$\kappa_t = \frac{\sigma_{u'(c)|t}\, \sigma_{Q_{t+1}|t}}{\mathbb{E}_t[u'(c_{t+1})]} > 0 \tag{24}$$

where $\sigma_{u'(c)|t}$ is the standard deviation of marginal utility of consumption, conditional on date-t information, etc..

Martingale Properties

Corollary

(Martingale Properties of Equilibrium Bitcoin Prices:) *Suppose $B_t > 0$ and $Q_t > 0$ for all t. Impose ass. 2. If and only if for all t, marg. util. of cons. and Bitcoin price are positively correlated at $t + 1$, given t info, the Bitcoin price is a supermartingale and strictly falls in expectation,*

$$Q_t > \mathbb{E}_t[Q_{t+1}] \tag{25}$$

If and only if marginal utility and the Bitcoin price are always neg. corr.,

$$Q_t < \mathbb{E}_t[Q_{t+1}] \tag{26}$$

If and only if marginal utility and the Bitcoin price are always uncorr., the Bitcoin price is a martingale,

$$Q_t = \mathbb{E}_t[Q_{t+1}] \tag{27}$$

Bitcoin Price Convergence

Theorem

(Bitcoin Price Convergence Theorem.) *Suppose that* $B_t > 0$ *and* $Q_t > 0$ *for all* t*. Impose assumption 2. For all* t *and conditional on information at date* t*, suppose that marginal utility* $u'(c_{t+1})$ *and the Bitcoin price* Q_{t+1} *are either always nonnegatively correlated or always non-positively correlated. Then the Bitcoin price* Q_t *converges almost surely pointwise as well as in* L^1 *norm to a (random) limit* Q_∞*,*

$$Q_t \to Q_\infty \text{ a.s.} \quad \text{and } \mathbb{E}[| Q_t - Q_\infty |] \to 0 \tag{28}$$

Proof.

Q_t or $-Q_t$ is a bounded supermartingale. Apply Doob's martingale convergence theorem. □

Outline

Scenario 1 - Conventional approach

Assume that Bitcoin prices move independently of central bank policies. Impose assumption 2. Then

Proposition

(Conventional Monetary Policy:)
The equilibrium Dollar quantity is given as

$$D_t = y_t - Q_t B_t \tag{29}$$

The central bank's transfers are

$$\tau_t = y_t - Q_t B_t - z_{t-1} \tag{30}$$

Scenario 1 - Conventional approach

Proposition

(Dollar Stock Evolution:)
Tomorrow's expected Dollar quantity equals today's Dollar quantity corrected for deviation from expected production, purchasing power of newly produced Bitcoin and correlation

$$\mathbb{E}_t[D_{t+1}] = D_t - (y_t - \mathbb{E}_t[y_{t+1}]) - A_t Q_t + \kappa_t \, B_{t+1} \cdot corr_t(u'(c_{t+1}), Q_{t+1})$$

Likewise, the central bank's expected transfers satisfy

$$\mathbb{E}_t[\tau_{t+1}] = -(y_t - \mathbb{E}_t[y_{t+1}]) - A_t Q_t + \kappa_t \, B_{t+1} \cdot corr_t(u'(c_{t+1}), Q_{t+1})$$

If the Bitcoin price is a martingale, then

$$\begin{aligned} \mathbb{E}_t[D_{t+1}] &= D_t - (y_t - \mathbb{E}_t[y_{t+1}]) - A_t Q_t \\ \mathbb{E}_t[\tau_{t+1}] &= - (y_t - \mathbb{E}_t[y_{t+1}]) - A_t Q_t \end{aligned}$$

Scenario 2 - Unconventional approach

- Unconventional view, but compatible with equilibrium: the Central Bank can maintain the price level $P_t \equiv 1$ independently of the transfers she sets.
- Further, assume that she sets transfers independently of production.
- Note that

$$Q_t = \frac{y_t - D_t}{B_t} \tag{31}$$

- Intuitively, the causality is in reverse compared to scenario 1: now central bank policy drives Bitcoin prices.
- However, the process for the Dollar stock cannot be arbitrary.
  - To see this, suppose that $y_t \equiv \bar{y}$ is constant. We already know that Q_t must then be a martingale. Suppose B_t is constant as well. Equation (31) now implies that D_t must be a martingale too.

Scenario 2 - Unconventional approach

Proposition

(Submartingale Implication:)
If the Dollar quantity is set independently of production, the Bitcoin price process is a submartingale, $\mathbb{E}_t[Q_{t+1}] \geq Q_t$.

Scenario 2 - Unconventional approach

Suppose that production y_t is iid. Let F denote the distribution of y_t, $y_t \sim F$. The distribution G_t of the Bitcoin price is then given by

$$G_t(s) = \mathbb{P}(Q_t \leq s) = F(B_t s + D_t). \tag{32}$$

Proposition

(Bitcoin Price Distribution:)

In "scenario 2", if Bitcoin quantity or Dollar quantity is higher, high Bitcoin price realizations are less likely in the sense of first order stochastic dominance.

Scenario 2 - Unconventional approach

Compare two economies with $y_t \sim F_1$ vs $y_t \sim F_2$,iid.

Definition

- Economy 2 is **more productive** than economy 1, if F_2 first order stochastically dominates F_1.
- Economy 2 has **more predictable production** than economy 1, if F_2 second order stochastically dominates F_1.

Proposition

(Bitcoins and Productivity)
Assume "scenario 2". In more productive economies or economies with higher predictability of production, the Bitcoin price is higher in expectation.

Outline

Constructing an equilibrium: an example.

- Suppose $\theta_t \in \{L, H\}$, each with probability $1/2$.
- Let m_t be iid, $m_t = m(\theta_t)$, with $m(L) \leq m(H)$ and $\mathbb{E}[m_t] = (m_L + m_H)/2 = 1$. Pick $0 < \beta < 1$ such that $m(L) > \beta$.
- At date t and for $\epsilon(\theta^t) = \epsilon_t(\theta_t)$, consider two cases

  Case A: $\epsilon_t(H) = 2^{-t}$, $\epsilon_t(L) = -2^{-t}$

  Case B: $\epsilon_t(H) = -2^{-t}$, $\epsilon_t(L) = 2^{-t}$.
- Pick $Q_0 > \xi + (m(H) - m(L))/2$. Set

$$Q_{t+1} = Q_t + \epsilon_{t+1} - \frac{\text{cov}_t(m_{t+1}, \epsilon_{t+1})}{E_t[m_{t+1}]}$$

- Fix some strictly concave $u(\cdot)$. Let $y_t = (u')^{-1}(m_t)$.
- Start with some initial B_0. With B_t and Q_t, equation (15) delivers new Bitcoin mining A_t and thus B_{t+1}.
- The No-Bitcoin-Speculation Theorem now implies the purchases $x_t = b_t = Q_t/B_t$ and $z_t = d_t = y_t - b_t$.
- Be careful with B_0, so that $b_t \leq y_t$ for all t. Or: fix "ex post".

Super-, sub-, non-martingale examples

Consider three constructions,

Always A: Always impose case A, i.e. $\epsilon_t(H) = 2^{-t}$, $\epsilon_t(L) = -2^{-t}$. "Always A" results in supermartingale $Q_t > E_t[Q_{t+1}]$.

Always B: Always impose case B, i.e. $\epsilon_t(H) = -2^{-t}$, $\epsilon_t(L) = 2^{-t}$. "Always B" results in submartingale $Q_t < E_t[Q_{t+1}]$.

Alternate:

- In even periods, impose case A, i.e. $\epsilon_t(H) = 2^{-t}$, $\epsilon_t(L) = -2^{-t}$.
- In odd periods, impose case B, i.e. $\epsilon_t(H) = -2^{-t}$, $\epsilon_t(L) = 2^{-t}$.

This results in a price process that is neither a supermartingale nor a submartingale, but which one still can show to converge almost surely and in L_1 norm.

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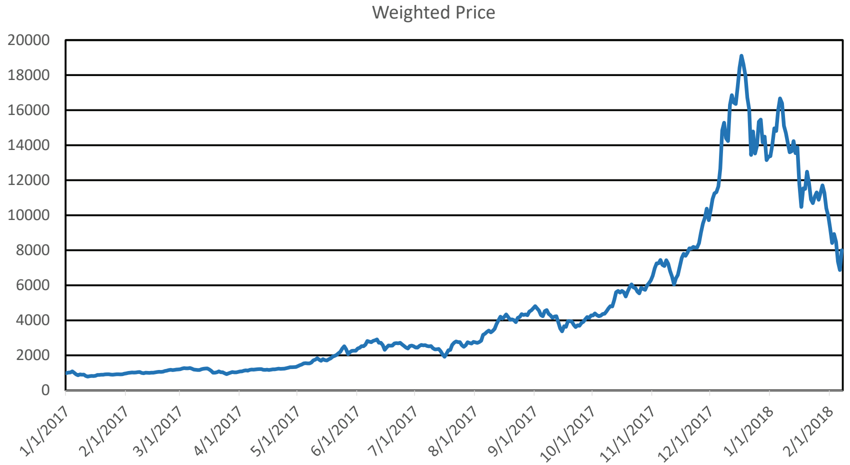

Data: quandl.com

“Bubble and bust” examples

- $\theta_t \in \{L, H\}$, but now $\mathbb{P}(\theta_t = L) = p < 0.5$.
- Suppose that $m(L) = m(H) = 1$.
- Pick some $\underline{Q} > 0$ as well as some $Q^* > \underline{Q}$.
- Pick some $Q_0 \in [\underline{Q}, Q^*]$. If $Q_t < Q^*$, let

$$Q_{t+1} = \begin{cases} \frac{Q_t - p\underline{Q}}{1-p} & \text{if } \theta_t = H \\ \underline{Q} & \text{if } \theta_t = L \end{cases}$$

  If $Q_t \geq Q^*$, let $Q_{t+1} = Q_t$.
- Therefore Q_t will be a martingale and satisfies (22).
- If Q_0 is sufficiently far above $\bar{Q}$ and if p is reasonably small, then typical sample paths will feature a reasonably quickly rising Bitcoin price Q_t, which crashes eventually to $\underline{Q}$ and stays there, unless it reaches the upper bound Q^* first.

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Recap and Conclusions.

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- A central bank keeps real value of Dollars constant...
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