Central Bank Balance Sheet Policies Without Rational Expectations*

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Abstract

We study the effects of central bank balance sheet policies—namely, quantitative easing and foreign exchange interventions—in a model where people form expectations through the level-k thinking process, consistent with experimental evidence on the behavior of people in strategic environments. We emphasize two main theoretical results. First, under a broad set of conditions, central bank interventions are effective under level-k thinking, while they are neutral in the rational expectations equilibrium. Second, while these interventions have a first-order effect on asset prices, they have only a second-order effect on aggregate output. Finally, we empirically show that forecast errors about future asset prices are predictable by balance sheet interventions, a property that differentiates our channel from the alternatives, such as portfolio-balance and signaling channels.

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1 Introduction

The balance sheets of central banks are among the most important stabilization policy tools (Bernanke, 2012; Draghi, 2015; Yellen, 2016). Prominent examples of their use are quantitative easing (QE), which has gained popularity during the recent financial crisis, and foreign exchange (FX) interventions, used widely by emerging countries and by some small open advanced economies.

Despite their popularity, central bank balance sheet policies are not yet well understood. From an empirical perspective, identifying a causal effect of these policies is challenging, as they are usually implemented in response to economic events, thereby creating an endogeneity problem. There is nonetheless evidence, which we discuss in the literature review below, in favor of the effects of QE and FX interventions on asset prices. There is still uncertainty, however, about whether these policies have any effect on real economic activity.

From a theoretical perspective, a wide class of standard macroeconomic models predicts that balance sheet policies are irrelevant, provided that the assets involved are valued for their pecuniary returns only. As noted by Wallace (1981), this *irrelevance result* is like the celebrated Modigliani-Miller proposition applied to central banks. The intuition behind this result relies on two key consequences of rational expectations. First, investors correctly forecast government behavior. When a central bank acquires, for example, private risky assets and, in exchange, creates reserves or sells short-term public bonds, investors correctly understand that any gains or losses incurred on the central bank's portfolio will be directly transferred to the fiscal authority and indirectly, through taxes, back to investors. As a result, investors reduce their individual demand for risky assets so as to hedge against this new tax risk. Second, investors also correctly forecast other investors' behavior. When an investor correctly believes that all other investors form their expectations rationally, she expects a decrease in total demand for risky assets, which exactly compensates the higher demand from the central bank. As a consequence, every investor anticipates that future asset prices will be unaffected by the new policy. Taken together, the two effects imply that current prices will be unchanged, making central bank interventions irrelevant.

In this paper, we argue that balance sheet policies can be effective when people fail to foresee all the future consequences of such policies. Specifically, we solve a non-linear dynamic stochastic general equilibrium model where we relax the assumption of rational expectations. The model is tractable enough to yield all results in closed-form. We show that, with a simple deviation from rational expectations, central bank interventions cease to be irrelevant and, instead, have a sizable impact on asset prices. Importantly, we also show that the effect of such policies on real economic activity is more limited. We then verify empirically some of our theoretical predictions.

We replace rational expectations with the level-k thinking process of belief formation. Laboratory experiments have repeatedly demonstrated that standard equilibrium analysis, which is based on rational expectations, fails to predict people's behavior, especially when players are confronted with novel strategic situations. Instead, the evidence suggests that a process known as level-k thinking is a better description of how players form beliefs about their opponents and, hence, make decisions. More specifically, this process assumes that agents form higher-order beliefs—i.e., beliefs about beliefs ... about the behavior of others—only up to some finite level k, either due to the complexity of the economic environment or because they believe that other agents are less sophisticated.

Level-k thinking can be particularly relevant in macroeconomic settings, especially when people are confronted with new policies—such as QE. The novelty of these policies implies that data on their effects is likely to be scarce, making it harder for households and firms to predict their consequences as well as the behavior of others. Agents must then form beliefs about endogenous variables with little guidance by past experience and by policymakers, thus, they are unlikely to hold rational expectations. Level-k thinking is then a more plausible description of their behavior: it does not require agents to have any experience with the new policy and, yet, it allows them to be forward-looking.

We construct beliefs under level-k thinking as follows. First of all, all agents are assumed to be perfectly aware of current balance sheet policies as well as of their own income and asset positions. However, each agent is characterized by a "level of thinking," which determines her expectations about the effects of balance sheet policies on future endogenous variables, such as transfers from the central bank to the treasury, taxes, and asset prices. More specifically, expectations are constructed according to an iterative procedure. First, "level-1 thinking" posits that, after observing the policy change, agents do not update their expectations about future endogenous variables. As a result, "level-1 thinkers" make consumption and portfolio decisions under their old expectations; in particular, they do not hedge against the future tax risk, as required for the irrelevance result to hold. Next, level-2 thinkers believe that the economy is populated only by level-1 thinkers. Thus, upon observing a policy change, they expect future variables to coincide with the equilibrium outcomes of an economy populated only by such agents. Notice that, unlike level-1 thinkers, these more sophisticated agents do revise their expectations following a policy intervention. However, we show that this revision is not enough to reach rational expectations. Proceeding recursively, we can define the expectations and, hence, the behavior of level-k thinkers, for any finite k. Having characterized the expectations of every agent, we compute the equilibrium of an economy populated by agents with different levels of thinking. We call the resulting notion of equilibrium *reflective* equilibrium.[1]

Our first main result shows that, when agents are level-k thinkers, balance sheet policies have an impact on asset prices. Thus, the irrelevance result under rational expectations is overturned. The intuition is as follows. Since agents do not hold rational expectations about future endogenous variables, they underestimate the tax risk emanating from policy interventions and incorrectly forecast the behavior of future assets prices. As a result, they demand lower risk premia, which boosts asset prices and makes balance sheet policies effective. Interestingly, even when all the agents correctly understand the tax risk, which is the case in our model when agents are level-2 thinkers or higher, balance sheet policies may still be effective. This happens because even sophisticated agents—i.e., those with a high level of thinking—fail to form all the higher-order beliefs correctly: while these agents predict the tax risk correctly, they believe that other agents will have incorrect beliefs, or that other agents believe that other agents will have incorrect beliefs, and so on.

Perhaps surprisingly, the strength of balance sheet policies can increase with average level of sophistication of agents in the economy, defined as the average level-k across all agents. This apparently counterintuitive result occurs because an increase in the sophistication has two opposing effects. On the one hand, more sophisticated agents foresee the fiscal consequences of balance sheet policies, bringing the policy closer to full neutrality. On the other hand, more sophisticated agents become endogenously more forward-looking, increasing the strength of persistent balance sheet interventions. When the average level of sophistication of agents tends to infinity, however, the reflective equilibrium converges to the rational expectations equilibrium.

To derive our results on risk premia and asset prices in the most transparent way, we first focus on an endowment economy, which we call "the simple model". However, the model only features *real* assets. We then extend the simple model to explicitly account for the fact that a large share of QE policies took the form of purchases of long-term *nominal* public bonds financed by creating *nominal* reserves (or selling short-term public bonds). We show that, while these alternative policies are once again irrelevant under rational expectations, they impact the prices and, hence, the returns on long-term bonds in the reflective equilibrium of our model. Moreover, in a two-country extension of the simple model, we show that also purchases of foreign bonds paid by selling domestic bonds—a policy known as "sterilized" FX intervention—have an impact on the exchange rate in the reflective equilibrium, while they are completely neutral in the rational expectations

[1] Angeletos and Lian (2017) point out that a notion of reflective equilibrium "smooths out" some of the unappealing properties of level-k equilibria (i.e., a reflective equilibrium with a degenerate distribution over k).

equilibrium. The intuition behind these two results relies on the fact that long-term nominal bonds and foreign bonds may be safe in the currency of their denomination, however, they carry inflation and foreign exchange risks (and possible a default risk) making them risky in real terms. As a result, when purchasing long-term or foreign bonds, the central bank takes a risk on its balance sheet, thus, the logic of our first result applies to these extensions.

Our second main result derives the response of aggregate output to the balance sheet policies. To do this, we extend the simple model to an environment with nominal rigidities in which output is determined by aggregate demand. We show that while central bank interventions have a first-order effect on asset prices, they only have a second-order effect on aggregate output. As in the simple model, when the central bank conducts balance sheet interventions, some agents fail to understand, or think that other agents will fail to understand, that the assets purchased by the central bank increase the risk of future taxes. As a result, they incorrectly infer that the amount of aggregate risk in the economy has fallen. They thus have a lower desire for precautionary saving and increase consumption demand and, in general equilibrium, output. Crucially, the increase in aggregate demand is proportional to change in the variance of tax risk, which is second order in the size of the policy intervention. This is in contrast to the effect on asset prices which, instead, is proportional to the supply of assets available to the private sector. Taken together, these results could rationalize the lack of conclusive empirical evidence about the effects of asset purchases on aggregate output, even though the empirical literature has been more successful at identifying the effects of such policies on asset prices. Importantly, our second main result warns us that the empirical studies that find effects on asset prices do not automatically imply that these policies have also a significant impact on the real economy.

The last part of the paper presents specific predictions of our model that can differentiate it from the models that stress other channels of balance sheet policies. Specifically, we focus on the behavior of forecast errors of asset prices after policy interventions. We show that individual and cross-sectional-average forecast errors are related to policy interventions. Predictable forecast errors are absent in the standard models that assume limited market participation or signaling by the central bank, but retain the assumption of full information rational expectations. Moreover, in models with incomplete information in which agents form expectations rationally, predictable forecast errors would arise only if agents had imperfect information about policy interventions. If, instead, the policy was well advertised and, as a result, agents had complete information about it, forecast errors would not be predictable. In contrast, in our model, agents are fully aware of the policy intervention, yet they make mistakes due to their inability to form rational expectations.

Finally, we empirically document the predictability of cross-sectional-average forecast errors by balance sheet policies. We focus on the mortgage market in the US and proxy balance sheet interventions with purchases of mortgages by government-sponsored enterprises (GSEs), such as Fannie Mae and Freddie Mac. This approach utilizes the fact that GSEs' purchases of mortgages resemble Fed's acquisition of mortgage backed securities, as argued in Fieldhouse, Mertens and Ravn (2018). We follow these authors, who identify "exogenous and unexpected" changes in mortgage purchases by the GSEs using a narrative approach in the spirit of Romer and Romer (2010). As predicted by our model, we first verify that these exogenous changes in mortgage purchases affect the conventional mortgage rate. We then use the Blue Chip Financial Forecasts survey data to show that exogenous purchases by the GSEs also predict conventional mortgage rate forecast errors. Using these empirical estimates together with our stylized model, we calculate that the number of level-1 thinkers (those who do not change their expectations after policy interventions) in the data is 86 percent.

Related literature. Our paper is related to Evans and Ramey (1992), García-Schmidt and Woodford (2019), and Farhi and Werning (2017), who use level-k thinking in macroeconomic models to study the effects of conventional monetary policy and forward guidance, but abstract from aggregate risk.[2] Our mechanism, instead, hinges entirely on aggregate risk: agents incorrectly believe that balance sheet policies "remove" a part of aggregate risk from the economy or that other agents believe that this is so, hence, they demand lower risk premia. Moreover, the focus of our paper is on balance sheet policies.

Level-k thinking that we use in this paper is related to other deviations from full information rational expectations in macroeconomics. Level-k thinking generates endogenous discounting of expectations about future endogenous variables, which can rationalize the exogenous discounting introduced in, for example, Gabaix (2016). Level-k thinking dampens the response to shocks of higher-order belief in a way similar to models with information frictions, such as noisy information (e.g., Angeletos and Huo, 2018), sticky information (e.g., Mankiw and Reis, 2002), and rational inattention (e.g., Maćkowiak and Wiederholt, 2009). Section 4 highlights the differences between our model with level-k thinking and models with information frictions. Last but not least, level-k thinking is a forward-looking process, thus, it is different from the so-called "inductive approaches" to

[2] The level-k thinking belief-formation process has been widely used in behavioral game theory to rationalize the behavior of subjects playing full-information games in various laboratory and field experiments (Stahl and Wilson, 1995; Nagel, 1995; Bosch-Domenech et al., 2002). Crawford et al. (2013) provide a recent review of level-k thinking in game theory. Experiments show that deviations from Nash-equilibrium behavior in many simple games are most stark on the first round of play, when agents face novel strategic environments, with no prior experience. In these games, subjects usually exhibit levels of thinking no higher than 3.

belief formation, such as statistical learning (Evans and Honkapohja, 2012; De Grauwe, 2012).

We contribute to the literature that studies the effectiveness of balance sheet policies. On the empirical side, there is evidence that these policies affect asset prices, but only mixed evidence of their impact on real activity. Using high-frequency financial data, Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2011), and Hancock and Passmore (2011) found that large-scale purchases of mortgage-backed securities (MBS) by the United States Federal Reserve affected mortgage market yields. Fieldhouse et al. (2018) show that purchases of mortgages by government-sponsored enterprises in the US, which resemble a policy of QE, affected mortgage rates, but not aggregate output or consumption. Dominguez and Frankel (1990, 1993) are early studies that estimate significant effects of sterilized FX interventions. More recently, Kearns and Rigobon (2005), Dominguez et al. (2013) and Kohlscheen and Andrade (2014), Chamon et al. (2017) find a significant effect of FX interventions on exchange rates using, respectively, a "natural experiment," high-frequency identification, and a synthetic control technique. Regarding real economic activity, using a sign-restriction VAR identification and a sample ranging from 2009M3 to 2014M5, Weale and Wieladek (2016) found a significant effect of QE on aggregate output in the US and UK. At the same time, Fieldhouse et al. (2018) estimate a small and insignificant effect of government asset purchases on real economic activity using narrative identification and a sample of US data ranging from 1967M1 to 2006M12. In Section 4, we extend the analysis in Fieldhouse et al. (2018) and show that balance sheet interventions not only affect asset prices, but also predict asset returns forecast errors of financial market experts. The use of forecast errors connects our exercise to a growing literature that studies the responses of forecast errors to various macroeconomic shocks, e.g., Coibion and Gorodnichenko (2012).

On the theory side of balance sheet interventions, an important starting point is the irrelevance result in Wallace (1981). Backus and Kehoe (1989) show the irrelevance of *sterilized* foreign-exchange interventions in an international setting. To deviate from the irrelevance result, the literature has proposed to add various frictions, such as incomplete information and market segmentation. The former friction generates the so-called "signaling" channel and the latter generates the "portfolio balance" channel. According to the signaling channel, changes in the composition of a central bank's balance sheet do not have a direct effect on the economy. Instead, they serve as a signal of the central bank's objectives or information about economic fundamentals (see Mussa, 1981).[3] The portfolio-balance channel posits that changes in the supplies of different assets affect as-

[3] Some papers proposed that central banks cannot commit to a desired monetary policy and use balance sheet policies as a costly signal about future intentions (see Jeanne and Svensson, 2007 and Bhattarai et al., 2015 for a discussion of FX and QE interventions, respectively).

set prices due to the segmentation of assets markets. Segmentation can occur because of fixed costs of entry or because of limited market participation by yet-unborn people as in models with overlapping generations. Kouri (1976) is an early paper and Gabaix and Maggiori (2015) is a more recent contribution that apply this idea to FX interventions. Curdia and Woodford (2011) study quantitative easing with market segmentation. Vayanos and Vila (2009) provide a framework to study asset purchases when markets are segmented. Finally, Krishnamurthy and Vissing-Jorgensen (2011) summarize the recent literature on quantitative easing.[4] In this paper, we propose a "bounded rationality" channel of balance sheet policies, derive the implications that can distinguish it from other prominent channels, and provide empirical support for it. Importantly, we demonstrate that the balance sheet policies can affect asset prices without having a big impact on real economic activity.

The rest of the paper is organized as follows. Section 2 presents the simple model and considers purchases of private risky assets by the central bank. Section 3 extends the model by allowing for endogenous output and derives the implications of balance sheet policies for aggregate output. Section 4 presents and tests the implications of the model. Section 5 concludes. Appendix A presents omitted proofs, Appendix B provides details of our empirical exercise, and the Online Appendix, which is currently attached to the paper, extends the model in various ways.

2 Balance Sheet Policies and Asset Prices

We now present a model that we will refer to as the "simple model." The model is a standard Lucas tree economy, with the exception that agents form their expectations according to the level-k thinking process. We use this model to investigate the effects on the asset prices of purchases of private risky assets by the central bank. The purchases of mortgage-backed securities by the Federal Reserve during the Great Recession (also referred to as QE1) are an example of such policies.

2.1 Assets, Agents, and Expectations

Time is discrete, infinite, and indexed by $t = 0, 1, 2, \ldots$. There are two assets in the economy. First, there is a one-period riskless asset, available in perfectly elastic supply, that

[4] Other recent contribution to FX interventions are Fanelli and Straub (2016), Amador et al. (2017), Cavallino (2017), and to QE policies are Chen et al. (2012), Silva (2016), Reis (2017), and Sterk and Tenreyro (2018). In Cui and Sterk (2018), central banks stimulate the economy by substituting illiquid assets for more liquid ones. Goncharov et al. (2017) propose a political economy explanation of the effects of balance sheet policies by noting that central bankers are averse to make losses under greater political pressure.

yields a real net return $r > 0$. Second, there is a risky asset, available in fixed supply $\overline{X}$, that entitles the owner to a stream of dividends $\{D_t\}$ and trades at price q_t in period t. The dividend on the risky asset is $D_t = \overline{D} + \epsilon_t^x$, where $\overline{D}$ is constant and ϵ_t^x is assumed to be random, independent over time and normally distributed, with zero mean and standard deviation σ_x.

Households. There are infinitely-lived households in the economy, who are identical except, perhaps, for their beliefs. We describe beliefs in detail below; for now, we use a tilde on top of the expectation operator to emphasize the fact that households may use a probability distribution over future variables that differs from their actual distribution. Households have constant absolute risk aversion (CARA) preferences with risk aversion γ and discount factor $e^{-\rho}$, with $\rho > 0$. At time 0, given beliefs and prices, households choose consumption $\{c_t\}$, investment in the safe asset $\{b_{t+1}\}$, and investment in the risky asset $\{x_{t+1}\}$, to maximize

$$\widetilde{\mathbb{E}}_0 \left[-\frac{1}{\gamma} \sum_{t=0}^{\infty} e^{-\rho t - \gamma c_t} \right], \tag{1}$$

subject to the flow budget constraint

$$c_t + b_{t+1} + q_t x_{t+1} \leq W_t - T_t + (1+r)b_t + (D_t + q_t)x_t, \tag{2}$$

where T_t is lump-sum taxes, W_t is an endowment process, for example, labor income, which, for simplicity, we assume to be non-stochastic (Section 3 relaxes this assumption).

It is worth commenting on our choice of preferences and distribution of shocks. It is well known that exponential preferences with Gaussian shocks deliver a closed-form solution to the agent's portfolio problem, even in the presence of uninsurable risk. In fact, the combination of CARA preferences and Gaussian shocks has been the workhorse assumption in the finance literature starting from the seminal contribution of De Long et al. (1990).[5] We take advantage of these properties in our paper.[6]

Government. The government consists of a treasury and a central bank. The treasury conducts fiscal policy and the central bank implements purchases of private risky assets. Government policies will be denoted by capital letters. The treasury controls real per capita lump-sum taxes $\{T_t\}$ and the *real* amount of one-period public bonds $\{B_{t+1}\}$. The central bank manages the real purchases of private risky assets $\{X_{t+1}\}$, the amount of

[5] Barberis et al. (2015) extend this type of model by adding a fraction of agents who form beliefs by extrapolating past realizations of endogenous variables into the future.

[6] Note, however, that the particular functional forms will not be responsible for the qualitative results that balance sheet policies are neutral in the REE and effective in the reflective equilibrium.

outstanding reserves $\{R_{t+1}\}$, which are perfect substitutes of safe assets, and the transfers to the treasury $\{Tr_t\}$. We let $\Pi_t \equiv \{T_t, B_{t+1}, X_{t+1}, R_{t+1}, Tr_t\}$ denote the collection of government policies.

It is worth noting that here central bank reserves are valued for their pecuniary returns only. This is in contrast to macroeconomic models that assume non-pecuniary benefits of reserves. Importantly, the results in this paper are also relevant to those models. First, many actual instances of balance sheet policies occurred during liquidity traps when non-pecuniary benefits of reserves approach zero. Second, often balance sheet policies change only the composition of central bank assets, as in sterilized interventions, without altering the size of the balance sheet. In our discussion below, we formally model balance sheet policies by assuming that a central bank creates reserves. Because here reserves are perfect substitutes of short-term public debt, the results will be identical if, instead, the central bank financed its purchases by selling treasury bonds.

The intertemporal budget constraint of the treasury is

$$(1+r)B_t = \sum_{s=0}^{\infty} \frac{T_{t+s}}{(1+r)^s} + \sum_{s=0}^{\infty} \frac{Tr_{t+s}}{(1+r)^s}, \tag{3}$$

where we assumed that public debt does not grow "too fast," i.e., $\lim_{s \to 0} B_{t+s+1}/(1+r)^s = 0$. The left-hand side of equation (3) represents treasury's outlays, consisting of repayment of outstanding debt, $(1+r)B_t$. The right-hand side is government income. The treasury finances itself by raising taxes and by receiving transfers from the central bank (which can be negative).

We assume that in period 0 the central bank announces, and commits to, the entire path of risky asset purchases $\{X_{t+1}\}$ and finances it by creating reserves such that[7]

$$R_{t+1} = q_t X_{t+1}. \tag{4}$$

We refer to these purchases as "quantitative easing." Equation (4) states that the central bank does not reinvest its profits, but it rebates them to the Treasury. This approximates the way the Federal Reserve in the US and the European Central Bank in the Eurozone operate. As a result, the budget constraint of the central bank simplifies to

$$Tr_t = (D_t + q_t)X_t - (1+r)R_t. \tag{5}$$

[7] Angeletos and Sastry (2018) show when it is optimal to the central bank to announce a policy target in a model without rational expectations.

Expectations. We allow household expectations to deviate from rational expectations. More precisely, consistent with the idea that households understand policy announcements, but may be unable (or may think that other agents may be unable) to solve for the rational expectations equilibrium of the economy, we make the following assumptions. First, for future *exogenous* variables—that is, $\{\epsilon_t^x\}$—we assume that expectations coincide with the *true* distribution of such variables.[8] Second, for future *endogenous* variables, we assume that expectations are described by a sequence of one-period-ahead conditional distributions. More precisely, letting Z_{t+1} denote the vector of endogenous variables at some future time $t+1$, we assume that, conditional on information at t, households expect Z_{t+1} to be distributed according to some cumulative distribution function $\widetilde{\phi}_t$.[9] Given these one-period-ahead conditional distributions, it is immediate to derive n-period-ahead distributions, for any n. In the model of this section, $Z_t = (q_t, T_t, Tr_t, B_{t+1}, X_{t+1}, R_{t+1})$. For convenience, we denote the sequence of one-period-ahead beliefs starting from period t as $\widetilde{\Phi}_t \equiv \{\widetilde{\phi}_s\}_{s \geq t}$.

We assume that households' expectations $\widetilde{\Phi}_t$ are such that the intertemporal budget constraint of the treasury (3) is satisfied. Specifically, after observing a change in current revenues or spending by the treasury, households correctly anticipate the changes in future revenues or spending that are required to satisfy the treasury's intertemporal budget constraint. This is a realistic assumption when, for example, the treasury follows a fiscal rule that ensures that the intertemporal budget constraint is satisfied and the households have had enough time and data to become familiar with such a rule. To keep the analysis as general as possible, we do not specify any particular fiscal rule and only assume that households' expectations are consistent with the intertemporal budget constraint (3). As a result, Ricardian equivalence (Ricardo, 1821; Barro, 1974) will hold when it comes to treasury's policies and, in particular, it will not matter whether the transfers from the central bank are rebated immediately to the households or at any other date in the future.[10]

Crucially, however, we do not require households' expectations to be consistent with the central bank's budget constraint (5). As a result, some households may fail to understand that the central bank sends any profits it collects on its risky portfolio to the treasury. This assumption formalizes the idea that, due to absence of historical data, households

[8] In a recent work, Gennaioli et al. (2015) show that, by over-weighing the future likelihood of events that occurred in the recent past, investors may neglect the risk of a financial crisis. Also, Bordalo et al. (2016) study the consequences of incorporating "diagnostic expectations" on the volatility and predictability of credit spreads. In contrast, we assume that agents' expectations about future shocks coincide with their true distribution. It would be interesting to combine "diagnostic expectations" with level-k thinking. We leave this exercise for future research.

[9] We use a "tilde" to stress that the distribution $\widetilde{\phi}_t$ can potentially differ from the distribution ϕ_t implied by the equilibrium.

[10] The main result of this section does not change if we assume that expectations do not satisfy the intertemporal budget constraint of the treasury.

are unable to forecast the future behavior of the central bank, when the latter moves into "uncharted territory," such as during the conduct of quantitative easing, when central bank profits become significant.

2.2 Beliefs and Equilibrium Concepts

When studying deviations from rational expectations, it is useful to start with a more general notion of equilibrium known as temporary equilibrium (Hicks, 1939; Lindahl, 1939; Grandmont, 1977; Woodford, 2013). A temporary equilibrium generalizes the standard notion of rational expectations equilibrium by relaxing the assumption that beliefs about endogenous variables must be consistent with the equilibrium distribution of such variables. Instead, in a temporary equilibrium, agents' beliefs are free to deviate from their equilibrium counterparts. More specifically, a temporary equilibrium takes as given household beliefs about future endogenous variables and requires only that *(i)* households optimize given these beliefs and that *(ii)* markets clear in every period.

Definition (Temporary Equilibrium)**.** Conditional on beliefs $\{\widetilde{\Phi}_t\}$, a temporary equilibrium is a collection of household choices $\{c_t, x_{t+1}, b_{t+1}\}$, government policies $\{\Pi_t\}$, and prices $\{q_t\}$ such that

1. Given beliefs and prices, households optimize for all t;
2. The risky-asset market clears for all t

$$x_{t+1} + X_{t+1} = \overline{X}, \tag{6}$$

3. The treasury and central bank's budget constraints (3), (4), and (5) are satisfied for all t.

Notice that the equilibrium definition does not feature the bonds market clearing condition, because we assumed that the supply of bonds is perfectly elastic and it satisfies any demand from households (at the net return r). Finally, the goods market clears by Walras' law.

In what follows, it will be convenient to write the equilibrium mapping more compactly. Let $\widetilde{\Phi}_t$ capture agents' expectations. By definition, a temporary equilibrium is a collection of (potentially stochastic) endogenous variables, which satisfy household optimality, market clearing, and budget constraints for every t. Equilibrium variables can then be described with a sequence of one-period ahead conditional distributions, which we denote with $\Phi_t \equiv \{\phi_s\}_{s\geq t}$. As a result, we can represent a temporary equilibrium as a mapping from beliefs to equilibrium distributions, which we formally write as

$$\Phi_t = \Psi(\widetilde{\Phi}_t, \{X_{t+1}\}). \tag{7}$$

In general, the sequence of future distributions Φ_t may differ from the original sequence of household beliefs $\widetilde{\Phi}_t$, except when agents hold rational expectations.[11]

Definition (Rational Expectations Equilibrium)**.** A REE is a temporary equilibrium that satisfies

$$\widetilde{\Phi}_t = \Phi_t, \text{ for all } t.$$

Note that REE beliefs are a fixed point of the mapping (7).[12]

Level-*k* process of belief formation. The definition of temporary equilibrium is silent about the origin of beliefs. We now consider a specific process of belief formation, known as level-*k* thinking, where *k* denotes the level of sophistication of an agent. By assumption, agents know the correct model of the economy, including the distribution of exogenous variables, and understand policy announcements. Instead, the process introduced in this section concerns agents' beliefs about future *endogenous* variables.

We start by assuming that, before the policy intervention, the economy is in its REE, that is, all agents hold rational expectations about future variables. We begin with level-1 agents, the lowest level of sophistication. We assume that, after the policy intervention in period $t = 0$, these agents do not change their beliefs about future endogenous variables. Formally, the beliefs of level-1 agents are $\widetilde{\Phi}_t^1 = \Phi_t^{SQ}$, for all periods $t \geq 0$, where the additional superscript denotes "level-1" beliefs and Φ_t^{SQ} denotes the distribution of endogenous variables in the REE *before* the policy intervention. We refer to this particular REE as the "status quo."[13] Having specified beliefs of level-1 agents, we can use the mapping (7) to obtain the distributions generated in the temporary equilibrium of an economy populated by level-1 thinkers. We then move to level-2 agents and assume that their

[11] The discrepancy between beliefs and equilibrium outcomes can in principle open the door to learning. The notion of temporary equilibrium, however, does not allow households to update their beliefs when observing equilibrium variables, such as prices. We make this assumption to emphasize the implications of non-rational expectations, which is the novel channel of this paper. We discuss this possibility of learning in Section 2.7.

[12] It is standard in macroeconomics to define rational expectations equilibrium by requiring that a perceived law of motion equals an actual law of motion of variables (Ljungqvist and Sargent, 2012). In our notation, the sequence $\widetilde{\Phi}_t$ represents the perceived law of motion and Φ_t is the actual law of motion. We do not summarize these laws of motion with functions or conditional distributions (as it is usually done in macroeconomics), but rather with a *sequence* of conditional distributions, because government policy $\{X_{t+1}\}$ may take a non-recursive form.

[13] Note that the beliefs of level-1 agents do not incorporate the effects of policy interventions, neither at the time when the policy is announced nor at any later date. This will not be the case if agents can update their beliefs over time, for example, through learning. We discuss this possibility in Section 2.7.

beliefs coincide with the temporary equilibrium distributions just obtained. Proceeding recursively, we can define the beliefs of level-k agents for any $k \geq 1$.[14]

Formally, level-k agents' beliefs are defined as follows. Given level-k thinkers' beliefs $\widetilde{\Phi}_t^k$, $k \geq 1$, we use (7) to obtain the distributions of endogenous variables in the temporary equilibrium, for all t. We then assume that these distributions coincide with the beliefs of level-$(k+1)$ thinkers. The entire process of belief formation is thus described by the following recursion:

$$\widetilde{\Phi}_t^{k+1} = \Psi(\widetilde{\Phi}_t^k, \{X_{t+1}\}), \tag{8}$$

for all $k \geq 1$ and $t \geq 0$.

Reflective equilibrium. Having defined the beliefs of level-k thinkers, for any k, we introduce the notion of equilibrium that we will use to investigate central bank interventions. We follow García-Schmidt and Woodford (2019) and consider an economy populated by households who are heterogeneous in their levels of sophistication k. In particular, the population is divided into different groups depending on their beliefs. Each group contains households with the same level of sophistication k and has a mass given by the probability density function $f(k) \geq 0$, with $\sum_{k=1}^{\infty} f(k) = 1$. One advantage of this approach is that the economy is not indexed by a discrete level of sophistication. Instead, by changing the mean of $f(k)$, we can vary the average level of sophistication in the economy and perform comparative statics in a continuous way.

Definition (Reflective Equilibrium)**.** A reflective equilibrium is a collection of beliefs $\{\widetilde{\Phi}_t^k\}_k$, household choices $\{c_t^k, x_{t+1}^k, b_{t+1}^k\}$, government policies $\{\Pi_t\}$, and prices $\{q_t\}$ such that

1. Given beliefs and prices, households optimize for all t;

2. The risky-asset market clears for all t

$$\sum_{k=1}^{\infty} f(k) x_{t+1}^k + X_{t+1} = \overline{X}; \tag{9}$$

3. The treasury and central bank budget constraints (3), (4), and (5) are satisfied for all t;

4. Beliefs are generated through the mapping (8), starting from $\widetilde{\Phi}_t^1 = \Phi_t^{SQ}$, for all t.

[14] Camerer et al. (2004) propose a related model of "cognitive hierarchy" in which level-k thinkers assume that the other players are not only level-$(k-1)$, like in this paper, but also level-$(k-2)$ and so on. This alternative assumption retains most of the tractability of level-k thinking, but outperforms it in some applications. We can incorporate this alternative into our model at the expense of tractability. The main results will be unchanged.

2.3 Equilibrium Effects of Risky Assets Purchases

We now solve the household problem and then derive the temporary equilibrium for a general sequence of balance sheet policies.

In what follows, it will be useful to impose more structure on beliefs. Specifically, in this section we assume that every element of $\widetilde{\Phi}_t$, for example, $\widetilde{\phi}_s$, $s \geq t$, is such that the vector of the following endogenous variables $z_{s+1} \equiv (q_{s+1}, Tr_{s+1})$ can be represented as a linear function of the contemporaneous shock:

$$z_{s+1} = \alpha_s + \beta_s \epsilon^x_{s+1}, \tag{10}$$

where α_s and β_s are (vectors of) deterministic functions of time. We use a subscript i, $i \in \{q, Tr\}$, to denote any element of α_s and β_s. Here, α_s represents the expected value of z_{s+1}, while β_s captures the expected sensitivity of z_{s+1} to the aggregate shock. Note that we do not restrict endogenous variables other than z_{s+1}, because they either do not enter the household problem directly—this is the case of B_{t+2}, X_{t+2}, and R_{t+2}—or they can be expressed through q_{s+1} and Tr_{s+1}—this is the case of taxes T_{t+1}.

While the assumption of linearity might seem restrictive at this stage, it turns out that it will be implied by all our notions of equilibrium. This result follows from the fact that the mapping (8) preserves linearity and that the initial condition is given by REE variables, which take the form of (10).

Since preferences are exponential, beliefs are linear, and shocks are normally distributed, the household demand for risky assets has a simple closed-form solution.

Lemma 1. *Suppose household beliefs satisfy (10). In equilibrium, household asset demand satisfies*

$$x_{t+1} = \frac{\widetilde{\mathbb{E}}_t(D_{t+1} + q_{t+1}) - (1+r)q_t}{\gamma \frac{r}{1+r}\sigma_x^2} - \beta_{Tr,t}. \tag{11}$$

This result, which we prove in Appendix A.1, is well known in the finance literature. The first term on the right-hand side of (11) shows that the demand for risky assets is proportional to their excess return and inversely proportional to the coefficient of absolute risk aversion γ times the volatility of excess returns. The second term captures a hedging motive coming from the fact that the return on the risky asset may correlate with net labor income, which, in our simple setting where income is non-stochastic, coincides with the correlation between asset returns and taxes. Importantly, this correlation may vary when, as it will be the case here, the central bank conducts balance sheet policies and households realize that assets returns will affect future transfers to the treasury and, hence, taxes. If, for example, households expect future transfers to be positively correlated to the return on risky assets or, equivalently, future taxes to be negatively correlated with the risky return, then the risky assets will be a bad hedge against future tax risk. As a result, demand (11) will be lower.

A convenient property of (11) is that it does not depend on the optimal choice of consumption nor on investment in the riskless asset. As a result, we can impose the market-clearing condition in the risky-asset market, i.e., equation (6), and solve for the endogenous price without reference to the other markets. Formally, using (10), we can express the risky-asset price at time t as

$$q_t = \frac{\overline{D} + \alpha_{q,t} - \frac{r}{1+r}\gamma\sigma_x^2(\overline{X} - X_{t+1} + \beta_{Tr,t})}{1+r}. \tag{12}$$

Equation (12) illustrates that the risky-asset price in the temporary equilibrium is, in general, a function of both government holdings of risky assets and of household beliefs about future prices and transfers.

Finally, equilibrium transfers and creation of reserves by the central bank are obtained by substituting the pricing equation (12) into equations (4) and (5), which characterize the behavior of the central bank. Taken together, equations (3)-(5) and (12) define the mapping in (7).

2.4 Neutrality under Rational Expectations

We next solve for the response of the economy to balance sheet policies in the REE, in which expectations are linear in fundamental shocks as in (10). By definition, in a REE subjective beliefs must be equal to equilibrium distributions. In particular, since the contemporaneous realization of the shock does not appear in equation (12), the equilibrium asset price must satisfy $\beta_{q,t} = 0$, implying

$$q_{t+1} = \widetilde{\mathbb{E}}_t q_{t+1} = \alpha_{q,t}.$$

The fact that the asset price is independent of the aggregate shock is not surprising, since neither demand nor supply of risky assets are stochastic. In addition, from the budget constraint of the central bank (5), we conclude that beliefs about transfers must satisfy $\beta_{Tr,t} = X_{t+1}$. With these two observations, we can rewrite equation (12) in a familiar way:

$$q_t = \frac{\overline{D} + q_{t+1} - \frac{r}{1+r}\gamma\sigma_x^2\overline{X}}{1+r}.$$

This standard asset pricing equation shows that the current price equals the discounted sum of expected dividends and future resale price minus the risk premium. We can then solve the above equation forward to obtain

$$q_t = \frac{1}{r}\left(\overline{D} - \gamma\frac{r}{1+r}\sigma_x^2\overline{X}\right) \equiv q^*. \tag{13}$$

The following proposition summarizes the key property of q^*.

Proposition 1. *In the REE, the price of the risky asset does not depend on balance sheet policies.*

Proposition 1 states that, when all the agents in the economy anticipate future central bank transfers correctly, balance sheet interventions are irrelevant. This result is similar to the one obtained by Wallace (1981). The intuition is simple. When expectations are rational, households correctly anticipate that future taxes will inherit the stochastic properties of the assets purchases by central bank and, thus, adjust their demand for such assets. In the end, equilibrium prices are unaffected.

2.5 Non-neutrality under Level-k Thinking

We now depart from rational expectations and assume that, following an announcement of interventions in period $t = 0$, households form expectations following the level-k process described in Section 2.2. As explained in that section, we assume that, before the announcement, the economy is in the REE. Thus, absent a policy change, households would correctly forecast the behavior of future prices and transfers. Formally, status-quo beliefs $\widetilde{\Phi}_t^{SQ}$ are a fixed point of (7) when $X_{t+1} = 0$, for all t.

Let q_t^k denote the asset price in the temporary equilibrium when all agents hold level-k beliefs. The pricing equation (12) implies

$$q_t^k = \begin{cases} \dfrac{\overline{D} + q^* - \frac{r}{1+r}\gamma\sigma_x^2(\overline{X} - X_{t+1})}{1+r}, & k = 1, \\ \dfrac{\overline{D} + q_{t+1}^{k-1} - \frac{r}{1+r}\gamma\sigma_x^2\overline{X}}{1+r}, & k > 1. \end{cases} \tag{14}$$

By assumption, this price coincides with level-$(k+1)$ agents' beliefs.

To understand equation (14), remember that, when solving their maximization problem, households need to form expectations about next period's asset price and transfers. The first line of equation (14) reflects the fact that, following a policy of asset purchases, level-1 agents do not revise their expectations. Instead, they think that the following period asset price and transfers will coincide with the status quo before the policy intervention, where $q_{t+1} = q^*$ and $Tr_{t+1} = 0$. In particular, the latter, which follows from equation

(3) with $X_{t+1} = 0$, $t \geq 0$, imply that level-1 agents fail to understand that transfers are now risky and, hence, so are taxes. Instead, level-1 behave as if the X_{t+1} units of risky assets purchased by the central bank have disappeared from the economy. In turn, since they expect their future consumption to be less risky, they ask for a lower risk premium and, as a result, the asset price increases. Moving to the second line of equation (14), we see that level-k thinkers, for $k > 1$, expect next period's asset price to coincide with the asset price computed in the previous iteration, q_{t+1}^{k-1}. On the transfer side, these agents incorporate the budget constraint of the central bank into their expectations and, thus, correctly anticipate future tax risk. As a result, they fully hedge against such risk, thus, X_{t+1} disappears from the pricing equation.

The iterative process implied by (14) is depicted in Figure 1. The horizontal axis represents time and the vertical axis plots the level of sophistication k. A bold dot corresponding to time t and level k represents the temporary equilibrium price q_t^k. The diagram shows visually that, if one wants to compute, for example, the asset price at time 0 in a temporary equilibrium with level-5 agents, one has to "iterate diagonally" and compute the asset price at time 1 in a temporary equilibrium with level-4 agents, q_1^4, the asset price at time-2 in a temporary equilibrium with level-3 agents, q_2^3, and so on.

Importantly, these iterations always stop at the point where the temporary equilibrium with level-1 agents is reached. Beyond that point, agents will no longer revise their expectations following a policy announcement. In fact, Figure 1 suggests that the economy displays a form of *endogenous discounting*. To see this, suppose that the economy is populated by level-k agents, with $k \leq 5$, and that, at time 0, the central bank announces that it will purchase risky assets in period 6. From (14), level-k agents do not react to events happening more than k periods ahead. Therefore, the announcement of the government will have no effect on asset prices at $t = 0$. To see this more formally, we can iterate equation (14) to compute the temporary equilibrium price at time t:

$$q_t^k = q^* + \frac{r}{1+r}\gamma\sigma_x^2 \frac{X_{t+k}}{(1+r)^k}. \tag{15}$$

As suggested by Figure 1, equation (15) depends only on asset purchases k-periods ahead.

Having characterized beliefs for any level of sophistication, we turn to the reflective equilibrium, which considers an economy populated by agents who are heterogeneous in their level of sophistication. Using Lemma 1, the market-clearing condition in the reflective equilibrium can be written explicitly in period t as

$$X_{t+1} + \sum_{k=1}^{\infty} f(k) \left[\frac{\overline{D} + \alpha_{q,t}^k - q_t(1+r)}{\gamma \frac{r}{1+r}\sigma_x^2} - \beta_{Tr,t}^k \right] = \overline{X}. \tag{16}$$

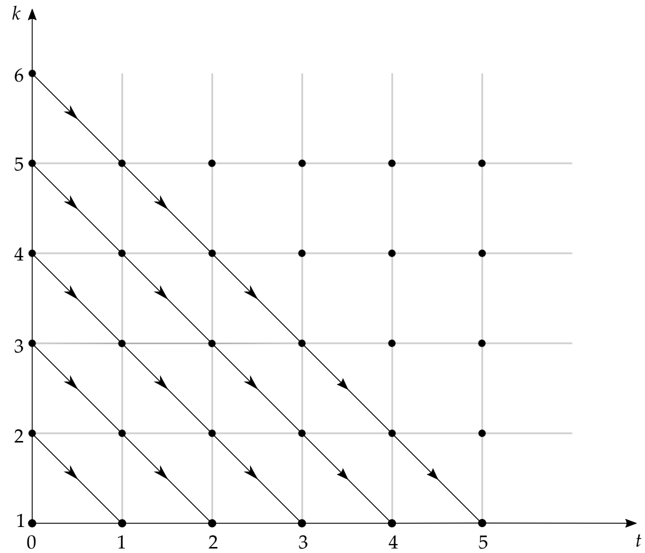

Figure 1: The price q_t^k of the risky asset in the temporary equilibrium where agents hold level-k beliefs. The horizontal axis plots time and the vertical axis plots the level of sophistication k. Every bold dot represents q_t^k corresponding to a given pair (t, k). The arrows point to the direction of "diagonal iterations" required to compute the price q_t^k.

Using the beliefs of level-k households obtained above, we can then solve (16) for the price of the risky asset.

Proposition 2. *The asset price in the reflective equilibrium depends on the entire future path of risky asset purchases* $\{X_{t+1}\}$ *and satisfies*

$$q_t = q^* + \frac{r}{1+r}\gamma\sigma_x^2 \sum_{k=1}^{\infty} f(k)\frac{X_{t+k}}{(1+r)^k}. \tag{17}$$

Proposition 2 shows that, in the reflective equilibrium, balance sheet policies are an effective tool for controlling the price of risky assets. Moreover, since the risk-free real return is fixed at r, any change in the risky asset price can be immediately interpreted as a change in the risk premium required by investors. This is not surprising: the reason for why prices change is that some households fail to understand how asset risk translates into future tax risk, thus, they underestimate the total amount of risk in the economy.

An important consequence of equation (17) is that the average level of sophistication of the agents in the economy—defined as the mean of $f(k)$—has two counteracting effects on the strength of balance sheet policies. First, a lower average level of sophistication implies that fewer agents internalize the future fiscal consequences of balance sheet policies. This effect tends to make balance sheet policies stronger. Second, following our discussion of Figure 1, a lower average level of sophistication implies a higher endogenous discounting, thus, more agents disregard the effects of balance sheet policies in the distant future. This effect tends to make balance sheet policies weaker.

To illustrate these two effects formally, we compute the risky-asset price following the announcement, at time 0, of a one-time purchase of risky assets in the following period, i.e., $X_2 > 0$ and $X_{t+1} = 0$, $t \geq 0$, $t \neq 1$. For the sake of specificity, we also assume that $f(k)$ is the probability distribution function of an exponential distribution, i.e., $f(k) = (1-\lambda)\lambda^{k-1}$, $\lambda \in [0,1)$. With this distribution, the average level of sophistication in the economy $\overline{k}$ is $1/(1-\lambda)$. From equation (17), the risky asset price at time 0 equals

$$q_0 - q^* = \frac{r}{1+r} \cdot \frac{\gamma \sigma_x^2}{(1+r)^2} \left(\frac{1}{\overline{k}} - \frac{1}{\overline{k}^2} \right) X_2.$$

The nonlinear effect of $\overline{k}$ on the price is clear. If $\overline{k} = 1$, then the policy is completely ineffective because the discounting effect is so strong ($\overline{k} = 1$ if and only if all agents are level-1) that households do not react even to policies implemented in the very near future. As $\overline{k}$ increases above 1, the endogenous discounting effect becomes weaker and the policy gains power, provided that $\overline{k}$ is below 2. The policy strength peaks at $\overline{k} = 2$ and then declines again to zero as $\overline{k}$ approaches infinity, that is, when the equilibrium approaches the REE.

2.6 Purchases of Domestic and Foreign Nominal Public Bonds

So far, we have studied our simple model, in which the central bank acquires private real risky assets. Balance sheet policies, however, are not confined to purchases of private assets. For example, in November 2010, the Federal Reserve announced the purchase of $600 billion worth of US Treasury securities (this operation was dubbed "QE2"), and in September 2011, disclosed a plan to purchase long-term public bonds by selling short-term bonds (this intervention was dubbed "Operation Twist," after a similar policy action implemented in 1961).[15] Moreover, there are many historical cases of central banks around the world intervening in the foreign-exchange market by purchasing foreign public bonds. In some cases, such purchases are financed by selling domestic public bonds (i.e., the so-called "sterilized" foreign exchange interventions).

Can our simple model also shed light on the effectiveness of these policies? The answer to this question is yes. One has to just reinterpret the total supply $\overline{X}$ as a real stock of domestic or foreign public bonds. With a non-trivial risk of default, domestic and foreign public bonds are risky. Even when default risk is negligible, public bonds (usually

[15] See the November 3, 2010 FOMC statement for the details on QE2, the September 21, 2011 FOMC statement for further information about the 2011 Operation Twist, and Alon and Swanson (2011) for the discussion of the 1961 Operation Twist.

fixed-income securities) are risky in *real* terms in the presence of inflation or exchange rate risk. As a result, the insights obtained in the simple model are applied to purchases of domestic and foreign public bonds. Of course, one can object that our simple model does not explicitly feature nominal variables, long-term domestic public bonds or foreign public bonds, and exchange rates; however, those are not central for the main mechanism of this paper.

We enrich the simple model with nominal variables and long-term public bonds in Appendix C.1. In particular, we assume that long-term bonds carry inflation risk. The alternative assumption of default risk would not alter the main message. Moreover, in Appendix C.2, we build a two-country extension of the simple model to study the effects of sterilized FX interventions. These two extensions formally demonstrate that the simple-model logic carries over to these richer environments.

2.7 Discussion of Alternative Assumptions

We now discuss how the results we have obtained so far change when we make two additional assumptions, namely, *(i)* a fraction of agents hold rational expectations and *(ii)* agents use a simple learning mechanism.

Presence of rational expectations agents. In the simple model, we assumed that all agents are level-k thinkers. A natural question is whether the presence of households who hold correct expectations about future endogenous variables can restore the neutrality result of balance sheet policies. The answer to this question is negative. Intuitively, if the presence of such agents was enough to guarantee the neutrality of balance sheet policies then, following any such policy, the price of risky assets would remain at its REE value. In this case, as we have seen in the proof of the REE benchmark, the demand for risky assets by rational-expectations agents would drop exactly by the amount of the government intervention. At the same time, however, the demand for risky assets by level-k thinkers would drop by less than the government intervention, because some of these agents fail to hedge against the tax risk. As a result, the market-clearing condition would not be satisfied. We formalize this logic in Appendix C.3.

Interestingly, the presence of rational-expectations agents can amplify the effects of balance sheet policies on the risky-asset price. Intuitively, rational-expectations agents are fully forward looking relative to any level-k thinker with finite k. As a result, when a balance sheet policy persists over time, rational-expectations agents take into account the entire future path of the policy.

Equilibrium unraveling and long-run neutrality. Laboratory and field experiments provide some evidence that people "learn" to play the Nash equilibrium after several repetitions of simple one-shot games (Nagel, 1995). In the simple model, we assumed that households always initiate their belief-formation process from the status quo, which corresponds to the REE without policy actions. As a result, if the policy is stationary over time, the model produces stationary outcomes and, in particular, there is no convergence towards the REE.

In Appendix C.4, we consider a simple process of "learning." Specifically, we assume that, after observing the policy, the level of sophistication of the agents increases over time. We show that this simple process leads to a form of "equilibrium unraveling," whereby the economy converges to the REE and balance sheet policies become ineffective. Interestingly, as the power of balance sheet policies fades away, it is no longer enough for the government to keep its policy constant to exert a constant effect on the risky-asset price. Instead, asset purchases need to grow over time to counteract the unraveling effect. This logic suggests that, when a policy of asset purchases is repeated over time, subsequent rounds may turn out to be less effective.

3 Balance Sheet Policies and Aggregate Output

The ultimate goal of balance sheet policies is usually to affect real economic activity, such as aggregate output. The previous section presented the simple model, where both labor income and dividends were exogenous. That model allowed us to study the impact of these policies only on asset prices and taxes. In this section, we bridge the gap between asset prices and the rest of the economy and investigate whether balance sheet policies are an effective tool for stimulating the economy or, instead, their strength remains confined to asset markets.

To answer this question we extend our simple model so that aggregate output is endogenous due to the presence of nominal price rigidities. This choice is motivated by two reasons. First, many central banks around the world routinely use dynamic stochastic general equilibrium models that hinge on nominal rigidities to analyze the effects of various disturbances and policy decisions. Second, a strand of the empirical literature has showed that such models are successful at capturing some key properties of the data. We build a model that is both simple enough to be solved analytically and rich enough to deliver important insights into the effects of balance sheet policies.[16] Specifically, we assume that prices are infinitely rigid. This assumption makes the analysis very tractable by al-

[16] Qiu (2018) quantifies the effects of conventional monetary policies in a calibrated New Keynesian model, extended with level-k thinking, but without aggregate risk.

lowing us to abstract from the firms' price-setting decision and, hence, the Phillips curve. Importantly, when prices cannot change, the short-term nominal interest rate coincides with the real interest rate. In addition, we assume that the central bank follows a simple interest rate rule: it keeps the nominal interest rate constant over time.[17] This assumption is motivated by the fact that balance sheet policies—such as quantitative easing—have been implemented precisely when the nominal interest rate was constrained at zero. Together, these two assumptions reduce the standard three-equation New-Keynesian model to a model that can be characterized by just one Euler equation.

3.1 Agents and Markets

We now lay out the model details and emphasize the differences with the simple model.

Households. Given beliefs and prices, the household maximizes

$$\widetilde{\mathbb{E}}_0 \left[-\frac{1}{\gamma} \sum_{t=0}^{\infty} e^{-\rho t + B_t - \gamma c_t} \right], \tag{18}$$

subject to a flow budget constraint, which is identical to the one in the simple model, represented by equation (2). We let $B_t \equiv \sum_{s=0}^{t} \epsilon_{s-1} - \epsilon_{-1}$ and assume that $\{\epsilon_t\}$ are independent over time and normally distributed, with zero mean and variance σ^2. The shock ϵ_t affects the subjective discount factor between periods t and $t+1$, which equals $e^{-(\rho-\epsilon_t)}$: a high realization of ϵ_t increases future marginal utility of consumption and makes households willing to postpone their current consumption.[18] Finally, we let Y_t denote aggregate output and assume that dividends per unit of risky assets and endowment ("labor income") are given by, respectively, $D_t = \delta Y_t / \overline{X}$ and $W_t = (1-\delta) Y_t$, $\delta \in (0, 1]$. By assuming that a part of endogenous output is distributed as endowment, we effectively avoid introducing the labor market explicitly as in Caballero and Farhi (2017).

Relative to the simple model in Section 2, there are two key differences. First, we assume that the households' discount factor is stochastic. Changes in the discount factor will lead to changes in consumption demand, which, in turn, due to price stickiness, will affect output in equilibrium. Second, in this section, a fraction $1-\delta$ of total output is paid out as labor income, while the remaining fraction is distributed as dividends to the owners of the risky asset. In equilibrium, therefore, there will be an endogenous

[17] Instead of the two assumptions of completely sticky prices and a constant nominal interest rate, we could alternatively assume that the central bank sets a constant real interest rate. This would result in identical model predictions, without requiring prices to be fully rigid.

[18] Note that neither the shock ϵ_t nor its variance σ^2 bear the index x, so as to explicitly differentiate them from the stochastic component of the process for dividends in the simple model.

correlation between labor income and capital income, which, for simplicity, we assumed away in the model of Section 2.

As before, we capture households' expectations with a sequence of one-period-ahead distributions $\widetilde{\Phi}_t \equiv \{\widetilde{\phi}_s\}_{s \geq t}$, where $\widetilde{\phi}_s$ is the distribution, conditional on information available at time s, of the vector of endogenous variables Z_{s+1}. In the extended model, $Z_t \equiv (Y_t, q_t, T_t, Tr_t, B_{t+1}, X_{t+1}, R_{t+1})$. As in the simple model, level-k thinkers will form expectations in a recursive way, starting from a status-quo distribution $\widetilde{\Phi}_t^{SQ}$ that corresponds to the linear REE before the intervention.

Government. The government is the same as the one in the simple model. In particular, the intertemporal budget constraint of the treasury is still given by (3). Moreover, we maintain the assumption that the central bank finances its stream of purchases by creating reserves, i.e., equation (4) continues to hold. The only difference is that now dividends are a share of total output, thus, equation (5) becomes

$$Tr_t = \left(\frac{\delta Y_t}{\overline{X}} + q_t \right) X_t - (1+r) R_t. \tag{19}$$

Markets. There are markets for assets, both risky and safe, and for goods. The supply of risky assets is exogenous and normalized to $\overline{X}$. The supply of safe bonds is determined by the government. In the equilibria we consider, output and the asset price adjust so as to clear the goods market, the risky-asset market, and, by Walras' law, the safe-asset market. Finally, notice that the real interest rate is constant as in Section 2, but for a different reason. In the simple model, we assumed an infinitely elastic supply of safe bonds. Instead, in this section, the constant interest rate follows from our assumption of sticky prices combined with the constant target for the nominal interest rate.

3.2 Consumption Demand and Aggregate Output

This section presents the solution to the household problem, extends the notion of reflective equilibrium to a model with endogenous output, and derives the main result, that is, the output effects of balance sheet policies.

Household problem. Given current aggregate endogenous variables and beliefs about the future values of these variables, the household chooses consumption and investment in risky and safe assets to maximize (18), conditional on the budget constraint (2). We solve this problem recursively by guessing, and then verifying, the household value function $V_t(a_t, \epsilon_t)$, where a_t denotes the sum of individual financial wealth, n_t =

$(1+r)[(1-\delta)\,Y_{t-1} - T_{t-1} + n_{t-1} - c_{t-1} - q_{t-1}x_t] + (\delta Y_t/\overline{X} + q_t)x_t$, and "human capital" h_t, which we define as the expected discounted sum of future disposable income. The value function may directly depend on time since beliefs about future endogenous variables may be time-dependent.

To streamline the exposition, we relegate the solution to the individual problem to Appendix A.2 and present here only the optimal choices for consumption and risky assets. Given linear beliefs about future endogenous variables, the optimal investment plan satisfies

$$x_{t+1} = \frac{\widetilde{\mathbb{E}}_t(\delta Y_{t+1}/\overline{X} + q_{t+1}) - (1+r)q_t}{\frac{r}{1+r}\gamma\left(\delta\beta_{Y,t}\sigma/\overline{X}\right)^2} - \frac{\beta_{Tr,t}}{\delta\beta_{Y,t}/\overline{X}} - \frac{(1-\delta)\,\beta_{Y,t}}{\delta\beta_{Y,t}/\overline{X}} + \frac{1}{\gamma r\delta\beta_{Y,t}/\overline{X}}, \tag{20}$$

for $t \geq 0$. Equation (20) states that the demand for risky assets is affected by four factors, which correspond to the four terms on the right-hand side of the equation. The first two terms are similar to those in equation (11), the risky-asset demand in the simple model. They imply that the household's demand for risky assets responds positively to the expected excess return on these assets, negatively to risk aversion and to the variance of excess returns (which equals $(\delta\beta_{Y,t}\sigma/\overline{X})^2$ in this case), and negatively to the covariance between transfers and risky-asset returns.

The third and fourth terms in equation (20) are specific to the current model. The third term is proportional to the covariance between labor income $(1-\delta)Y_{t+1}$ and the asset payoff in period $t+1$. Naturally, households reduce their exposure to risky assets when their labor income covaries positively with asset returns. Finally, the last term is due to the simultaneous impact that the aggregate shock ϵ_{t+1} has on future marginal utility, through its direct presence in the objective function (18), and on the risky-asset return. Risky assets yield high payoffs precisely in states when marginal utility is high.[19] This tends to increase the demand for such assets.

Let's now consider consumption. Household's optimal consumption plan is

$$\begin{aligned} c_t = &-\frac{\log(1+r)-\rho}{\gamma r} + \frac{r}{1+r}a_t - \frac{\epsilon_t}{\gamma(1+r)} + \frac{r}{1+r}\widetilde{\mathbb{E}}_t\sum_{j=t}^{\infty}\frac{\delta Y_{j+1}/\overline{X} + q_{j+1} - (1+r)q_j}{(1+r)^{j-t}}x_{j+1} \\ &-\frac{\gamma\sigma^2}{2}\sum_{j=t}^{\infty}\frac{1}{(1+r)^{j-t}}\left[\frac{r}{1+r}\left(\frac{\delta\beta_{Y,j}}{\overline{X}}x_{j+1} + (1-\delta)\,\beta_{Y,j} - \beta_{T,j} - \frac{1}{\gamma r}\right)\right]^2, \end{aligned} \tag{21}$$

for $t \geq 0$, where the value x_{s+1}, $s \geq t$, is the optimal risky-asset choice (20).

Consumption demand consists of five terms, the first of which is just a constant.

[19] This is clear from the cross-partial derivative, with respect to a_t and ϵ_t, of the household's value function $V_t(a_t, \epsilon_t) = -\exp[-\gamma a_t r/(1+r) + \epsilon_t/(1+r) + \vartheta_t]/\gamma$, which we derive in Appendix A.2.

The second term summarizes the effect of the contemporaneous level of total household wealth a_t on consumption c_t, by stating that the household consumes a constant fraction $r/(1+r)$ of total wealth. The third term captures the fact that the shock ϵ_t increases the desire to save or, equivalently, reduces the incentive to consume. The fourth term adds the future excess returns that the household expects to collect by investing in risky assets. As for total wealth, the household consumes only a fraction $r/(1+r)$ of these returns. Finally, the last term represents the negative effect that aggregate risk has on consumption demand, through the precautionary saving behavior of the household. This term will be key for understanding the effect of balance sheet policies. Risk comes from four channels—dividends, labor income, taxes, and marginal utility—which correspond to the four terms inside the round brackets on the last line of equation (21).

Temporary, rational, and reflective equilibria. All the notions of equilibrium introduced in Section 2—temporary, rational-expectations, and reflective—extend naturally to the environment of this section. The main difference is that we need to impose that aggregate demand for consumption equals aggregate output in every period. For brevity, we do not repeat the definitions here.

We first characterize the REE benchmark. When solving for the response of the economy to policy changes in a REE, we need to take care of indeterminacy of equilibria. It is well-known that, when expectations are rational, a constant real interest rate may lead to multiple equilibria. In standard New-Keynesian models, local uniqueness of the equilibrium is achieved by requiring some form of the *Taylor principle* to hold (Woodford, 2003). Our assumption of a fixed nominal interest rate, together with fixed goods prices, does not satisfy this principle. To get around this complication and, at the same time, retain the tractability brought by our assumptions, we employ a simple equilibrium selection. We first solve the model for a REE in the absence of policy interventions, which will serve as the status quo in the construction of level-k beliefs. Since we are interested in the response of the economy to shocks, the specific equilibrium we choose to be a status quo will not matter for the subsequent analysis. Then, after the policy change takes place, we solve for the response of the economy by requiring that, in the long run, the REE after the policy change converges to the chosen REE before the change. This approach, which is illustrated in Farhi and Werning (2017), uniquely pins down the response of the economy to the policy intervention.

We now compute the asset price and aggregate output after a policy intervention. For convenience and without loss of generality, we select the particular REE in which output does not exhibit a time trend. Formally, to ensure the absence of a trend, we assume that the central bank chooses its interest rate so that $1 + r = \exp(\rho - \sigma^2/2)$. The following proposition generalizes the irrelevance result of Proposition 1.

Proposition 3. *In the REE, the risky-asset price and output are independent of asset purchases and are equal to, respectively,*

$$q^* = \frac{\delta}{r\overline{X}}\left(\overline{Y} - \frac{\sigma^2}{\gamma}\right) \tag{22}$$

and

$$Y_t^* = \overline{Y} - \frac{1}{\gamma}\epsilon_t, \tag{23}$$

for some constant $\overline{Y} > 0$.

The proof is in Appendix A.2. Note that the multiplicity we just described is parameterized by the value of $\overline{Y}$, which the proposition does not specify. For the asset price to be positive in equilibrium, $\overline{Y} - \sigma^2/\gamma$ needs to be positive, an assumption that we maintain from now on.[20] Equation (22) states that the REE asset price depends positively on the expected dividends and negatively on the variance of aggregate shocks. The output equation shows that a shock that increases the desire to save, i.e., a positive realization of ϵ_t, reduces current output.

We now present the main result of this section: the effect of asset purchases on output in the reflective equilibrium of the economy. It is convenient to express the main variables in deviation from their REE counterparts, given by (22) and (23). We thus set $\widehat{q}_t \equiv q_t - q^*$ and $\widehat{Y}_t \equiv Y_t - Y_t^*$.

Proposition 4. *Let $\{X_{t+1}\}$ be a sequence of balance sheet policies. In the reflective equilibrium, the deviations of the asset price and output from their REE values satisfy, respectively,*

$$\widehat{q}_t = \sum_{k=1}^{\infty} f(k)\, \widehat{q}_t^k$$

and

$$\widehat{Y}_t = \sum_{k=1}^{\infty} f(k)\, \widehat{Y}_t^k + \eta_1 \sum_{k=1}^{\infty} f(k)\, (\widehat{q}_t - \widehat{q}_t^k)^2 \overline{X}^2,$$

where $\widehat{q}_t^k$ and $\widehat{Y}_t^k$ are, respectively, the asset price and output (in deviation from their REE values) in the temporary equilibrium with level-k agents and satisfy

[20] The fact that the asset price can become negative is due to our assumptions of CARA preferences and normal distribution of shocks. The latter assumption implies that dividends can be negative, which, in turn, can cause the price of the asset to become negative as well. This inconvenience is the cost we pay to obtain closed-form solutions.

$$\widehat{q}_t^k = \begin{cases} \eta_2 \dfrac{X_{t+1}}{\overline{X}}, & k=1, \\ \dfrac{1}{1+r}\delta \widehat{Y}_{t+1}^{k-1}\dfrac{1}{\overline{\overline{X}}} + \dfrac{1}{1+r}\widehat{q}_{t+1}^{k-1}, & k \geq 2, \end{cases} \tag{24}$$

$$\widehat{Y}_t^1 = \begin{cases} \eta_3 \left(\dfrac{X_{t+1}}{\overline{X}}\right)^2, & k=1 \\ \displaystyle\sum_{j=0}^{\infty} \dfrac{r}{(1+r)^{j+1}} \left[\widehat{Y}_{t+j+1}^{k-1} + \dfrac{\eta_1}{1+r}(\widehat{q}_{t+j+1}^{k-1} - \widehat{q}_{t+j+1}^{k})^2 \overline{X}^2\right], & k \geq 2, \end{cases} \tag{25}$$

where η_1, η_2, and η_3 are positive scalars given in the proof.

The proof is in Appendix A.2. The two main messages of this proposition are that, in an economy populated by level-k thinkers, asset purchases have a *first-order* effect on the asset price and at most a *second-order* effect on aggregate output. We consider them in order.

First of all, the conclusion about the first-order effect on the asset price was already evident from the simple model (see the discussion preceding Proposition 2). In fact, iterating forward the second line of equation (24) and combining it with the first line of the same equation yields an expression for the asset price that resembles equation (17) in the simple model. The main difference is the first term in the second line of the pricing equation (24), which comes from the fact that output is endogenous and simply adds an extra, second-order, effect on the asset price, through an endogenous response of dividends.

Second, balance sheet interventions have at most a second-order effect on aggregate output. This is the main result of this section. To be precise, Proposition 4 shows that, depending on the duration of the balance sheet policy and on the level of sophistication of the households, changes in output are proportional to the square of the size of the intervention as well as to higher powers of it. This is why we say that interventions have *at most* a second-order effect on output. Importantly, equation (25) does not feature a term that is linear in the size of the intervention.

Moreover, the coefficient on the second-order term is small. Specifically, in Appendix A.2, we show that, when the economy is populated by level-1 thinkers, the percentage increase in output (normalized by the square of interventions) relative to percentage increase in prices (normalized by the interventions) is proportional, among other things, to the net interest rate r. Formally, we obtain

$$\frac{\frac{\widehat{Y}_t^1}{\mathbb{E}Y_t^*} \Big/ \left(\frac{X_{t+1}}{\overline{X}}\right)^2}{\frac{q_t^1}{q^*} \Big/ \left(\frac{X_{t+1}}{\overline{X}}\right)} = \frac{\delta}{2} \cdot \frac{r}{1+r} \cdot \frac{\overline{Y} - \frac{\sigma^2}{\gamma}}{\overline{Y}}.$$

We immediately see that all the terms in the product are smaller than one and, in addition, under standard calibrations, the net interest rate r is much smaller than one.

To gain intuition for the effect on output, observe that an intervention by the central bank does not change the current value of the households' financial wealth. Households sell their risky assets to the central bank, which issues just enough safe assets—i.e., reserves—to pay for them. Thus, households swap risky assets with safe assets in their portfolio. In fact, by Proposition 3, if agents held rational expectations, purchases would have no effect on output. However, since some households either do not understand the fiscal consequences of interventions or think others do not understand them, they behave as if they were facing less aggregate risk. Accordingly, they reduce their precautionary saving and increase aggregate demand, thereby raising output in equilibrium. This channel has, at most, a second-order effect on output because consumption demand is proportional to the *variance* of aggregate risk "removed" from the economy by the central bank's intervention.

This result can potentially rationalize the relative abundance of evidence on price effects of balance sheet policies and, at the same time, the lack of convincing empirical support for aggregate output effects.

3.3 Portfolio Balance and Signaling Channels with Level-k Thinking

The conclusion that balance sheet policies only have a second-order effect on output is a distinctive feature of our channel. Alternative popular explanations of non-neutrality of central bank balance sheet policies normally lead to first-order output effects. First, the portfolio balance channel, which arises when some households have limited access to asset markets, predicts that purchases have a first-order effect on output. Intuitively, when only some households can trade risky assets, while everybody can access the market for riskless debt, households excluded from the risky assets market cannot hedge against future tax risk, which breaks the irrelevance result. In addition, since typically all households form expectations rationally in this class of models, they will understand that the central bank will, on average, turn a profit on its intervention. If a part of these profits is rebated to the households excluded from the risky asset market, these agents' expected net present discounted income goes up, thus, they increase their current consumption by an amount that is proportional to the size of the intervention.

Second, the signaling channel relies on the assumption that current balance sheet interventions serve as a signal of future conventional monetary policy, which, in turn, affects current aggregate demand and output through, for example, the standard intertemporal Euler equation. It follows that, if the expected change in future conventional policy is proportional to the size of balance sheet interventions, then, once again, such policies will have a first-order effect on output.

Both of these alternative channels are, however, dampened by the presence of level-k thinking. The most straightforward way to see this is to consider the case in which all agents are level-1 thinkers.[21] First, in the case of portfolio balance channel, level-1 households would fail to anticipate higher future transfers from the government, which eliminates a first-order effect of the interventions on consumption demand and output. Second, in the case of signaling channel, level-1 agents would not be sophisticated enough to think about current balance sheet interventions implications for future conventional monetary policy, hence the strength of the signaling channel would be greatly diminished. In reality, the extent to which level-k can dampen the response of output will depend on the number of level-1 thinkers in the economy. In Section 4, we present some evidence hinting that this number can be quite large.

4 Testable Implications

In this last part of the paper, we highlight the differences between our model and the models based on alternative channels of balance sheet policies. We proceed in four steps. First, we present the key difference between our model and those that, while emphasizing other channels, maintain the assumption of full information rational expectations. Second, we contrast our model with those that assume information frictions. Third, we present evidence that support some of the predictions of our model. Finally, we look at this evidence through the lens of our model and conclude that 86 percent of the forecasters in our data are level-1 thinkers.

Full information rational expectations models. When households do not hold rational expectations, they make systematic mistakes. We can use the simple model (or any of its extensions) to derive closed-form expressions for the errors that agents make following central bank interventions. Importantly, these predictions can help us differentiate the mechanism in this paper from other mechanisms that maintain the assumption of *full information rational expectations*, but assume either market segmentation (i.e., the portfolio balance channel) or asymmetric information between the government and private agents (i.e., the signaling channel).[22]

It is instructive to present the asset price forecast errors in our model. Recall that a

[21]It is straightforward to extend our model to formalize this intuition. For brevity, we do not present these extensions here. They are, however, available upon request.

[22]We use the term "full information" to refer to the information available to agents in the economy.

level-k thinker assumes that the world is populated only by level-$(k-1)$ thinkers; thus, she expects the price of the risky asset at some future period s to be q_s^{k-1}. We denote her forecast error with $u_{t,s}^k \equiv q_s - \widetilde{\mathbb{E}}_t^k q_s = q_s - q_s^{k-1}$. Proposition 2 gives us an expression for q_s and equation (15) provides the price q_s^{k-1}. We thus have the following expression for the *average* forecast error:

$$\overline{u}_{t,s} \equiv \sum_{k=1}^{\infty} f(k) u_{t,s}^k = \frac{r}{1+r} \gamma \sigma_x^2 \sum_{k=1}^{\infty} f(k) \left[1 - \frac{f(k+1)}{f(k)}\right] \frac{X_{t+k}}{(1+r)^k}. \tag{26}$$

This formula indicates that the size of the average forecast error depends on the size of the interventions. This relation between the average forecast error and the size of balance sheet policies is a peculiar prediction of our mechanism.

Heterogeneous information rational expectations models. There is, however, an alternative class of models in which agents form expectations rationally but possess heterogeneous information. Some of these models can potentially generate non-neutrality of balance sheet policies together with predictable forecast errors (both individually and on average across agents). For example, if some agents do not have accurate information about government interventions, they will make predictable forecast errors from the view point of an econometrician who is perfectly aware of the policy implementation.[23]

The two types of models are, however, not observationally equivalent. A crucial difference between them is that, in models with heterogeneous information, individual agents hold rational expectations *conditional* on their information sets. More formally, models with heterogeneous information predict that an agent's forecast error is orthogonal to any variable contained in the agent's information set. For example, if agents are aware of the policy of asset purchases, which is publicly announced, then this policy should not predict future forecast errors. On the contrary, level-k agents use their information "incorrectly" and, hence, make forecast errors that are predictable even with their information. To distinguish the two types of models empirically, therefore, one would need a proxy for individual information about government interventions.[24]

[23]Prominent examples of information frictions are noisy information (Lucas, 1972; Woodford, 2001; Angeletos and La'O, 2010), sticky information (Mankiw and Reis, 2002; Reis, 2006a,b), and rational inattention (Sims, 2003; Maćkowiak and Wiederholt, 2009).

[24]Using individual forecast updates to proxy for individual information sets, Bordalo et al. (2018) provide evidence on the predictability of individual forecast errors by such variables, which points to non-rational expectations. Moreover, in several papers, laboratory experiments mimic macroeconomic situations. For example, Kneeland (2016) studies coordinated attack games, such as currency attacks, in a laboratory setting and concludes that a model with level-k thinking fits the responses of subjects to public information better than a model with dispersed information and rational expectations. Giamattei (2015) runs an experiment in which price setters respond to the central bank's attempt to reduce inflation. He shows that subjects' price choices are better approximated by a model with level-k thinking, rather than

In addition, one can argue that, when it comes to forecasting financial variables by financial institutions, professional forecasters are likely to pay a great deal of attention to government interventions reducing the extent of heterogeneous information about the interventions. In such a case, if incomplete information was the main friction, it would be unlikely that government interventions could predict forecast errors.

Predictability of Forecast Errors in the data. We present evidence that balance sheet interventions predict forecast errors. We focus on average forecast errors of conventional mortgage rates in the US and show that they respond significantly to "exogenous and unexpected" purchases of mortgages by quasi-government agencies, also known as government-sponsored enterprises (GSEs), such as Fannie Mae and Freddie Mac.

We follow Fieldhouse, Mertens and Ravn (2018) (henceforth referred to as "FMR"), who argue that the purchases of mortgages by GSEs resemble the purchases of private risky assets by the Federal Reserve in the recent financial crisis. The authors use a narrative approach to identify "exogenous and unexpected" shocks to GSE's balance sheets. They then document a significant reaction of mortgage rates after these shocks. We use the data and specification employed by the authors with the only difference that we regress mortgage rates forecast errors instead of mortgage rates themselves. To construct forecast errors, we use a survey of expectations by major financial institutions collected in the Blue Chip Financial Forecast (BCFF) database.

Because we adopt the exact econometric approach of FMR, for brevity, we present here only the responses of conventional mortgage rates and their forecast errors to the shocks. Appendix B contains all of the econometric details.

The left panel of Figure 2 shows the impulse response of the mortgage rate after an expansion of the GSEs balance sheet. By presenting this plot, we confirm that the main conclusion in FMR does not change much when we use our restricted data sample due to a shorter sample of forecast data that we use. The right panel of Figure 2 presents the response of mortgage rate forecast errors at various horizons along with one- and two-standard-error confidence intervals. Consistently with the predictions of our model, forecast errors react negatively and significantly to the GSEs' mortgage purchases, which suggests that forecasters tend to under-react to news about such interventions.

Next, we use these empirical results and our model to compute the average level of sophistication of agents in the data.[25] Specifically, we first assume that all of the reaction of forecast errors can be attributed to the bounded rationality channel of this paper. Then

with rational expectations.

[25]We implicitly assume that the BCFF survey participants have the same average level of sophistication as the agents who price mortgages. We leave the alternative assumption of different levels of sophistication to future research.

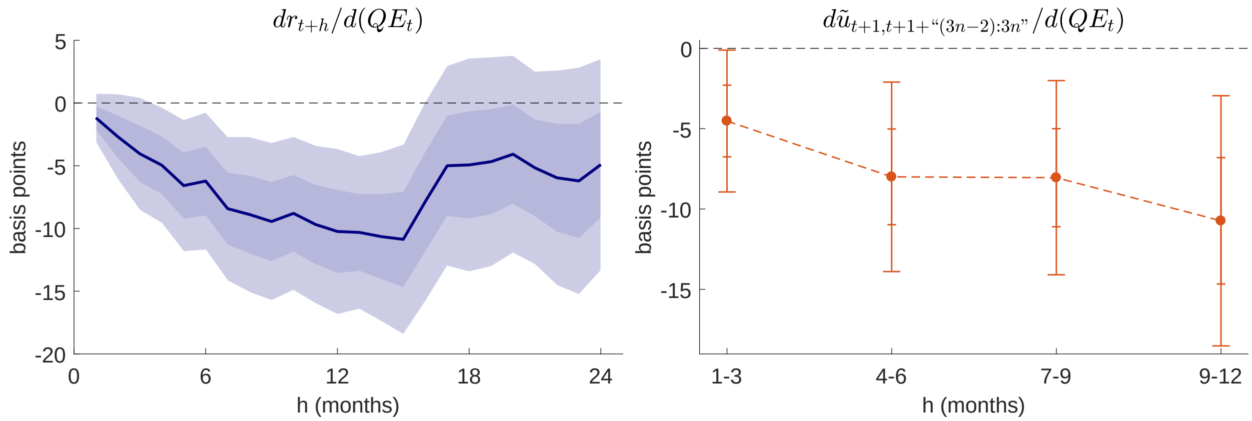

Figure 2: The left panel presents the conventional mortgage-rate impulse response function to an exogenous change in the GSEs' purchases of mortgages. Formally, the notation QE_t refers to the term multiplying $\gamma_h^{(2)}$ on the right-hand side of equation (B.2) in Appendix B. The right panel shows the response of the conventional mortgage-rate forecast errors, at various forecasting horizons, to an exogenous change in the GSEs' purchases of mortgages. The labels on the horizontal axis in the right panel represent the forecast horizon. For example, "1-3 months" refers to one-quarter-ahead forecasts. The appendix states all variable definitions explicitly. In both panels, the confidence intervals show the one and two Newey and West (1987) standard deviation error bands.

we use equations (17) and (26), together with the assumption that the distribution of levels of sophistication $f(k)$ is exponential with mean of $\bar{k}$, to show that the average level of sophistication $\bar{k}$ equals the ratio of the impulse responses:

$$\bar{k} = \frac{\partial q_s / \partial X_{t+1}}{\partial \bar{u}_{t,s} / \partial X_{t+1}}. \tag{27}$$

The key property of (27) is that it is independent of the specific process of balance sheet policies $\{X_{t+1}\}$, thus, we can use it to estimate $\bar{k}$, independently of the exact details of the asset purchase programs in our data. In addition, the same formula holds true if we replace the price of the risky asset in the numerator with the one-period return on the risky asset. Therefore, if we identify the conventional mortgage rate in our data with the one-period return on the risky assets in our theoretical model, then we can use equation (27) to get an estimate of $\bar{k}$.

Guided by our theory, we compute $\bar{k}$ by taking the impulse response of the moving average of the realized mortgage rates and dividing it by the responses of the forecast errors at the same horizon from the right panel of Figure 2.[26] The estimates of $\bar{k}$ for each horizon are presented in Appendix Table A1 and the average across the four horizons is 1.17. This number implies that 86 percent of the agents in the sample consists of level-1 thinkers, who do not change their forecasts after the policy intervention, while only 14

[26]See Appendix B for more details.

percent achieves higher levels of thinking. This is quite a low estimate of $\bar{k}$, especially considering that our sample is made up of major financial institutions. At the same time, when interpreting the numbers in this exercise, one may want to keep in mind that we assumed that all the agents are perfectly aware of the policy interventions. If, on the contrary, some agents do not react to new policies simply because they are not aware of them, then these agents would be incorrectly classified as being of the lowest level of thinking.

5 Conclusion

In this paper, we showed that the assumption of rational expectations is essential for the irrelevance result, whereby balance sheet policies (e.g., quantitative easing) are irrelevant, both for asset prices and for the real activity. Once we assume that agents form expectations according to the level-k thinking process, balance sheet policies become effective policy tools. Moreover, we showed that these policies have a first-order effect on asset prices, while only a second-order effect on aggregate output. This result can potentially explain the lack of convincing evidence about the output effects of these policies. Next, we contrasted the predictions of our model with the alternative models that emphasize other channels of balance sheet policies. Finally, we tested one of the main implications of our channel, namely, that forecast errors should be predictable by balance sheet policies. Specifically, we show that identified exogenous and unexpected purchases of mortgages by quasi-government agencies predict average forecast errors of asset prices (i.e., mortgage rates) at different horizons.

There are many important directions ahead. For example, we have studied the response of the economy to an exogenous path of asset purchases. A crucial step will be to understand when the central bank finds it optimal to use balance sheet policies as a stabilization tool in an environment when people form expectations as in our paper.

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Appendix

A Proofs

This part of our appendix details the proofs omitted from the main text in Sections 2 and 3.

A.1 Balance Sheet Policies and Asset Prices

We start from the proofs of the claims in Section 2.

Household Problem and Proof of Lemma 1.

We conjecture that the asset price is a non-stochastic function of time, that is, $\beta_{q,t} = 0$, $t \geq 0$, thus, $\widetilde{\mathbb{E}}_t q_{t+1} = \alpha_{q,t}$. We later verify that this is indeed the case in all the equilibria we consider.

We solve the household problem (1)-(2) recursively. In particular, the value function at time t, $V_t(a_t)$, is a solution to the Bellman equation

$$V_t(a_t) = \max_{c_t, x_{t+1}} u(c_t) + e^{-\rho} \widetilde{\mathbb{E}}_t [V_{t+1}(a_{t+1})], \tag{A.1}$$

subject to

$$n_{t+1} = (1+r)(W_t - T_t + n_t - c_t - q_t x_{t+1}) + (D_{t+1} + q_{t+1}) x_{t+1}, \tag{A.2}$$

where n_t is the household financial wealth at the beginning of time t, that is, before income and consumption. We conjecture that the value function $V_t(a_t)$ depends on time t, and total wealth a_t, defined as the sum of financial wealth n_t and human capital h_t, which is the expected discounted sum of future disposable income:

$$h_t \equiv \widetilde{\mathbb{E}}_t \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} \left(W_{t+j} - T_{t+j} \right).$$

We guess and verify that

$$V_t(a) = -\frac{1}{\gamma} \exp\left(-\gamma A a + \vartheta_t\right),$$

where ϑ_t is a deterministic function of time. We can then use standard properties of Normal distributions to rewrite the problem (A.1)-(A.2) as

$$V_t(a_t) = \max_{c_t, x_{t+1}} \left[-\frac{1}{\gamma} \exp\left(-\gamma c_t\right) - \frac{1}{\gamma} \exp\left(-\rho - \gamma A \widetilde{\mathbb{E}}_t a_{t+1} + \frac{1}{2} \gamma^2 A^2 \widetilde{\mathbb{V}}_t a_{t+1} + \widetilde{\mathbb{E}}_t \vartheta_{t+1} \right) \right].$$

The conditional expectation of the following period's total wealth is

$$\widetilde{\mathbb{E}}_t a_{t+1} = (1+r)\left[W_t - T_t + n_t - c_t - q_t x_{t+1}\right] + \left(\overline{D} + \alpha_{q,t}\right) x_{t+1} + \widetilde{\mathbb{E}}_t h_{t+1},$$

where $\overline{D} = \widetilde{\mathbb{E}}_t \left[D_{t+1}\right]$, since agents are assumed to know the true distribution of exogenous variables.

Similarly, the conditional variance of the following period's total wealth is

$$\begin{aligned}
\widetilde{\mathbb{V}}_t a_{t+1} =& \widetilde{\mathbb{V}}_t \left[(1+r)(W_t - T_t + n_t - c_t - q_t x_{t+1}) + (D_{t+1} + q_{t+1})x_{t+1} + h_{t+1}\right] \\
=& \widetilde{\mathbb{V}}_t \left[D_{t+1}x_{t+1} + \widetilde{\mathbb{E}}_{t+1} \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} \left(W_{t+1+j} - T_{t+j+1}\right)\right] \\
=& \widetilde{\mathbb{V}}_t \left[D_{t+1}x_{t+1} - (1+r)B_{t+1} + \widetilde{\mathbb{E}}_{t+1} \sum_{j=0}^{\infty} \frac{Tr_{t+j+1}}{(1+r)^j}\right] \\
=& \sigma_x^2 (x_{t+1} + \beta_{Tr,t})^2,
\end{aligned}$$

where the second to last line uses the treasury's budget constraint (3). The first order condition with respect to consumption, c_t, is

$$\exp\left(-\gamma c_t\right) = A(1+r)\exp\left(-\rho - \gamma A \widetilde{\mathbb{E}}_t a_{t+1} + \frac{1}{2}\gamma^2 A^2 \widetilde{\mathbb{V}}_t a_{t+1} + \vartheta_{t+1}\right) \tag{A.3}$$

and the first order condition with respect to investment in the risky asset, x_{t+1}, is

$$-(1+r)q_t + \overline{D} + \alpha_{q,t} - \gamma A \sigma_x^2 \left(x_{t+1} + \beta_{Tr,t}\right) = 0. \tag{A.4}$$

The latter equation determines the demand for the risky asset

$$x_{t+1} = \frac{\overline{D} + \alpha_{q,t} - (1+r)q_t}{\gamma A \sigma_x^2} - \beta_{Tr,t}. \tag{A.5}$$

Taking logs of both sides of equation (A.3) and using the expressions for $\widetilde{\mathbb{E}}_t a_{t+1}$ and $\widetilde{\mathbb{V}}_t a_{t+1}$, we rewrite (A.3) as

$$\begin{aligned}
-\gamma c_t =& \log\left[A(1+r)\right] - \rho - \gamma A \left[(1+r)\left(W_t - T_t + n_t - c_t - q_t x_{t+1}\right) + \left(\overline{D} + \alpha_{q,t}\right)x_{t+1} + \widetilde{\mathbb{E}}_t h_{t+1}\right] \\
&+ \frac{1}{2}\gamma^2 A^2 \sigma_x^2 \left(x_{t+1} + \beta_{Tr,t}\right)^2 + \widetilde{\mathbb{E}}_t \vartheta_{t+1}.
\end{aligned}$$

We next solve this equation for consumption:

$$\begin{aligned}
c_t =& -\frac{\log\left[A(1+r)\right] - \rho}{\gamma\left[A(1+r)+1\right]} + \frac{A(1+r)}{A(1+r)+1}a_t + \frac{A}{A(1+r)+1}\left(\overline{D} + \alpha_{q,t} - (1+r)q_t\right)x_{t+1} \\
&- \frac{1}{2} \cdot \frac{A^2\gamma}{A(1+r)+1}\sigma_x^2\left(x_{t+1} + \beta_{Tr,t}\right)^2 - \frac{1}{\gamma\left[A(1+r)+1\right]}\widetilde{\mathbb{E}}_t \vartheta_{t+1},
\end{aligned}$$

where we took into account that $h_t = W_t - T_t + \widetilde{\mathbb{E}}_t h_{t+1}/(1+r)$.

For our guess to be true, equation (A.1) must be satisfied by our conjectured value function, for any value of a_t and t, that is,

$$\begin{aligned}
&-\frac{1}{\gamma}\exp\left(-\gamma A a_t + \vartheta_t\right) \\
=& \max_{c_t, x_{t+1}} -\frac{1}{\gamma}\exp\left(-\gamma c_t\right) - \frac{1}{\gamma}\exp\left(-\rho - \gamma A \widetilde{\mathbb{E}}_t a_{t+1} + \frac{1}{2}\gamma^2 A^2 \widetilde{\mathbb{V}}_t a_{t+1} + \widetilde{\mathbb{E}}_t \vartheta_{t+1}\right).
\end{aligned}$$

Using (A.3), the latter becomes

$$-\frac{1}{\gamma}\exp\left(-\gamma A a_t+\vartheta_t\right)=-\frac{1}{\gamma}\exp\left\{\log\left[\frac{A(1+r)+1}{A(1+r)}\right]-\gamma c_t\right\},\tag{A.6}$$

where, with slight abuse of notation, we use c_t to denote optimal consumption. Because equation (A.6) must hold for any value of a_t, the constant A must satisfy $A=\frac{r}{1+r}$. The asset demand (A.5) is thus

$$x_{t+1}=\frac{\widetilde{\mathbb{E}}_t(D_{t+1}+q_{t+1})-(1+r)q_t}{\gamma\frac{r}{1+r}\sigma_x^2}-\beta_{Tr,t},\tag{A.7}$$

which is equation (11) in the main text. Similarly, consumption becomes

$$\begin{aligned}c_t=&-\frac{\log(r)-\rho}{\gamma(1+r)}+\frac{r}{1+r}a_t+\frac{r}{(1+r)^2}\left(\overline{D}+\alpha_{q,t}-(1+r)q_t\right)x_{t+1}\\&-\frac{1}{2}\cdot\frac{1}{1+r}\left(\frac{r}{1+r}\right)^2\gamma\sigma_x^2\left(x_{t+1}+\beta_{Tr,t}\right)^2-\frac{1}{\gamma(1+r)}\widetilde{\mathbb{E}}_t\vartheta_{t+1}\end{aligned}$$

and, finally,

$$\begin{aligned}\vartheta_t=&\log\left(\frac{1+r}{r}\right)+\frac{\log(r)-\rho}{1+r}-\gamma\frac{r}{(1+r)^2}\left(\overline{D}+\alpha_{q,t}-(1+r)q_t\right)x_{t+1}\\&+\frac{1}{2}\cdot\frac{1}{1+r}\left(\frac{r}{1+r}\right)^2\gamma^2\sigma_x^2\left(x_{t+1}+\beta_{Tr,t}\right)^2+\widetilde{\mathbb{E}}_t\vartheta_{t+1}.\end{aligned}$$

The latter can be iterated forward to get

$$\begin{aligned}\vartheta_t=&\log\left(\frac{1+r}{r}\right)+\frac{\log(1+r)-\rho}{r}-\gamma\frac{r}{(1+r)^2}\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left(\overline{D}+\alpha_{q,t+j}-(1+r)\alpha_{q,t+j-1}\right)x_{t+j+1}\\&+\frac{1}{2(1+r)}\left(\frac{r}{1+r}\right)^2\gamma^2\sigma_x^2\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left(x_{t+j+1}+\beta_{Tr,t+j}\right)^2,\end{aligned}$$

where, to simplify notation, we used $\alpha_{q,t-1}=q_t$. Note that, in the last two equations, we used x_{t+j+1} to denote the optimal investment in risky assets. Since (A.7) is a deterministic function of time, so is ϑ_t, therefore, our conjecture is verified.

Proof of Proposition 1.

In the REE, the market-clearing condition (6) holds for $t\geq 0$. In addition, households take the budget constraints of both branches of the government into account when forming expectations about the future. Thus, we immediately obtain $\beta_{Tr,t}=X_{t+1}$, $t\geq 0$. Plugging the latter and (6) into the asset demand (A.7) and solving for the asset price gives (we add an asterisk to denote REE objects):

$$q_t^*=\frac{\overline{D}+q_{t+1}^*-\gamma\frac{r}{1+r}\sigma_x^2\overline{X}}{1+r},$$

where we used the fact that, since expectations are rational and the price is deterministic, the expectation of next period's asset price coincides with its realized value. The unique non-explosive solution of the above equation is (13) and is independent of asset purchases.

Proof of Proposition 2.

We first prove (14). We proceed recursively, starting with level-1 agents. At time t, level-1 agents solve (A.1) under the belief that the economy will be in the REE without asset purchases. In particular, since the transfers are zero in the absence of asset purchases, level-1 agents believe that $\beta^1_{Tr,s} = 0$, for all $s \geq t$. The value function in (A.1) conveniently encapsulates the agent's beliefs about the future. We can then use the risky-asset demand (A.7), together with market clearing (6) and (13), to derive the first line of (14).

Now, let q_t^{k-1} be asset price in the TE of an economy populated by level-$(k-1)$ agents, with $k > 1$. Importantly, agents with $k > 1$ take into account the intertemporal budget constraints of the entire government when forming their expectations. As a result, $\beta_{Tr,s} = X_{s+1}$, $s \geq t$. Moreover, they expect the future asset price to coincide with the asset price in an economy with level-$(k-1)$ agents, that is, $\widetilde{\mathbb{E}}_t^k [q_{t+1}] = q_{t+1}^{k-1}$. Plugging the latter into (A.7) and imposing market clearing (6) gives the second line of (14).

Equation (16) then follows immediately by using the results above with (A.7) and the market-clearing condition (9).

A.2 Balance Sheet Policies and Aggregate Output

This section of the Appendix contains the proofs omitted from Section 3.

Household Problem Solution

As we did in the proof of Lemma 1, we look for equilibria in which beliefs are linear in fundamental shocks and we conjecture that the asset price is a non-stochastic function of time, that is, $\beta_{q,t} = 0$, $t \geq 0$, thus, $\widetilde{\mathbb{E}}_t q_{t+1} = \alpha_{q,t}$. We later verify that this is indeed the case in all the equilibria we consider.

Notice that, using the definition of B_t,

$$\widetilde{\mathbb{E}}_0 \left\{ -\frac{1}{\gamma} \sum_{t=0}^{\infty} e^{-\left[\sum_{s=0}^{t}(\rho-\epsilon_{s-1})-(\rho-\epsilon_{-1})\right]-\gamma c_t} \right\} = \widetilde{\mathbb{E}}_0 \left[-\frac{e^{-\gamma c_0}}{\gamma} - \frac{e^{-\gamma c_1}}{\gamma} e^{-(\rho-\epsilon_0)} - \frac{e^{-\gamma c_2}}{\gamma} e^{-(2\rho-\epsilon_0-\epsilon_1)} - \dots \right],$$

thus, we can write the recursive version of the household problem as

$$V_t(a_t, \epsilon_t) = \max_{c_t, x_{t+1}} \left\{ u(c_t) + e^{\epsilon_t - \rho} \widetilde{\mathbb{E}}_t \left[V_{t+1}(a_{t+1}, \epsilon_{t+1}) \right] \right\}, \tag{A.8}$$

subject to

$$\begin{aligned} n_{t+1} &= (1+r)\left[(1-\delta)Y_t - T_t + n_t - c_t - q_t x_{t+1}\right] + \left(\frac{\delta}{\overline{X}} Y_{t+1} + q_{t+1}\right) x_{t+1}, \\ h_{t+1} &\equiv \widetilde{\mathbb{E}}_{t+1} \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} \left[(1-\delta) Y_{t+j+1} - T_{t+j+1}\right], \\ a_t &= n_t + h_t. \end{aligned}$$

We guess and later verify that

$$V_t(a, \epsilon) = -\frac{1}{\gamma} \exp\left(-\gamma C \epsilon - \gamma A a + \vartheta_t\right),$$

where ϑ_t is a deterministic function of time. We use standard properties of Normal distributions to rewrite the problem as

$$V_t\left(a_t, \epsilon_t\right) = \max_{c_t, x_{t+1}} \left\{ -\frac{1}{\gamma} \exp\left(-\gamma c_t\right) - \frac{1}{\gamma} \exp\left(\epsilon_t - \rho - \gamma A \widetilde{\mathbb{E}}_t a_{t+1} + \frac{\gamma^2}{2} \widetilde{\mathbb{V}}_t (C\epsilon_{t+1} + A a_{t+1}) + \widetilde{\mathbb{E}}_t \vartheta_{t+1} \right) \right\}.$$

Note that it is not the variance of future wealth that matters for the household's optimal choice, but the variance of a linear combination of the future preference shock and future wealth.

The conditional expectation of one-period-ahead total wealth is

$$\widetilde{\mathbb{E}}_t a_{t+1} = (1+r)\left[(1-\delta) Y_t - T_t + n_t - c_t - q_t x_{t+1}\right] + \left(\frac{\delta}{\overline{X}} \alpha_{Y,t} + \alpha_{q,t}\right) x_{t+1} + \widetilde{\mathbb{E}}_t h_{t+1}.$$

Similarly, the conditional variance of $C\epsilon_{t+1} + Aa_{t+1}$ is

$$\begin{aligned} \widetilde{\mathbb{V}}_t\left(C\epsilon_{t+1} + A a_{t+1}\right) =& \widetilde{\mathbb{V}}_t \left[C\epsilon_{t+1} + A \frac{\delta}{\overline{X}} \left(\alpha_{Y,t} + \beta_{Y,t}\epsilon_{t+1}\right) x_{t+1} + A h_{t+1} \right] \\ =& \sigma^2 \left[C + A \frac{\delta \beta_{Y,t}}{\overline{X}} x_{t+1} + A\left(1-\delta\right) \beta_{Y,t} + A \beta_{Tr,t} \right]^2, \end{aligned}$$

where we used (3) to compute the sensitivity of h_{t+1} to ϵ_{t+1}.

The first order condition with respect to c_t is

$$\exp\left(-\gamma c_t\right) = A\left(1+r\right) \exp\left(\epsilon_t - \rho - \gamma A \widetilde{\mathbb{E}}_t a_{t+1} + \frac{1}{2} \gamma^2 \widetilde{\mathbb{V}}_t \left[C\epsilon_{t+1} + A a_{t+1}\right] + \widetilde{\mathbb{E}}_t \vartheta_{t+1} \right) \tag{A.9}$$

and with respect to investment in the risky asset x_{t+1} is

$$\frac{\delta}{\overline{X}} \alpha_{Y,t} + \alpha_{q,t} - (1+r)\, q_t - \gamma \sigma^2 \frac{\delta \beta_{Y,t}}{\overline{X}} \left[C + A \frac{\delta \beta_{Y,t}}{\overline{X}} x_{t+1} + A\left(1-\delta\right) \beta_{Y,t} + A \beta_{Tr,t} \right] = 0. \tag{A.10}$$

The latter equation determines the demand for the risky asset

$$x_{t+1} = \frac{\delta \alpha_{Y,t}/\overline{X} + \alpha_{q,t} - (1+r)\, q_t}{\gamma \left(\delta \beta_{Y,t}\sigma/\overline{X}\right)^2 A} - \frac{(1-\delta)\,\beta_{Y,t}}{\delta \beta_{Y,t}/\overline{X}} - \frac{\beta_{Tr,t}}{\delta \beta_{Y,t}/\overline{X}} - \frac{C/A}{\delta \beta_{Y,t}/\overline{X}}. \tag{A.11}$$

Taking logs of both sides of equation (A.9) and using the expressions for $\widetilde{\mathbb{E}}_t a_{t+1}$ and $\widetilde{\mathbb{V}}_t(C\epsilon_{t+1} + Aa_{t+1})$, we rewrite (A.3) as

$$\begin{aligned} c_t =& -\frac{1}{\gamma[A(1+r)+1]} \left\{ \log[A(1+r)] + \epsilon_t - \rho \right\} + \frac{A(1+r)}{A(1+r)+1} a_t \\ &+ \frac{A}{A(1+r)+1} \left[\frac{\delta}{\overline{X}} \alpha_{Y,t} + \alpha_{q,t} - (1+r) q_t \right] x_{t+1} \\ &- \frac{\gamma \sigma^2}{2[A(1+r)+1]} \left[C + A \frac{\delta \beta_{Y,t}}{\overline{X}} x_{t+1} + A\left(1-\delta\right) \beta_{Y,t} + A \beta_{Tr,t} \right]^2 - \frac{1}{\gamma[A(1+r)+1]} \widetilde{\mathbb{E}}_t \vartheta_{t+1}. \end{aligned}$$

For our guess to be true, our conjectured value function must satisfy equation (A.8), for all values of ϵ_t, a_t, and t:

$$-\frac{1}{\gamma}\exp\left(-\gamma C\epsilon_t-\gamma A a_t+\vartheta_t\right)$$
$$=\max_{c_t,x_{t+1}}\left\{-\frac{1}{\gamma}\exp\left(-\gamma c_t\right)-\frac{1}{\gamma}\exp\left[\epsilon_t-\rho-\gamma A\widetilde{\mathbb{E}}_t a_{t+1}+\frac{1}{2}\gamma^2\widetilde{\mathbb{V}}_t(C\epsilon_{t+1}+Aa_{t+1})+\widetilde{\mathbb{E}}_t\vartheta_{t+1}\right]\right\},$$

Using (A.9), the latter becomes

$$-\frac{1}{\gamma}\exp\left(-\gamma C\epsilon_t-\gamma A a_t+\vartheta_t\right)=-\frac{1}{\gamma}\exp\left\{\log\left[\frac{A(1+r)+1}{A(1+r)}\right]-\gamma c_t\right\},$$

where, with slight abuse of notation, we use c_t to denote optimal consumption. Therefore, the constants A and C must satisfy

$$A=\frac{r}{1+r},\ C=-\frac{1}{\gamma(1+r)}.$$

The asset demand (A.11) is thus

$$x_{t+1}=\frac{\delta\alpha_{Y,t}/\overline{X}+\alpha_{q,t}-(1+r)\,q_t}{\frac{r}{1+r}\gamma\left(\delta\beta_{Y,t}\sigma/\overline{X}\right)^2}-\frac{(1-\delta)\,\beta_{Y,t}}{\delta\beta_{Y,t}/\overline{X}}-\frac{\beta_{Tr,t}}{\delta\beta_{Y,t}/\overline{X}}+\frac{1}{\gamma r\delta\beta_{Y,t}/\overline{X}},\tag{A.12}$$

which is equation (20) in the main text. Similarly, consumption becomes

$$\begin{aligned}c_t=&-\frac{1}{\gamma(1+r)}\left[\log r+\epsilon_t-\rho\right]+\frac{r}{1+r}a_t+\frac{r}{(1+r)^2}\left[\frac{\delta}{\overline{X}}\alpha_{Y,t}+\alpha_{q,t}-(1+r)\,q_t\right]x_{t+1}\\&-\frac{\gamma\sigma^2}{2(1+r)}\left(\frac{r}{1+r}\right)^2\left[\frac{\delta\beta_{Y,t}}{\overline{X}}x_{t+1}+(1-\delta)\,\beta_{Y,t}+\beta_{Tr,t}-\frac{1}{\gamma r}\right]^2-\frac{1}{\gamma\,(1+r)}\widetilde{\mathbb{E}}_t\vartheta_{t+1}\end{aligned}\tag{A.13}$$

and, finally,

$$\begin{aligned}\vartheta_t=&\log\left(\frac{1+r}{r}\right)+\frac{\log r-\rho}{1+r}-\gamma\frac{r}{(1+r)^2}\left(\frac{\delta}{\overline{X}}\alpha_{Y,t}+\alpha_{q,t}-(1+r)\,q_t\right)x_{t+1}\\&+\frac{\gamma^2\sigma^2}{2(1+r)}\left(\frac{r}{1+r}\right)^2\left[\frac{\delta\beta_{Y,t}}{\overline{X}}x_{t+1}+(1-\delta)\,\beta_{Y,t}+\beta_{Tr,t}-\frac{1}{\gamma r}\right]^2+\frac{1}{1+r}\widetilde{\mathbb{E}}_t\vartheta_{t+1}.\end{aligned}$$

We can then iterate the latter forward to get

$$\begin{aligned}\vartheta_t=&\frac{1+r}{r}\log\left(\frac{1+r}{r}\right)+\frac{\log r-\rho}{r}-\gamma\frac{r}{(1+r)^2}\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left[\frac{\delta}{\overline{X}}\alpha_{Y,t+j}+\alpha_{q,t+j}-(1+r)\,\alpha_{q,t+j-1}\right]x_{t+j+1}\\&+\frac{\gamma^2\sigma^2}{2\,(1+r)}\left(\frac{r}{1+r}\right)^2\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left[\frac{\delta\beta_{Y,t+j}}{\overline{X}}x_{t+j+1}+(1-\delta)\,\beta_{Y,t+j}+\beta_{Tr,t+j}-\frac{1}{\gamma r}\right]^2,\end{aligned}$$

where, to simplify notation, we used $\alpha_{q,t-1}=q_t$. Note that the last formula does not feature the expectation operator $\widetilde{\mathbb{E}}_t$ because the coefficients α's and β's are deterministic. Using the expression for ϑ_t, we can further rewrite (A.13) as

$$c_t = -\frac{\log(1+r)-\rho}{\gamma r} - \frac{1}{\gamma(1+r)}\epsilon_t + \frac{r}{1+r}a_t + \frac{r}{(1+r)^2}\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left[\frac{\delta}{\overline{X}}\alpha_{Y,t+j} + \alpha_{q,t+j} - (1+r)\,\alpha_{q,t+j-1}\right]x_{t+j+1}$$
$$- \frac{\gamma\sigma^2}{2(1+r)}\left(\frac{r}{1+r}\right)^2\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left[\frac{\delta\beta_{Y,t+j}}{\overline{X}}x_{t+j+1} + (1-\delta)\,\beta_{Y,t+j} + \beta_{Tr,t+j} - \frac{1}{\gamma r}\right]^2. \quad \text{(A.14)}$$

Notice that, in the last two equations, we used x_{t+j+1} to denote the optimal investment in risky assets. Since (A.12) is a deterministic function of time, so is ϑ_t, therefore, our conjecture is verified.

Proof of Proposition 3

To simplify notation, we henceforth drop the "star" from Y_t and q_t, which we use in the main text to denote REE values.

We begin with equation (23). As before, when solving the household problem, we conjecture that, conditional on $t \geq 0$, the subset of endogenous variables $z_{t+1} \equiv (Y_{t+1}, q_{t+1}, Tr_{t+1})$ satisfies (10). In addition, we conjecture that $\beta_{q,t} = 0$, $t \geq 0$. We now compute the REE and verify that these conjectures hold.

First, both the goods and the asset market clear, that is, $c_t = Y_t$ and $X_{t+1} + x_{t+1} = \overline{X}$, $t \geq 0$. Second, with these asset-market clearing conditions, we can express total wealth of the households as

$$a_t = (1+r)\,B_t + \left(\frac{\delta}{\overline{X}}Y_t + q_t\right)(\overline{X} - X_t) + \mathbb{E}_t\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left[(1-\delta)\,Y_{t+j} - T_{t+j}\right].$$

Taking into account the clearing conditions, consumption demand (A.14), the optimal choice of risky assets (A.12), and the above expression for total wealth, we can write the goods market clearing conditions as

$$Y_t = -\frac{\log(1+r)-\rho}{\gamma r} - \frac{1}{\gamma(1+r)}\epsilon_t$$
$$+ \frac{r}{1+r}\left\{(1+r)\,B_t + \left(\frac{\delta}{\overline{X}}Y_t + q_t\right)(\overline{X} - X_t) + \mathbb{E}_t\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left[(1-\delta)\,Y_{t+j} - T_{t+j}\right]\right\}$$
$$+ \frac{r}{(1+r)^2}\mathbb{E}_t\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left[\frac{\delta}{\overline{X}}Y_{t+j+1} + q_{t+j+1} - (1+r)\,q_{t+j}\right](\overline{X} - X_{t+j+1})$$
$$- \frac{\gamma\sigma^2}{2(1+r)}\left(\frac{r}{1+r}\right)^2\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left[\frac{\delta\beta_{Y,t+j}}{\overline{X}}(\overline{X} - X_{t+j+1}) + (1-\delta)\,\beta_{Y,t+j} + \beta_{Tr,t+j} - \frac{1}{\gamma r}\right]^2. \quad \text{(A.15)}$$

Households take into account the overall budget constraint of the government when forming expectations about the future. Thus, iterating (3) forward, gives

$$(1+r)\,B_t = \sum_{j=0}^{\infty}\frac{1}{(1+r)^j}T_{t+j} + \left(\frac{\delta}{\overline{X}}Y_t + q_t\right)X_t$$
$$+ \sum_{j=0}^{\infty}\frac{1}{(1+r)^{j+1}}\left[\frac{\delta}{\overline{X}}Y_{t+j+1} + q_{t+j+1} - (1+r)\,q_{t+j}\right]X_{t+j+1}. \quad \text{(A.16)}$$

Combining (A.15) and (A.16), we obtain

$$Y_t = -\frac{\log(1+r)-\rho}{\gamma r} - \frac{1}{\gamma(1+r)}\epsilon_t + \frac{r}{1+r}\left[Y_t + \mathbb{E}_t \sum_{j=0}^{\infty} \frac{1}{(1+r)^{j+1}} Y_{t+j+1}\right]$$
$$- \frac{\gamma\sigma^2}{2(1+r)}\left(\frac{r}{1+r}\right)^2 \sum_{j=0}^{\infty} \frac{1}{(1+r)^j}\left[\frac{\delta\beta_{Y,t+j}}{\overline{X}}(\overline{X} - X_{t+j+1}) + (1-\delta)\beta_{Y,t+j} + \beta_{Tr,t+j} - \frac{1}{\gamma r}\right]^2.$$

From (19), Tr_{t+j+1} is a linear function of Y_{t+j+1}, q_{t+j+1}, and, by (4), of q_{t+j}. Given our conjecture on Y_{t+j+1} and q_{t+j+1}, it is immediate to see that Tr_{t+j+1} is a linear function of ϵ_{t+j+1}. In particular, $\beta_{Tr,t+j} = \delta\beta_{Y,t+j}X_{t+j+1}/\overline{X}$. Thus,

$$Y_t = -\frac{\log(1+r)-\rho}{\gamma r} - \frac{1}{\gamma(1+r)}\epsilon_t + \frac{r}{1+r}\left(Y_t + \mathbb{E}_t \sum_{j=0}^{\infty} \frac{1}{(1+r)^{j+1}} Y_{t+j+1}\right)$$
$$- \frac{\gamma\sigma^2}{2(1+r)}\left(\frac{r}{1+r}\right)^2 \sum_{j=0}^{\infty} \frac{1}{(1+r)^j}\left[\beta_{Y,t+j} - \frac{1}{\gamma r}\right]^2$$

or, solving for Y_t,

$$Y_t = -\frac{1+r}{r} \cdot \frac{\log(1+r)-\rho}{\gamma} - \frac{1}{\gamma}\epsilon_t + \frac{r}{1+r}\mathbb{E}_t \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} Y_{t+j+1}$$
$$- \frac{\gamma\sigma^2}{2}\left(\frac{r}{1+r}\right)^2 \sum_{j=0}^{\infty} \frac{1}{(1+r)^j}\left[\beta_{Y,t+j} - \frac{1}{\gamma r}\right]^2.$$

For our conjecture on Y_t to be true, we must have $\beta_{Y,t} = -1/\gamma$, for all $t \geq 0$. Therefore,

$$Y_t = -\frac{1}{\gamma}\epsilon_t + \frac{r}{1+r}\mathbb{E}_t \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} Y_{t+j+1},$$

where we also used the condition $1 + r = \exp(\rho - \sigma^2/2)$ to eliminate the constant. As in the case of Proposition 1, we face a multiplicity problem. We then let $\mathbb{E}_t Y_{t+j} = \overline{Y}$, for some $\overline{Y}$ and all $j > 0$, and obtain (23).

Finally, we can simply combine the market-clearing condition (6) with (A.12) to get

$$\overline{X} = \frac{\delta\overline{Y} + q_{t+1}\overline{X} - (1+r)q_t\overline{X}}{\frac{r}{1+r} \cdot \frac{\delta^2\sigma^2}{\gamma}}\overline{X} - \frac{1-\delta}{\delta}\overline{X} - \frac{1}{r\delta}\overline{X},$$

whose unique non-explosive solution is (22). Finally, since (22) is a deterministic function of time, our initial conjecture is verified.

Proof of Proposition 4

We first compute the temporary equilibrium with level-k agents. For each equilibrium we consider, we also need to verify our conjecture that, conditional on $t \geq 0$, the vector of endogenous variables $z_{t+1} \equiv (Y_{t+1}, q_{t+1}, Tr_{t+1})$ satisfies (10) and, in addition, $\beta_{q,t} = 0$, $t \geq 0$.

We proceed recursively, starting from the first line of the pricing equation (24). At time t, level-1 agents solve (A.8) under the belief that the economy will be in the REE without asset purchases. By Proposition

3, in the REE, z_{t+1} satisfies (10) and the asset price (22) depends only on time, therefore, beliefs of level-1 agents will also satisfy the same properties and our conjecture is verified. In particular, since transfers are zero in the absence of asset purchases, level-1 agents believe $\alpha_{Tr,t+j} = \beta_{Tr,t+j} = 0$, $j \geq 0$. For the same reason, $\alpha_{Y,t+j} = \overline{Y}$ and $\alpha_{Y,t+j} = q^*$, $j \geq 0$. Substituting the latter into (A.12) yields the asset demand

$$x_{t+1} = \frac{\delta \overline{Y} + q^* \overline{X} - (1+r)\, q_t \overline{X}}{\frac{\sigma^2}{\gamma} \cdot \frac{r}{1+r} \delta^2} \overline{X} - \frac{1-\delta}{\delta} \overline{X} - \frac{1}{r\delta} \overline{X}. \tag{A.17}$$

The asset price in the first line of equation (24) is the value q_t that sets (A.17) equal to the supply $\overline{X} - X_{t+1}$, with $\eta_2 = \sigma^2 \delta^2 / [\gamma (1+r) \overline{X}]$. Notice that q_t^1 is a deterministic function of time.

Consider now the output equation (25). Substituting level-1 beliefs into (A.14) and using the definition of a_t yields

$$\begin{aligned} c_t = &- \frac{\log(1+r) - \rho}{\gamma r} - \frac{1}{\gamma(1+r)} \epsilon_t + \frac{r}{1+r} n_t + \frac{r}{1+r} (1-\delta) \left(Y_t^1 + \frac{1}{r} \overline{Y} \right) - \frac{r}{1+r} \widetilde{\mathbb{E}}_t^1 \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} T_{t+j} \\ &+ \frac{r}{(1+r)^2} \left[\frac{\delta}{\overline{X}} \overline{Y} + q^* - (1+r)\, q_t^1 \right] x_{t+1} + \frac{r}{(1+r)^2} \left[\frac{\delta}{\overline{X}} \overline{Y} + q^* - (1+r)\, q^* \right] \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} x_{t+j+1} \\ &- \frac{\sigma^2}{2\gamma(1+r)} \left(\frac{r}{1+r} \right)^2 \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} \left(\frac{\delta}{\overline{X}} x_{t+j+1} + 1 - \delta + \frac{1}{r} \right)^2, \end{aligned}$$

where the superscript "1" on Y_t and q_t denotes the fact that the realized current price and output are, by definition, the price and output in the level-1 equilibrium. Since level-1 agents expect the economy to be in the REE, from (A.12), they expect their demand to be equal to $x_{t+j+1} = \overline{X}$, $j > 0$. Thus,

$$\begin{aligned} c_t = &- \frac{\log(1+r) - \rho}{\gamma r} - \frac{1}{\gamma(1+r)} \epsilon_t + \frac{r}{1+r} n_t + \frac{r}{1+r} (1-\delta) \left(Y_t^1 + \frac{1}{r} \overline{Y} \right) - \frac{r}{1+r} \widetilde{\mathbb{E}}_t^1 \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} T_{t+j} \\ &+ \frac{r}{(1+r)^2} \left[\frac{\delta}{\overline{X}} \overline{Y} + q^* - (1+r)\, q_t^1 \right] x_{t+1} + \frac{1}{(1+r)^2} \delta \frac{\sigma^2}{\gamma} \\ &- \frac{\sigma^2}{2\gamma(1+r)} \left(\frac{r}{1+r} \right)^2 \left(\frac{\delta}{\overline{X}} x_{t+1} + 1 - \delta + \frac{1}{r} \right)^2 - \frac{\sigma^2}{2\gamma(1+r)} \frac{1}{r}, \end{aligned}$$

where we also used (22). A level-1 agent observes current transfers from the central bank and, in addition, expects future transfers to be zero. Hence, from the government budget constraint,

$$\begin{aligned} c_t = &- \frac{\log(1+r) - \rho}{\gamma r} - \frac{1}{\gamma(1+r)} \epsilon_t + \frac{r}{1+r} n_t + \frac{r}{1+r} (1-\delta) \left(Y_t^1 + \frac{1}{r} \overline{Y} \right) + \frac{r}{1+r} \left[Tr_t - (1+r)\, B_t \right] \\ &+ \frac{r}{(1+r)^2} \left[\frac{\delta}{\overline{X}} \overline{Y} + q^* - (1+r)\, q_t^1 \right] x_{t+1} + \frac{1}{(1+r)^2} \delta \frac{\sigma^2}{\gamma} \\ &- \frac{\sigma^2}{2\gamma(1+r)} \left(\frac{r}{1+r} \right)^2 \left(\frac{\delta}{\overline{X}} x_{t+1} + 1 - \delta + \frac{1}{r} \right)^2 - \frac{\sigma^2}{2\gamma(1+r)} \frac{1}{r}, \end{aligned}$$

or, after further simplifications,

$$c_t = -\frac{\log(1+r)-\rho}{\gamma r} - \frac{1}{\gamma(1+r)}\epsilon_t + \frac{r}{1+r}\left[n_t - (1+r)B_t\right] + \frac{r}{1+r}(1-\delta)Y_t^1 + \frac{1}{1+r}\overline{Y} + \frac{r}{1+r}Tr_t - \frac{r}{1+r}q_t^1\overline{X}$$
$$+ \frac{r}{(1+r)^2}\left[\delta\overline{Y} + q^*\overline{X} - (1+r)q_t^1\overline{X}\right]\frac{x_{t+1}-\overline{X}}{\overline{X}} - \frac{\sigma^2}{2\gamma(1+r)}\left(1+\frac{r}{1+r}\delta\frac{x_{t+1}-\overline{X}}{\overline{X}}\right)^2 - \frac{\sigma^2}{2\gamma(1+r)}\frac{1}{r}.$$

Finally, if we rearrange the equilibrium condition in the asset market (A.17), we can rewrite the expression above as

$$\begin{aligned} c_t = &-\frac{1}{r}\left[\frac{\log(1+r)-\rho}{\gamma} + \frac{1}{2}\frac{\sigma^2}{\gamma}\right] - \frac{1}{\gamma(1+r)}\epsilon_t + \frac{r}{1+r}\left[n_t-(1+r)B_t\right] + \frac{r}{1+r}(1-\delta)Y_t^1 \\ &+ \frac{1}{1+r}\overline{Y} + \frac{r}{1+r}Tr_t - \frac{r}{1+r}q_t^1\overline{X} + \frac{1}{2}\frac{r}{(1+r)^2}\cdot\frac{\sigma^2}{\gamma}\delta\frac{r}{1+r}\delta\frac{X_{t+1}}{\overline{X}}\cdot\frac{X_{t+1}}{\overline{X}}. \end{aligned} \tag{A.18}$$

From market clearing at $t-1$, households hold $x_t = \overline{X} - X_t$ units of risky assets, all the reserves created by the central bank, and all the bonds issued by the treasury, thus, $n_t = (1+r)(B_t + R_t) + (\delta Y_t^1/\overline{X} + q_t^1)(\overline{X} - X_t)$. In addition, equation (4) requires that reserves equal the value of assets purchased by the central bank and that all the profits or losses are transferred to the treasury, that is, $R_t = q_{t-1}^1 X_t$ and $Tr_t = [\delta Y_t^1/\overline{X} + q_t^1 - (1+r)q_{t-1}^1]X_t$. Therefore, using the latter expressions and imposing clearing in the goods market, we can rewrite (A.18), in equilibrium, as follows:

$$Y_t^1 = \overline{Y} - \frac{1}{\gamma}\epsilon_t - \frac{1+r}{r}\left[\frac{\log(1+r)-\rho}{\gamma} + \frac{\sigma^2}{2\gamma}\right] + \frac{1}{2}\frac{r^2}{(1+r)^2}\cdot\frac{\sigma^2}{\gamma}\delta^2\left(\frac{X_{t+1}}{\overline{X}}\right)^2,$$

which, using (23) and $1+r = \exp(\rho - \sigma^2/2)$, gives the first line of equation (25), with $\eta_3 = r^2\sigma^2\delta^2/[2\gamma(1+r)^2]$. Notice that output is a linear function of the shock.

Finally, in equilibrium, transfers satisfy (19), which together with (4) and (24), imply that Tr_t is a linear function of ϵ_t.

We now move to the second line of the pricing equation (24). Suppose that the vector of endogenous variables z_{t+1} resulting from the temporary equilibrium with level-$(k-1)$ agents satisfies (10) and, in addition, that q_t^{k-1} is a deterministic function of time. Level-k agents, for $k > 1$, form beliefs under the assumption that the economy is populated by level-$(k-1)$ agents, thus, our conjecture about z_{t+1} is verified. In particular, $\alpha_{q,t} = q_{t+1}^{k-1}$, $t \geq 0$. Also, these households incorporate (19) into their beliefs, hence, $\beta_{Tr,t} = -\delta X_{t+1}/(\gamma\overline{X})$, $t \geq 0$. We can thus rewrite the asset demand (A.12) as follows:

$$x_{t+1} = \frac{\delta\alpha_{Y,t} + q_{t+1}^{k-1}\overline{X} - (1+r)q_t\overline{X}}{\frac{\sigma^2}{\gamma}\cdot\frac{r}{1+r}\delta^2}\overline{X} - \frac{1-\delta}{\delta}\overline{X} - X_{t+1} - \frac{1}{r\delta}\overline{X}. \tag{A.19}$$

The asset price (24) is then the value q_t that sets demand (A.19) equal to the net supply $\overline{X} - X_{t+1}$. Notice that q_t^k is a deterministic function of time.

We finally consider the second line of the output equation (25). We first substitute level-k beliefs into

(A.14) and use the definition of a_t:

$$c_t = -\frac{\log(1+r)-\rho}{\gamma r} - \frac{1}{\gamma(1+r)}\epsilon_t + \frac{r}{1+r}n_t + \frac{r}{1+r}(1-\delta)\left[Y_t^k + \frac{1}{1+r}\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\alpha_{Y,t+j}\right]$$
$$- \frac{r}{1+r}\widetilde{\mathbb{E}}_t^k \sum_{j=0}^{\infty}\frac{1}{(1+r)^j}T_{t+j} + \frac{r}{(1+r)^2}\left[\frac{\delta}{\overline{X}}\alpha_{Y,t} + q_{t+1}^{k-1} - (1+r)\,q_t^k\right]x_{t+1}$$
$$+ \frac{r}{(1+r)^2}\sum_{j=1}^{\infty}\frac{1}{(1+r)^j}\left[\frac{\delta}{\overline{X}}\alpha_{Y,t+j} + q_{t+j+1}^{k-1} - (1+r)\,q_{t+j}^{k-1}\right]x_{t+j+1}$$
$$- \frac{\sigma^2}{2\gamma(1+r)}\left(\frac{r}{1+r}\right)^2\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left[\frac{\delta}{\overline{X}}x_{t+j+1} + \frac{\delta}{\overline{X}}X_{t+j+1} + 1 - \delta + \frac{1}{r}\right]^2.$$

We can use the treasury's budget constraint (3) to simplify the last equation as follows:

$$c_t = -\frac{\log(1+r)-\rho}{\gamma r} - \frac{1}{\gamma(1+r)}\epsilon_t + \frac{r}{1+r}\left[n_t - (1+r)B_t\right] + \frac{r}{1+r}(1-\delta)\,Y_t^k + \frac{r}{(1+r)^2}\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\alpha_{Y,t+j}$$
$$+ \frac{r}{1+r}Tr_t - \frac{r}{(1+r)}q_t^k\overline{X} + \frac{r}{(1+r)^2}\left[\frac{\delta}{\overline{X}}\alpha_{Y,t} + q_{t+1}^{k-1} - (1+r)\,q_t^k\right]\left(x_{t+1} + X_{t+1} - \overline{X}\right)$$
$$+ \frac{r}{(1+r)^2}\sum_{j=1}^{\infty}\frac{1}{(1+r)^j}\left[\frac{\delta}{\overline{X}}\alpha_{Y,t+j} + q_{t+j+1}^{k-1} - (1+r)\,q_{t+j}^{k-1}\right]\left(x_{t+j+1} + X_{t+j+1} - \overline{X}\right)$$
$$- \frac{\sigma^2}{2\gamma(1+r)}\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left[1 + \frac{r}{1+r}\frac{\delta}{\overline{X}}\left(x_{t+j+1} + X_{t+j+1} - \overline{X}\right)\right]^2.$$

Taking the expectation, conditional on time $t+j-1$, $j>0$, of (A.19) gives

$$\delta\alpha_{Y,t+j} + q_{t+j+1}^{k-1}\overline{X} - (1+r)\,q_{t+j}^{k-1}\overline{X} = \frac{\sigma_x^2}{\gamma}\delta\left(1 + \frac{r}{1+r}\delta\frac{x_{t+j+1} + X_{t+j+1} - \overline{X}}{\overline{X}}\right),$$

where we used the fact that $\widetilde{\mathbb{E}}_{t+j-1}^k q_{t+j} = q_{t+j}^{k-1}$. Similarly, from (A.12) with $q_t = q_t^k$,

$$\delta\alpha_{Y,t} + q_{t+1}^{k-1}\overline{X} - (1+r)\,q_t^k\overline{X} = \frac{\sigma^2}{\gamma}\delta\left(1 + \frac{r}{1+r}\delta\frac{x_{t+1} + X_{t+1} - \overline{X}}{\overline{X}}\right).$$

We can then rewrite the above expression for c_t as

$$c_t = -\frac{1}{\gamma(1+r)}\epsilon_t + \frac{r}{1+r}\left[n_t - (1+r)B_t\right] + \frac{r}{1+r}(1-\delta)\,Y_t^k + \frac{r}{(1+r)^2}\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\alpha_{Y,t+j}$$
$$+ \frac{r}{1+r}Tr_t - \frac{r}{1+r}q_t^k\overline{X} + \frac{1}{2}\cdot\frac{1}{1+r}\left(\frac{r}{1+r}\right)^2\frac{\sigma^2}{\gamma}\delta^2\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left(\frac{x_{t+j+1} + X_{t+j+1} - \overline{X}}{\overline{X}}\right)^2,$$

where we used $1+r = \exp(\rho - \sigma^2/2)$ to eliminate the constant term.

To further simplify the expression above, we can again take the expectation, conditional on time $t+j-1$, $j>0$, of (A.19) and use the expressions for the asset price in the level-k equilibrium in equation (24) to get

$$\frac{\overline{X} - X_{t+j+1} - x_{t+j+1}}{\overline{X}} = \frac{(1+r)^2}{\frac{\sigma^2}{\gamma} r\delta^2}(q_{t+j}^{k-1} - q_{t+j}^{k})\overline{X}.$$

Therefore, the optimal level of consumption becomes

$$\begin{aligned} c_t = &-\frac{1}{\gamma(1+r)}\epsilon_t + \frac{r}{1+r}[n_t - (1+r)B_t] + \frac{r}{1+r}(1-\delta)Y_t^k + \frac{r}{(1+r)^2}\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\alpha_{Y,t+j} \\ &+ \frac{r}{1+r}Tr_t - \frac{r}{1+r}q_t^k\overline{X} + \frac{1}{2}\cdot\frac{1}{1+r}\cdot\frac{(1+r)^2}{\frac{\sigma^2}{\gamma}\delta^2}\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}(q_{t+j}^{k-1} - q_{t+j}^{k})^2\overline{X}^2. \end{aligned} \tag{A.20}$$

As in the case with level-1 agents, we have $n_t = (1+r)(R_t + B_t) + (\delta Y_t^k/\overline{X} + q_t^k)(\overline{X} - X_t)$, $R_t = q_{t-1}^k X_t$, and $Tr_t = [\delta Y_t^k/\overline{X} + q_t^k - (1+r)\,q_{t-1}^k]X_t$. Substituting the latter expressions into (A.20), we can rewrite equilibrium in the goods market as

$$Y_t^k = \overline{Y} - \frac{1}{\gamma}\epsilon_t + \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left[(\alpha_{Y,t+j} - \overline{Y}) + \frac{1}{2}\cdot\frac{1+r}{\frac{\sigma^2}{\gamma}\delta^2}\left(q_{t+j+1}^{k-1} - q_{t+j+1}^{k}\right)^2\overline{X}^2\right].$$

Since $\alpha_{Y,t} - \overline{Y} = \widetilde{\mathbb{E}}_t^k Y_{t+1} - \overline{Y} = Y_{t+1}^{k-1} - Y_{t+1}^{*}$, we obtain the output equation (25), with $\eta_1 = \gamma(1+r)^2/(2\sigma^2\delta^2)$. Notice that output is a linear function of the shock ϵ_t.

Finally, in equilibrium, transfers satisfy (19), which together with (4) and (24), imply that Tr_t is a linear function of ϵ_t.

Reflective Equilibrium. In the reflective equilibrium, the asset and goods markets clear under the assumption that the economy is populated by agents with different levels of sophistication. Importantly, each agent holds the same beliefs as in the temporary equilibrium with level-k agents, therefore, our conjecture about z_{t+1} is immediately satisfied.

We begin with the asset price. The total demand for the asset is obtained by the weighted sum of (A.17) and (A.19), for all k, with weights given by $f(k)$:

$$\frac{\overline{X} - X_{t+1}}{\overline{X}} = \frac{1}{\frac{\sigma^2}{\gamma}\cdot\frac{r}{1+r}\delta^2}\sum_{k=1}^{\infty} f(k)\left[\delta\alpha_{Y,t}^k + \tilde{q}_{t+1}^k\overline{X} - (1+r)q_t\overline{X}\right] - \sum_{k=1}^{\infty} f(k)\frac{\beta_{Tr,t}^k}{\delta\beta_{Y,t}^k} - \frac{1-\delta}{\delta} - \frac{1}{r\delta},$$

where, from above, $\alpha_{Y,t}^1 = \overline{Y}$, $\beta_{Tr,t}^1 = 0$, $\beta_{Tr,t}^k/\delta\beta_{Y,t}^k = -X_{t+1}/\overline{X}$, and $\tilde{q}_{t+1}^k = q_{t+1}^{k-1}$. Adding and subtracting q_t^k and rearranging yields

$$\frac{(1+r)\overline{X}}{\frac{\sigma^2}{\gamma}\cdot\frac{r}{1+r}\delta^2}\sum_{k=1}^{\infty} f(k)(q_t - q_{t+1}^k) = \sum_{k=1}^{\infty} f(k)\left\{\frac{1}{\frac{\sigma^2}{\gamma}\cdot\frac{r}{1+r}\delta^2}\left[\delta\alpha_{Y,t}^k + q_{t+1}^{k-1}\overline{X} - (1+r)\,q_t^k\overline{X}\right] - \frac{\beta_{Tr,t}^k}{\delta\beta_{Y,t}^k} - \frac{1+r}{r\delta} + \frac{X_{t+1}}{\overline{X}}\right\}.$$

The expression for the asset price in the reflective equilibrium, i.e., $\widehat{q}_t = \sum_{k=1}^{\infty} f(k)\,\widehat{q}_t^k$, follows from the fact that, by definition, q_t^k is the price that clears the market when agents are level-k, that is, q_t^k sets the expression in the round bracket on the right-hand side equal to zero.

We turn to output. As for the asset price, total demand for consumption is obtained by the weighted sum of (A.18) and (A.20), for all k:

$$\sum_{k=1}^{\infty} f(k)\, c_t^k = -\frac{1}{\gamma(1+r)}\epsilon_t + \frac{r}{1+r}\left[n_t - (1+r)B_t\right] + \frac{r}{1+r}(1-\delta)\, Y_t + \frac{r}{(1+r)^2}\sum_{k=1}^{\infty} f(k) \sum_{j=0}^{\infty} \frac{1}{(1+r)^j}\alpha_{Y,t+j}^k$$
$$+ \frac{r}{1+r}Tr_t - \frac{r}{1+r}q_t\bar{X} + \frac{1}{2}\cdot\frac{1}{1+r}\frac{(1+r)^2}{\frac{\sigma_X^2}{\gamma}\delta^2}\sum_{k=1}^{\infty} f(k)\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left(q_{t+j}^{k-1} - q_{t+j}^k\right)^2 \overline{X}^2, \qquad \text{(A.21)}$$

where $Y_t = \sum_{k=1}^{\infty} f(k)\, Y_t^k$. Also, aggregate financial wealth n_t is given by

$$n_t = (1+r)\sum_{k=1}^{\infty} f(k)\,(1+r)\,(R_t^k + B_t^k) + \left(\frac{\delta}{\overline{X}}Y_t + q_t\right)\sum_{k=1}^{\infty} f(k)\, x_t^k$$
$$= (1+r)\sum_{k=1}^{\infty} f(k)\,(1+r)\,(R_t^k + B_t^k) + \left(\frac{\delta}{\overline{X}}Y_t + q_t\right)\left(\overline{X} - X_t\right),$$

where the last line used the risky-asset market-clearing at time $t-1$. Notice that we are allowing holdings of reserves and safe bonds to differ across agents with different levels of sophistication. The goods market-clearing condition requires $Y_t = \sum_{k=1}^{\infty} f(k)\, c_t^k$. Therefore, using the latter condition, together with (4) and (19), into (A.21), we can rewrite equilibrium in the goods market as

$$Y_t = -\frac{1}{\gamma}\epsilon_t + \frac{r}{1+r}\sum_{k=1}^{\infty} f(k)\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\alpha_{Y,t+j}^k + \frac{1}{2}\cdot\frac{(1+r)^2}{\frac{\sigma_X^2}{\gamma}\delta^2}\sum_{k=1}^{\infty} f(k)\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left(q_{t+j}^{k-1} - q_{t+j}^k\right)^2\overline{X}^2$$
$$= \sum_{k=1}^{\infty} f(k)\,\widehat{Y}_t^k + \eta_1\sum_{k=1}^{\infty} f(k)\left(\widehat{q}_t - \widehat{q}_t^k\right)^2\overline{X}^2,$$

which proves the result.

Sizes of the effects. As a bonus to this proof, we compare the sizes of the first order effect of prices and the second-order effect on output under level-1 thinking:

$$\frac{\widehat{Y}_t^1/\mathbb{E}Y_t^*}{q_t^1/q^*} = \frac{\eta_3\left(\frac{X_{t+1}}{\overline{X}}\right)^2/\overline{Y}}{\eta_2\frac{X_{t+1}}{\overline{X}}/\frac{\delta}{r}\left(\overline{Y} - \frac{\sigma^2}{\gamma}\right)\frac{1}{\overline{X}}}$$
$$= \frac{\eta_3}{\eta_2}\cdot\frac{X_{t+1}}{\overline{X}}\cdot\frac{\frac{\delta}{r}\left(\overline{Y} - \frac{\sigma^2}{\gamma}\right)\frac{1}{\overline{X}}}{\overline{Y}}$$
$$= \frac{\delta}{2}\cdot\frac{r}{1+r}\cdot\left(1 - \frac{\sigma^2}{\gamma\overline{Y}}\right)\cdot\frac{X_{t+1}}{\overline{X}}.$$

Note that this ratio is proportional to the net interest rate r, which is usually much smaller than one. As a result, not only the effect on output is second order, but the coefficient that controls this second-order effect is also very small.

B Predictability of Forecast Errors in the Data

In this appendix, we provide details of the empirical exercise that we only briefly described in the main text. As we mention in the main text, we follow Fieldhouse, Mertens and Ravn (2018). The authors provide a comprehensive description of the institutional details of the operations of the GSEs. Here, we briefly describe some of the details that are relevant for understanding our empirical results.

The GSEs have been routinely buying mortgages from mortgage issuers since their incorporation in the 1960s. They finance their purchases with debt securities that command a "liquidity and safety" premium similar to the one of Treasury securities. Although most of these purchases are motivated by the cyclical developments in the mortgage market (e.g., stimulating housing starts in recessions), some purchases are related to non-cyclical regulatory events (e.g., those invoked by a desire to increase homeownership among lower-income households or by concerns regarding structural budget deficits). FMR use narrative records to identify the motivation behind any considerable change in the GSEs' mortgage purchasing behavior and construct a list of major regulatory events that are not related to cyclical considerations (the paper contains a detailed discussion of the construction of these narrative events). We call these events "exogenous."

To quantify the impact of these exogenous events, FMR use various sources to obtain an estimate of the projected impact, denoted by m_t, of the agencies' capacity to purchase mortgages during the first year following the moment when a policy is publicly announced. Therefore, m_t can be thought of as news about future purchases by the GSEs following the exogenous events. The approach in FMR is similar to the one used in the literature on the effects of fiscal policies (see, for example, Ramey and Zubairy (2018), who used news about military spending as an instrument for government spending). We take m_t directly from FMR.

Empirical strategy. To estimate the effect of the asset purchases by the GSEs, we follow FMR and use the Jordà (2005) local projections method, implemented by two-stage least squares (2SLS). Specifically, in the first stage, we project the cumulative commitments $\sum_{j=0}^{h} p_{t+j}$ to purchase mortgages by the GSEs over $h+1$ months, expressed in constant dollars, on the non-cyclical narrative instrument m_t, also expressed in constant dollars, and a host of controls:

$$\frac{\sum_{j=0}^{h} p_{t+j}}{X_t} = \alpha_h^{(1)} + \gamma_h^{(1)} \frac{m_t}{X_t} + \varphi_h^{(1)}(L) Z_{t-1} + u_{t+h}^{(1)}. \tag{B.1}$$

We express the left-hand side variable as well as m_t on the right-hand side as ratios of X_t, a deterministic trend in real personal income obtained by fitting a third-degree polynomial of time to the log of personal income, deflated by the core personal consumption expenditures (PCE) price index. In equation (B.1), we also control for lagged values of the left-hand side variable, lagged growth rates of the core PCE price index, a nominal house price index, total mortgage debt, the log level of real mortgage originations, housing starts, and lags of several interest rate variables: the 3-month T-bill rate, the 10-year Treasury rate, the conventional mortgage interest rate, and the BAA-AAA corporate bond spread. The superscript (1) denotes first-stage regression coefficients and errors.

In the second stage, we estimate

$$y_{t+h} = \alpha_h^{(2)} + \gamma_h^{(2)} \left(\frac{12}{8} \times \frac{\sum_{j=0}^{7} p_{t+j}}{\widetilde{X}_t} \right) + \varphi_h^{(2)}(L)\, Z_{t-1} + u_{t+h}^{(2)}, \tag{B.2}$$

where y_{t+h} is any variable of interest in month $t+h$—such as the realized mortgage rate, or the mortgage rate forecast error—and $\widetilde{X}_t$ is a long-run trend in annualized mortgage originations. Since, in the first stage, we estimate the reaction of the GSEs' cumulative commitments at various horizons, we pick a specific horizon of eight months to use as an indicator of policy actions. The reason for this choice is that the F-statistics of the first stage is maximized at this horizon. By doing this, we again follow FMR. We estimate $\gamma_h^{(2)}$ by 2SLS, i.e., we replace the term multiplying $\gamma_h^{(2)}$ in (B.2) with its predicted value in the first stage

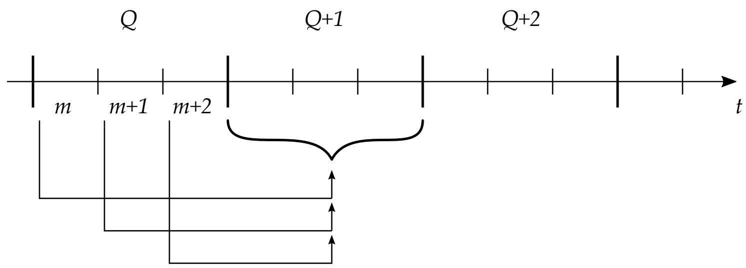

Figure A1: This figure depicts the definition of the monthly Blue Chip forecasts. The diagram shows that, when the participants are asked to forecast a certain variable over quarter $Q+1$ in months m and $m+1$, the forecast horizon varies.

(B.1). The regressions on both stages include twelve lags of the dependent variables.

Data. We use data from October 1982 to December 2006. The choice of the starting date is dictated by the availability of the forecast data. The choice of the end date avoids using data from the Great Recession when the GSEs faced a particularly turbulent experience, which culminated in their conservatorship by the government in September 2008. All data sources, except for data on forecasts, are identical to those used in FMR. We list them in Section B.1 of this Appendix.

To measure mortgage rate forecasts, we use a survey of expectations by major financial institutions collected in the Blue Chip Financial Forecast (BCFF) database. The Blue Chip Financial Forecasts dataset is proprietary; it can either be purchased directly from the official website or obtained through the institutions subscribed to this dataset. The BCFF contains monthly surveys of around forty financial institutions that forecast major financial indicators, including mortgage rates at horizons up to six quarters. The surveys are usually conducted in the last few days of a month and released on the first date of the following month. We focus on the median forecast across forecasters at any point in time.

Importantly, the Blue Chip survey asks participants to forecast the average value of a variable over the current and future calendar quarters. As a result, there is no fixed forecast horizon at a monthly frequency. For example, a January forecast of the mortgage rate r_t over the second quarter of a particular year is a three-/five-month-ahead forecast, while a February forecast of the same variable is a two-/four-month-ahead forecast. This property of the Blue Chip forecasts is illustrated in Figure A1.

We thus employ the following definition of forecast errors about next-quarter average mortgage rate:

$$\widetilde{u}_{t,t+\text{``1:3''}} \equiv \frac{r_{t+3-mod(t+2,3)} + r_{t+4-mod(t+2,3)} + r_{t+5-mod(t+2,3)}}{3} - f_t^{\text{``1:3''}},$$

where $f_t^{\text{``1:3''}}$ is the median forecast of next-quarter mortgage rate at time t in the BCFF database. Note that $t=1$ corresponds to January 1982, $t=2$ to February 1982, and so on. The notation "1:3" emphasizes the fact that the horizon of this forecast varies from one to three months and $mod(t+2,3)$ is the remainder of the division of $t+2$ by 3. Similarly, we define forecast errors of mortgage rates in the subsequent quarters as

$$\widetilde{u}_{t,t+\text{``}(3n-2):3n\text{''}} \equiv \frac{\sum_{i=0}^{2} r_{t+3n+i-mod(t+2,3)}}{3} - f_t^{\text{``}(3n-2):3n\text{''}}, \tag{B.3}$$

where $n = 1,2,3,4$.

Null hypothesis. As long as forecasters working for financial institutions are aware of significant purchases by the GSEs and hold rational expectations, the forecast errors $\widetilde{u}_{t,t+\text{``}(3n-2):3n\text{''}}$ should not be predictable by such purchases. To test this, we can simply use $\widetilde{u}_{t+1,t+1+``(3n-2):3n''}$ in place of y_{t+h} in equation (B.2) and verify whether the coefficient $\gamma_1^{(2)}$ is statistically different from zero. Note that, by regressing the forecast errors based on the information available to forecasters at the beginning of month $t+1$ on the GSEs' purchases in month t, we avoid the possibility that these interventions were not implemented before forecasters were asked to predict future prices.

Results. We begin by estimating the effects of the GSEs' exogenous mortgage purchases on mortgage yields in our sample from October 1982 to December 2006. In doing this, we confirm that the main conclusion in FMR does not change much when we use our restricted data sample. The left panel of Figure 2 shows the impulse response function of the conventional mortgage rate—i.e., the coefficients $\gamma_h^{(2)}$ in the second stage equation (B.2) when the dependent variable is r_{t+h}—following an exogenous increase in the GSEs' purchases by 1 percent of trend originations. Our results are only slightly different from those in FMR. One notable difference between our results and those reported in FMR is the value of the first-stage F-statistics. While the authors estimate the F-statistics to be higher than ten in their longer sample, the value of F-statistics is just slightly above five in our smaller sample. However, quantitatively, the results reported in Figure 2 are close to those presented in Figure VII of FMR, suggesting that the weak instrument bias is small.

Next, we turn to the estimation of the response of mortgage-rate forecast errors to purchases by GSEs. The right panel of Figure 2 presents the estimates of coefficients $\gamma_1^{(2)}$ in equation (B.2) when the dependent variable is $\widetilde{u}_{t+1,t+1+``(3n-2):3n''}$, $n = 1,2,3,4$, along with one- and two-standard-error confidence intervals. Consistently with the predictions of our model, forecast errors react negatively and significantly to the GSEs' mortgage purchases, which suggests that forecasters tend to under-react to news about such interventions. Moreover, under the additional assumption that forecasters working for financial institutions are aware of significant purchases by the GSEs, imperfect information models would fail to predict the under-reaction in the forecast errors.

We repeat our analysis for the "nowcast" error. We define the "nowcast" error using equation (B.3) where n is set to zero and $f_t^{''-2:0''}$ denotes the "nowcast"—the current-calendar-quarter average forecast of mortgage rates. It is clear that, when the nowcast is released in the beginning of the first month of a quarter, it is effectively a forecast of the mortgage rate during the whole quarter ahead. On the other hand, the nowcast released in the last month of the quarter is likely to depend on the data that has become available during the first part of the quarter that is being nowcasted. As a result, our measure of the nowcast error $\widetilde{u}_{t+1,t+1+``-2:0''}$ is an average between a true nowcast and a forecast at a short horizon. Hence, we expect our nowcast error to still be predictable, but perhaps to a smaller degree than the forecast errors at more distant horizons. Consistent with this logic, we find that the point estimate of $\gamma_1^{(2)}$ is -0.8 basis points with the standard deviation of 1.3 basis points.

Finally, to deduce the average level of sophistication of forecasters in the data, we first estimate the responses of the moving average of the realized mortgage rates, defined as $\sum_{i=0}^{2} r_{t+3n+i-mod(t+2,3)}/3$, which corresponds to the first term on the right-hand side of equation (B.3), at horizons $n = 1,2,3,4$ (the results are not explicitly shown here). Then, in accordance with the formula for the average level of sophistication in equation (27) of the main text, we divide the responses of forecast errors, depicted in the right panel of Figure 2, by the responses of the moving average of the realized mortgage rates. Note that equation (27) also holds for the variables that are defined similarly to the monthly forecasts in the BCFF data. Table A1 presents the results. The average sophistication different over horizons, but in all of the cases it is between 1 and 1.4. The average value over four horizons is 1.17.

	Forecast horizon in months				
	1-3	4-6	7-9	10-12	mean
$\widehat{\bar{k}}$	1.23	1.03	1.35	1.03	1.17

Table A1: The average level of sophistication of agents in the economy obtained from equation (27), for different horizons, and the mean value across all horizons. Because we use the BCFF dataset where the participants are asked to forecast variables for future *calendar* quarters, we do not have fixed forecasting horizons when we use monthly data. As a result, we introduce the notation where "1-3" denotes the forecast for the next calendar quarter, "4-6" denotes the forecast for the quarter after the next calendar quarter, and so on.

B.1 Data Sources

All variables used in the empirical part of the paper are monthly and identical to those in Fieldhouse et al. (2018) (FMR) except for forecasts of mortgage returns. For convenience we list all of the data sources here.

- **Agency purchase commitments** are computed by FMR in summing purchases by Fannie Mae, Freddie Mac, and the Federal Reserve.
- **The noncyclical narrative policy indicator** m_t is computed in FMR.
- **Personal income** is from NIPA (series PI in the FRED database).
- **The core PCE price index** is from NIPA (series PCEPILFE in the FRED database).
- **Nominal house price index** is the Freddie Mac house price index.
- **Total mortgage debt** are from the Financial Accounts of the United States and additional computations in FMR.
- **Residential mortgage originations** are computed by FMR from various sources and available from the authors.
- **Housing starts** are from the Census Bureau (series HOUST in the FRED database).
- **The 3-month T-bill rate** is from the Federal Reserve Release (FRSR) H.15 (series TB3MS in the FRED database).
- **The 10-year Treasury rate** is from the Federal Reserve Release (FRSR) H.15 (series GS10 in the FRED database).
- **The BAA-AAA corporate bond spread** is obtained by taking the difference in the Moody's seasoned BAA and AAA yields (series BAA and AAA in the FRED database).
- **The conventional mortgage rate** is the 30-year fixed-rate conventional conforming mortgage rate. It is measured as monthly average commitment rate from the Freddie Mac primary mortgage market survey.
- **Mortgage rate forecast** is the Blue Chip Forecasts of home mortgage rate which is defined as the 30-year fixed-rate conventional conforming mortgage rate. The Blue Chip reports note that "Interest rate definitions are the same as those in FRSR H.15."

Online Appendix

C Extensions of the Simple Model

In this online appendix, we extend the simple model in four ways. First, we explicitly introduce public long-term nominal bonds and show that balance sheet interventions, such as the so-called "Operation Twist", affect bond prices in the presence of inflation risk. Second, we extend the simple model to a two-country setting and discuss sterilized FX interventions. Third, we derive the consequences of the presence of the fraction of agents who form their expectations rationally. Finally, we study a simple learning mechanism.

C.1 A Model with Public Long-term Nominal Bonds

To study the effects of long-term public bonds purchases, we extend the simple model of Section 2 in two ways. First, we add a nominal friction in the form of a utility service from money balances, which is necessary to generate a demand for money. Second, we introduce nominal long-term bonds. There are thus four assets in the economy: *(i)* a riskless real asset, which pays a net return $r > 0$ and is available in perfectly elastic supply; *(ii)* money, which is issued by the central bank; *(iii)* a one-period nominal bond, which pays a continuously compounded nominal interest rate i_t and is issued by the treasury; and *(iv)* a nominal long-term bond, which pays one unit of currency every period, trades at price q_t in real terms, and is issued by the treasury. We assume that each long-term bond is a perpetuity that matures with probability $\delta \in [0, 1]$ in every period, independently of the other bonds. The expected time to maturity of a long-term bond is thus equal to $1/\delta$ in every period. Finally, for the sake of simplicity, we do not consider private risky assets in this extension (none of the results are affected by this simplification).

The only source of aggregate risk in the economy is given by shocks to the money supply. We abstract from default in this extension. In particular, we assume that money supply follows the stochastic process $\log M_{t+1} = \log \overline{M} + \epsilon_t^m - \epsilon_{t-1}^m (1 + v) / v$, where v and $\overline{M}$ are positive parameters. The disturbances $\{\epsilon_t^m\}$ are assumed to be i.i.d. and normally distributed with zero mean and standard deviation σ_m. The specific form of the money supply—i.e., the presence of the lagged shock ϵ_{t-1}^m and the parameter v that also appears in the household preferences—allows us to streamline the analysis. Under these assumptions, in fact, there is no inflation risk between two consecutive periods, making one-period nominal bonds riskless and long-term bonds risky in real terms. It is straightforward to solve the model under alternative processes for money supply that lead to a one-period-ahead inflation risk. However, this clutters the exposition without adding any important economic insights.

Households. Household preferences are

$$-\frac{1}{\gamma} \widetilde{\mathbb{E}}_0 \sum_{t=0}^{\infty} \exp \left[-\rho t - \gamma \left(c_t - \frac{m_t \left[\log \left(m_t / \overline{m} \right) - 1 \right]}{v} \right) \right], \tag{C.1}$$

where $\overline{m}$ is a positive constant. Preferences are assumed to depend on real money balances m_t, which is a standard way of introducing the demand for money in macroeconomic models. The particular functional form assumed here simplifies the analysis by making money demand independent of the consumption choice. Note that utility is increasing in m_t, for $m_t \leq \overline{m}$, and decreasing in m_t, for $m_t > \overline{m}$. We thus restrict our analysis to the case with $m_t \leq \overline{m}$.

Each household chooses safe real assets s_{t+1}, short-term nominal bonds b_{t+1} (expressed in units of period-t consumption), long-term nominal bonds d_{t+1} (expressed in units of period-t consumption), real money balances m_{t+1}, and consumption c_t, so as to maximize (C.1), subject to the budget constraint

$$\begin{aligned} &P_t c_t + P_t s_{t+1} + P_t b_{t+1} + P_t q_t d_{t+1} + P_t m_{t+1} \\ \leq &P_t \left(W_t - T_t\right) + P_t(1+r)s_t + e^{i_{t-1}} P_{t-1} b_t + \left[1 + (1-\delta) P_{t-1} q_t\right] d_t + P_{t-1} m_t, \end{aligned} \quad \text{(C.2)}$$

where T_t are real per capita taxes and P_t is the nominal price level. As described in Section 2, households' expectations are captured by the sequence of one-period ahead distributions $\widetilde{\Phi}_t \equiv \{\widetilde{\phi}_s\}_{s \geq t}$, where $\widetilde{\phi}_s$ is a distribution, conditional on information available at time s, of the vector of endogenous variables Z_{s+1}. Here, $Z_t \equiv (p_t, i_t, q_t, \overline{T}_t, Tr_t, R_{t+1})$, where $p_t \equiv \log P_t$, R_{t+1} and $\overline{T}_t$ are part of government policies, which we define below. As in the simple model, level-k thinkers will form expectations in a recursive way, starting from a status-quo distribution $\widetilde{\Phi}_t^{SQ}$ that corresponds to the linear REE before the intervention.

Government. The government consists again of the treasury and the central bank. The former sets real per capita taxes, the *real* amount of one-period (short-term) nominal bonds, the *real* amount of nominal long-term bonds, and the real safe short-term bonds (those that pay the interest rate r). Without any loss of generality and to simplify notation, we assume that the outstanding real amounts of short- and long-term bonds are held constant at $\overline{B}$ and $\overline{D}$, thus, the fiscal authority simply replaces maturing bonds with newly issued bonds. In addition, the fiscal authority receives transfers $\{Tr_t\}$ from the monetary authority in every period. The treasury's per-period budget constraint is thus

$$\overline{D} + e^{i_{t-1}} P_{t-1} \overline{B} + (1+r) S_t P_t = P_t T_t + P_t q_t \delta \overline{D} + P_t \overline{B} + S_{t+1} P_t + P_t Tr_t. \quad \text{(C.3)}$$

The left-hand side represents payments on long-term and two types of short-term bonds. Recall that each unit of long-term bonds $\overline{D}$ pays one unit of currency every period. The right-hand side sums up all sources of revenue: taxes, replacement of matured long-term bonds (i.e., issuance of new long-term bonds), issuance of nominal and real short-term bonds, and transfers from the monetary authority. Finally, note that we are implicitly assuming that the original quantity $\overline{D}$ of long-term bonds was issued at some date before period t.

The central bank controls the nominal money supply $\{M_{t+1}\}$, the *real* amount of one-period interest-paying reserves $\{R_{t+1}\}$, and purchases of long-term public bonds $\{D_{t+1}\}$. Since reserves and short-term public bonds will be perfect substitutes in equilibrium, they will pay the same interest rate i_t. For simplicity, we assume that only cash M_t, which we refer to as "money," provides utility benefits to households.The budget constraint of the monetary authority is

$$M_t + e^{i_{t-1}} P_{t-1} R_t + P_t q_t D_{t+1} + P_t Tr_t = M_{t+1} + P_t R_{t+1} + \left[1 + (1-\delta) P_t q_t\right] D_t.$$

The left-hand side represents outlays consisting of repayment to money holders, payments on reserves, purchases of long-term bonds, and transfers to the fiscal authority. The right-hand side represents central bank revenues consisting of issuance of money, creation of reserves, and income from coupons and sales of long-term bonds.

To save on notation and without loss of generality, we assume that central bank bond holdings before the intervention are zero. Moreover, again without loss of generality, we consider only balance sheet policies consisting of purchases of long-term bonds entirely financed by creation of reserves. We again refer to such policies as "quantitative easing." Formally, we require

$$R_{t+1} = q_t D_{t+1}, \tag{C.4}$$

which implies that the central bank's budget constraint simplifies into

$$P_t Tr_t = M_{t+1} - M_t + [1 + (1-\delta) P_t q_t - e^{i_{t-1}} P_{t-1} q_{t-1}] D_t. \tag{C.5}$$

Finally, note that, in this section, $\overline{D}$ and D_t denote total supply and purchases of long-term bonds, respectively, and not average and realized dividends, as in the baseline model of Section 2.

Household optimization. We begin by writing the Bellman equation for the household problem:

$$V_t(a_t, m_t) = \max_{c_t, b_{t+1}, d_{t+1}, m_{t+1}} \left\{ -\frac{1}{\gamma} \exp\left[-\gamma \left(c_t - \frac{m_t \left[\log\left(m_t/\overline{m}\right) - 1\right]}{v} \right) \right] + e^{-\rho} \widetilde{\mathbb{E}}_t v_{t+1}(a_{t+1}, m_{t+1}) \right\},$$

subject to

$$\begin{aligned} P_{t+1} n_{t+1} =& P_{t+1} (1+r) (W_t - T_t + n_t - c_t - b_{t+1} - m_{t+1} - q_t d_{t+1}) + e^{i_t} P_t b_{t+1} + P_t m_{t+1} \\ &+ [1 + (1-\delta) P_{t+1} q_{t+1}] d_{t+1}, \end{aligned} \tag{C.6}$$

where n_t is financial wealth, $h_t \equiv \widetilde{\mathbb{E}}_t \sum_{j=0}^{\infty} \left(W_{t+j} - T_{t+j}\right) / (1+r)^j$ is the expected net present discounted value of non-financial wealth, which we call *human capital*, and $a_t \equiv n_t + h_t$ is the total wealth of the households. Let also $\overline{T}_t \equiv \widetilde{\mathbb{E}}_t \sum_{s=0}^{\infty} T_{t+s}/(1+r)^s$ denote the present discounted value, at rate r, of taxes. Also note that $a_t = n_t + W_t - T_t + \widetilde{\mathbb{E}}_t h_{t+1}/(1+r)$.

We first rewrite (C.6) in real units by dividing both sides by P_{t+1}:

$$\begin{aligned} n_{t+1} =& (1+r)(W_t - T_t + n_t - c_t) + \left[e^{i_t} \frac{P_t}{P_{t+1}} - (1+r) \right] b_{t+1} \\ &+ \left[\frac{1}{P_{t+1}} + (1-\delta) q_{t+1} - (1+r) q_t \right] d_{t+1} + \left[\frac{P_t}{P_{t+1}} - (1+r) \right] m_{t+1}. \end{aligned}$$

We then take a first-order Taylor expansion around $(i_t, \pi_{t+1}, p_{t+1}) = (r, 0, 0)$:

$$n_{t+1} = (1+r)(W_t - T_t + n_t - c_t) + \begin{pmatrix} b_{t+1}, & d_{t+1}, & m_{t+1} \end{pmatrix} \mathcal{R}_{t+1}.$$

where $\mathcal{R}_{t+1} \equiv (i_t - \pi_{t+1} - r, 1 - p_{t+1} + (1-\delta) q_{t+1} - (1+r) q_t, -\pi_{t+1} - r)'$ is the vector of excess returns on $\begin{pmatrix} b_{t+1}, & d_{t+1}, & m_{t+1} \end{pmatrix}$. Our strategy of log-linearizing the budget constraint and treating it as exact follows Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006). This approach yields an approximation to the true solution that allows for an analytic characterization.

We conjecture that the distributions $\{\widetilde{\phi}_t\}$ are such that, conditional on information at time t, the vector of endogenous variables Z_{t+1} is linear in the underlying shocks of the economy. This conjecture will be verified in all the equilibria we consider below. Specifically,

$$x_{t+1} = \alpha_{x,t} + \beta_{x,t} \epsilon_{t+1}^m + \xi_{x,t} \epsilon_t^m, \tag{C.7}$$

for some, possibly time-varying, coefficients $\alpha_{x,t}$, $\beta_{x,t}$, and $\xi_{x,t}$, for all $x \in \{p, q, i, Tr, \overline{T}, \vartheta\}$.

We guess and verify that

$$V_t(a,m) = -\frac{1}{\gamma} e^{-\gamma\left[A(a+\vartheta_t) - A_m \frac{m[\log(m/\overline{m})-1]}{v}\right]},$$

where ϑ_t is a deterministic function of time, which summarizes a number of endogenous variables taken as given by the households.

Standard properties of Normal distributions imply

$$\begin{aligned} V_t(a_t, m_t) = \max_{\substack{c_t, m_{t+1}, \\ b_{t+1}, d_{t+1}}} & -\frac{1}{\gamma} \exp\left(-\gamma\left\{c_t - \frac{m_t[\log(m_t/\overline{m})-1]}{v}\right\}\right) \\ & -\frac{1}{\gamma} \exp\left[-\rho - \gamma A \widetilde{\mathbb{E}}_t(a_{t+1} + \vartheta_{t+1}) + \frac{1}{2}\gamma^2 A^2 \widetilde{\mathbb{V}}_t(a_{t+1}) + A_m \gamma \frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}\right], \end{aligned}$$

where the conditional moments of a_{t+1} are

$$\begin{aligned} \widetilde{\mathbb{E}}_t(a_{t+1}) =& (1+r)(W_t - T_t + n_t - c_t) + \left(i_t - \alpha_{p,t} - \xi_{p,t}\epsilon_t^m + p_t - r\right) b_{t+1} \\ & + [1 - \alpha_{p,t} - \xi_{p,t}\epsilon_t^m + (1-\delta)\left(\alpha_{q,t} + \xi_{q,t}\epsilon_t^m\right) - (1+r)q_t] d_{t+1} \\ & + \left(-\alpha_{p,t} - \xi_{p,t}\epsilon_t^m + p_t - r\right) m_{t+1} + \widetilde{\mathbb{E}}_t h_{t+1}, \\ \widetilde{\mathbb{V}}_t(a_{t+1}) =& \left(-\beta_{p,t} b_{t+1} - [\beta_{p,t} - (1-\delta)\beta_{q,t}] d_{t+1} - \beta_{p,t} m_{t+1} - \beta_{\overline{T},t} + \beta_{\vartheta,t}\right)^2 \sigma_m^2. \end{aligned}$$

The first-order condition with respect to c_t is

$$e^{-\gamma\left\{c_t - \frac{m_t[\log(m_t/\overline{m})-1]}{v}\right\}} = A(1+r) e^{-\rho - \gamma A \widetilde{\mathbb{E}}_t(a_{t+1}) + \vartheta_{t+1} + \frac{1}{2}\gamma^2 A^2 \widetilde{\mathbb{V}}_t(a_{t+1}) + A_m \gamma \frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}}, \tag{C.8}$$

and the first-order conditions with respect to m_{t+1}, b_{t+1}, and d_{t+1} are

$$\widetilde{\Sigma}_t \cdot \begin{pmatrix} b_{t+1} \\ d_{t+1} \\ m_{t+1} \end{pmatrix} = \frac{1}{A\gamma} \widetilde{\mathbb{E}}_t \mathcal{R}_{t+1} + \frac{1}{A\gamma} \begin{pmatrix} 0 \\ 0 \\ -\frac{A_m}{Av} \log\left(\frac{m_{t+1}}{\overline{m}}\right) \end{pmatrix} + \widetilde{cov}_t\left(\overline{T}_{t+1}, \mathcal{R}_{t+1}\right). \tag{C.9}$$

The variance-covariance matrix $\widetilde{\Sigma}_t \equiv \widetilde{\mathbb{V}}_t(\mathcal{R}_{t+1})$ is such that $(\widetilde{\Sigma}_t)_{1,1} = (\widetilde{\Sigma}_t)_{3,3} = (\widetilde{\Sigma}_t)_{1,3} = (\widetilde{\Sigma}_t)_{3,1} = (\beta_{p,t})^2\sigma_m^2$, $(\widetilde{\Sigma}_t)_{2,2} = [(1-\delta)\beta_{q,t} - \beta_{p,t}]^2\sigma_m^2$, $(\widetilde{\Sigma}_t)_{1,2} = (\widetilde{\Sigma}_t)_{2,1} = (\widetilde{\Sigma}_t)_{3,2} = (\widetilde{\Sigma}_t)_{2,3} = -\beta_{p,t}[(1-\delta)\beta_{q,t} - \beta_{p,t}]\sigma_m^2$. We use $(\widetilde{\Sigma}_t)_{m,n}$ to denote the (m,n)'th element of matrix $\widetilde{\Sigma}_t$. Note that $\widetilde{\Sigma}_t$ is not invertible because the return on money, the return on short-term bonds, and the one-period ahead return on long-term bonds have the same risk profile. Also, the last term in (C.9) equals

$$\widetilde{cov}_t\left(\overline{T}_{t+1}, \mathcal{R}_{t+1}\right) = \begin{pmatrix} -\beta_{p,t} \\ (1-\delta)\beta_{q,t} - \beta_{p,t} \\ -\beta_{p,t} \end{pmatrix} \left(\beta_{\overline{T},t}\right) \sigma_m^2.$$

The first and the third raw of equations (C.9) imply

$$m_{t+1} = \overline{m} e^{-\frac{A}{A_m} v i_t}.$$

We now verify our guess for the value function. To do this, we evaluate the Bellman equation at the optimum and check if it holds for every values of state variables a_t and m_t. At the optimum, the Bellman equation is

$$-\frac{1}{\gamma}e^{-\gamma\left[A(a_t+\vartheta_t)-A_m\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right]}=-\frac{1}{\gamma}e^{-\gamma\left\{c_t-\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right\}}$$
$$-\frac{1}{\gamma}e^{-\rho-\gamma A\mathbb{E}_t(a_{t+1}+\vartheta_{t+1})+\frac{1}{2}\gamma^2A^2\mathbb{V}_t(a_{t+1})+A_m\gamma\frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}}.$$

Using the first-order condition for consumption, we can rewrite the latter as

$$-\frac{1}{\gamma}e^{-\gamma\left[A(a_t+\vartheta_t)-(A_m-1)\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right]}=-\frac{1}{\gamma}e^{-\gamma c_t}\frac{1+A(1+r)}{A(1+r)}.$$

Optimal consumption is obtained from equation (C.8):

$$[1+A(1+r)]\,c_t=\frac{m_t[\log(m_t/\overline{m})-1]}{v}-\frac{1}{\gamma}\log\left[A(1+r)\right]+\frac{\rho}{\gamma}+A(1+r)a_t+A\begin{pmatrix} b_{t+1}, & d_{t+1}, & m_{t+1}\end{pmatrix}\widetilde{\mathbb{E}}_t\mathcal{R}_{t+1}$$
$$+A\vartheta_{t+1}-\frac{1}{2}\gamma A^2\mathbb{V}_t(a_{t+1})-\frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}.$$

Combining the last two equations, we get

$$[1+A(1+r)]\left\{A\left(a_t+\vartheta_t\right)-\left(A_m-1\right)\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right\}$$
$$=\frac{m_t[\log(m_t/\overline{m})-1]}{v}-\frac{1}{\gamma}\log\left[A(1+r)\right]+\frac{\rho}{\gamma}+A(1+r)a_t+A\begin{pmatrix} b_{t+1}, & d_{t+1}, & m_{t+1}\end{pmatrix}\widetilde{\mathbb{E}}_t\mathcal{R}_{t+1}$$
$$+A\vartheta_{t+1}-\frac{1}{2}\gamma A^2\mathbb{V}_t(a_{t+1})-\frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}-[1+A(1+r)]\frac{1}{\gamma}\log\frac{1+A(1+r)}{A(1+r)}.$$

For our conjecture to be true, the coefficients multiplying a_t and m_t must be identical, that is,

$$a_t:[1+A(1+r)]\,A=A(1+r),$$
$$\frac{m_t[\log(m_t/\overline{m})-1]}{v}:-[1+A(1+r)]\,(A_m-1)=1,$$

which is true if and only if

$$A=A_m=\frac{r}{1+r}.$$

Finally, we can express ϑ_t as follows

$$\vartheta_t=\frac{1}{r\gamma}\left[\rho+\log\frac{r^r}{(1+r)^{1+r}}\right]+\frac{\vartheta_{t+1}+\begin{pmatrix} b_{t+1}, & d_{t+1}, & m_{t+1}\end{pmatrix}\widetilde{\mathbb{E}}_t\mathcal{R}_{t+1}}{1+r}-\frac{\gamma r}{2\,(1+r)^2}\widetilde{\mathbb{V}}_t(a_{t+1})$$
$$-\frac{1}{r}\cdot\frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}.$$

Temporary equilibrium. First, the market-clearing conditions in the asset markets in period t are

$$\begin{aligned}\overline{B}+q_tD_{t+1} &= b_{t+1},\\ D-D_{t+1} &= d_{t+1},\\ \frac{\overline{M}}{P_t}e^{\epsilon_t^m-\epsilon_{t-1}^m\frac{1+v}{v}} &= \overline{m}e^{-vi_t}.\end{aligned}$$

The money-market equilibrium condition implies the following relationship between the price level and the short-term interest rate:

$$p_t = \log\left(\overline{M}/\overline{m}\right)+vi_t+\epsilon_t^m-\epsilon_{t-1}^m\frac{1+v}{v}. \tag{C.10}$$

To streamline the analysis, we assume that the demand and supply of money are negligibly small. Specifically, we let $\overline{m}$ and $\overline{M}$ approach zero so that the ratio $\overline{m}/\overline{M}$ approaches one. This "cashless limit" is a standard assumption employed in, for example, the New Keynesian literature to eliminate the real effects of money supply above and beyond its effects on inflation and the nominal interest rate. We can thus abstract from money holdings when computing equilibria.

In the cashless limit, the market clearing conditions together with optimal choice of bonds imply

$$\left(\widetilde{\Sigma}_t\right)_{1:2,1:2}\cdot\begin{pmatrix}\overline{B}+q_tD_{t+1}\\ D-D_{t+1}\end{pmatrix}=\frac{1}{\frac{r}{1+r}\gamma}\widetilde{\mathbb{E}}_t\left(\mathcal{R}_{t+1}\right)_{1:2}+\widetilde{cov}_t\left(\overline{T}_{t+1},\left(\mathcal{R}_{t+1}\right)_{1:2}\right), \tag{C.11}$$

where $(\widetilde{\Sigma}_t)_{1:2,1:2}$ is the upper-left sub-matrix of $\widetilde{\Sigma}_t$, and $(\mathcal{R}_{t+1})_{1:2}$ is the vector containing the first two elements of $\mathcal{R}_{t+1}$. The second line of this equation can be solved for price q_t:

$$\begin{aligned}q_t =&\frac{1-\alpha_{p,t}-\xi_{p,t}\epsilon_t^m+(1-\delta)\left(\alpha_{q,t}+\xi_{q,t}\epsilon_t^m\right)}{(1+r)\left\{1-\frac{r\gamma\sigma_m^2}{(1+r)^2}D_{t+1}\beta_{p,t}\left[(1-\delta)\beta_{q,t}-\beta_{p,t}\right]\right\}}\\ &-\frac{r\gamma\sigma_m^2}{(1+r)^2}\cdot\frac{\left[-\beta_{p,t}\overline{B}+\left[(1-\delta)\beta_{q,t}-\beta_{p,t}\right]\left(\overline{D}-D_{t+1}\right)-\beta_{\overline{T},t}\right]\left[(1-\delta)\beta_{q,t}-\beta_{p,t}\right]}{1-\frac{r\gamma\sigma_m^2}{(1+r)^2}D_{t+1}\beta_{p,t}\left[(1-\delta)\beta_{q,t}-\beta_{p,t}\right]}.\end{aligned} \tag{C.12}$$

Similarly, the nominal interest rate on short-term bonds is obtained from the first line of (C.11):

$$i_t = r+\alpha_{p,t}+\xi_{p,t}\epsilon_t^m-p_t+RP_t, \tag{C.13}$$

where, for convenience, we let

$$RP_t\equiv\frac{r}{1+r}\gamma\left[\beta_{p,t}(\overline{B}+q_tD_{t+1})-\left[(1-\delta)\beta_{q,t}-\beta_{p,t}\right](D-D_{t+1})+\beta_{\overline{T},t})\right]\beta_{p,t}\sigma_m^2.$$

The price level p_t is obtained by combining C.10, taking into account that $\overline{M}/\overline{m}=1$, and (C.13):

$$\begin{aligned}p_t =& vi_t+\epsilon_t^m-\frac{1+v}{v}\epsilon_{t-1}^m\\ =&\frac{v}{1+v}(r+\alpha_{p,t})+\frac{1+v\xi_{p,t}}{1+v}\epsilon_t^m-\frac{1}{v}\epsilon_{t-1}^m+\frac{v}{1+v}RP_t.\end{aligned} \tag{C.14}$$

Next, we consider fiscal policy. The realized transfers from the central bank to the treasury can be computed using equation (C.5) at the cashless limit. We obtain

$$
\begin{aligned}
Tr_t &= \left[e^{-p_t} + (1-\delta)\, q_t - e^{i_{t-1}-\pi_t} q_{t-1}\right] D_t \\
&= \left[1 - p_t + (1-\delta)\, q_t - (1 + i_{t-1} - \pi_t) q_{t-1}\right] D_t,
\end{aligned} \tag{C.15}
$$

where the second line log-linearized around $(i_{t-1}, \pi_t, p_t) = (r, 0, 0)$. The treasury's budget constraint (C.3) implies the following realized taxes in period t:

$$
T_t = \frac{\overline{D}}{P_t} + e^{i_{t-1}-\pi_t}\overline{B} - \overline{B} + (1+r)S_t - S_{t+1} - q_t \delta \overline{D} - Tr_t.
$$

The expected discounted sum of taxes is then

$$
\overline{T}_t = (1+r)S_t - \widetilde{\mathbb{E}}_t \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s \left[Tr_{t+s} + \left(q_{t+s}\delta - e^{-p_{t+s}}\right)\overline{D} + \left(1 - e^{i_{t-1+s}-\pi_{t+s}}\right)\overline{B}\right]. \tag{C.16}
$$

Finally,

$$
\begin{aligned}
\vartheta_t =& \frac{1}{r\gamma}\left[\rho + \log \frac{r^r}{(1+r)^{1+r}}\right] + \frac{\vartheta_{t+1} + \left(\overline{B} + q_t D_{t+1}, \overline{D} - D_{t+1}\right) \widetilde{\mathbb{E}}_t \left(\mathcal{R}_{t+1}\right)_{1:2}}{1+r} \\
& - \frac{\gamma r}{2\,(1+r)^2} \widetilde{\mathbb{V}}_t(a_{t+1}).
\end{aligned} \tag{C.17}
$$

Rational expectations equilibrium. In the REE, the equilibrium distribution of endogenous variables must be equal to the agents' beliefs about these variables.

Specifically, for the price level, we need to make sure that the coefficients $\alpha^{REE}_{p,t}$, $\beta^{REE}_{p,t}$, and $\zeta^{REE}_{p,t}$ satisfy

$$
\begin{aligned}
p_{t+1} =& \frac{v}{1+v}(r + \alpha^{REE}_{p,t+1}) + \frac{1 + v\zeta^{REE}_{p,t+1}}{1+v}\epsilon^m_{t+1} - \frac{1}{v}\epsilon^m_t + \frac{v}{1+v} RP_t \\
\overset{REE}{=}& \alpha^{REE}_{p,t} + \beta^{REE}_{p,t}\epsilon^m_{t+1} + \zeta^{REE}_{p,t}\epsilon^m_t,
\end{aligned}
$$

for all realizations of the shocks. The latter is satisfied if and only if

$$
\begin{aligned}
\zeta^{REE}_{p,t} &= -\frac{1}{v}, \\
\beta^{REE}_{p,t} &= \frac{1 + v\zeta^{REE}_{p,t+1}}{1+v} = 0.
\end{aligned}
$$

Similarly, for the price of the perpetuity,

$$
\begin{aligned}
\zeta^{REE}_{q,t} &= 0, \\
\beta^{REE}_{q,t} &= \frac{-\zeta^{REE}_{p,t+1} + (1-\delta)\,\zeta^{REE}_{q,t+1}}{1+r} = \frac{1}{(1+r)v}.
\end{aligned}
$$

The short-term nominal interest rate must satisfy

$$\begin{aligned} i_{t+1} &= r + \alpha_{p,t+1}^{REE} + \xi_{p,t+1}^{REE}\epsilon_{t+1}^{m} - (\alpha_{p,t}^{REE} + \xi_{p,t}^{REE}\epsilon_t^m) + RP_{t+1} \\ &= \alpha_{i,t}^{REE} + \beta_{i,t}^{REE}\epsilon_{t+1}^{m} + \xi_{i,t}^{REE}\epsilon_t^m, \end{aligned}$$

implying

$$\begin{aligned} \xi_{i,t}^{REE} &= -\xi_{p,t}^{REE} = \frac{1}{v}, \\ \beta_{i,t}^{REE} &= \xi_{p,t+1}^{REE} - \beta_{p,t}^{REE} = -\frac{1}{v}. \end{aligned}$$

Note also that $RP_{t+1} = 0$ because $\beta_{p,t}^{REE} = 0$. This automatically implies that $\alpha_{i,t}^{REE} = r$ and $\alpha_{p,t}^{REE} = vr$.

In addition, the sensitivities of the central bank transfers are

$$\begin{aligned} \beta_{Tr,t}^{REE} &= \frac{1-\delta}{(1+r)v} D_{t+1} \\ \xi_{Tr,t}^{REE} &= 0. \end{aligned}$$

while the sensitivity of the expected sum of taxes are

$$\begin{aligned} \beta_{\overline{T},t}^{REE} &= \frac{1-\delta}{(1+r)v}(\overline{D} - D_{t+1}), \\ \xi_{\overline{T},t}^{REE} &= \frac{1}{v}(\overline{D} + \overline{B}). \end{aligned}$$

To simplify notation, we omit time subscripts from those coefficients that are constant over time. We thus have

$$\begin{aligned} \left(\xi_p^{REE}, \xi_i^{REE}, \xi_q^{REE}, \xi_{Tr}^{REE}, \xi_{\overline{T}}^{REE}\right) &= \left(-\frac{1}{v}, \frac{1}{v}, 0, 0, \frac{\overline{D}+\overline{B}}{v}\right), \\ \left(\beta_p^{REE}, \beta_i^{REE}, \beta_q^{REE}, \beta_{Tr,t}^{REE}, \beta_{\overline{T},t}^{REE}\right) &= \left(0, -\frac{1}{v}, \frac{1}{(1+r)v}, \frac{1-\delta}{(1+r)v} D_{t+1}, \frac{1-\delta}{(1+r)v}(\overline{D} - D_{t+1})\right). \end{aligned}$$

In the REE, the top left entry of the variance-covariance matrix of $\mathcal{R}_{t+1}$, which we denote with Σ^{REE}, is given by

$$(\Sigma^{REE})_{1,1} = \frac{(1-\delta)^2}{(1+r)^2 v^2}\sigma_m^2.$$

Finally, for the non-random part of the endogenous variables, we have

$$\begin{aligned} \alpha_{p,t}^{REE} =& \frac{v}{1+v}\left(\alpha_{p,t+1}^{REE} + r\right), \\ \alpha_{i,t}^{REE} =& r + \alpha_{p,t}^{REE} - \frac{v}{1+v}(r + \alpha_{p,t+1}^{REE}), \\ \alpha_{q,t} =& \frac{1 - vr + (1-\delta)\,\alpha_{q,t+1}}{1+r}, \\ \alpha_{Tr,t}^{REE} =& \left[1 - vr + (1-\delta)\,\alpha_{q,t}^{REE} - \alpha_{q,t-1}^{REE}(1+r)\right] D_{t+1}, \\ \alpha_{\overline{T},t}^{REE} =& (1+r)S_{t+1} - \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s \left[\alpha_{Tr,t+s}^{REE} - \overline{D} + \left(\alpha_{q,t+s}^{REE}\delta + \alpha_{p,t+s}^{REE}\right)\overline{D} - r\overline{B}\right]. \end{aligned}$$

We can then solve the system of equations above as follows:

$$\begin{aligned}
\alpha_p^{REE} &= vr, \\
\alpha_i^{REE} &= r, \\
\alpha_q^{REE} &= \frac{1-vr}{r+\delta}. \\
\alpha_{Tr,t}^{REE} &= 0, \\
\alpha_{T,t}^{REE} &= (1+r)S_{t+1} - \left[\frac{vr-1}{r+\delta} r\overline{D} - r\overline{B}\right]\frac{1+r}{r}.
\end{aligned}$$

Finally, using market-clearing conditons and taking the cashless limit, we can rewrite the recursion for ϑ_t as

$$\vartheta_t = \frac{1}{r\gamma}\left[\rho + \log\frac{r^r}{(1+r)^{1+r}}\right] + \frac{\vartheta_{t+1}}{1+r} + \frac{(\overline{D} - D_{t+2})\left(\frac{\delta vr - 1 + vr + vr^2}{r+\delta}\right)}{1+r}.$$

This expression shows that ϑ_t is a deterministic function time, thus, our initial conjecture is verified.

We summarize these results with the following proposition.

Proposition C.1. *In the cashless limit of the REE, the price level p_t^*,* the nominal interest rate i_t^**, and the price of the long-term bond q_t^* are given by*

$$p_t^* = rv - \frac{1}{v}\epsilon_{t-1}^m,$$
$$i_t^* = r - \frac{1}{v}\left(\epsilon_t^m - \epsilon_{t-1}^m\right),$$

and

$$q_t^* = \frac{1-vr}{r+\delta} + \frac{1}{v(1+r)}\epsilon_t^m.$$

In particular, they are all independent of balance sheet policies.

Level-k beliefs. Equations C.4, (C.12), (C.13)-(C.17) define a mapping $\Psi(\cdot)$ from beliefs about policy actions and endogenous variables into distributions over the endogenous variables Z_t.

By iterating $\Psi(\cdot)$, starting from $\widetilde{\Phi}_t^{SQ}$, we can compute the beliefs of level-k agents, for any $k \geq 1$. It is easy to see that some β's and ϑ's coincide with their REE counterparts: $\beta_{p,t}^k = \beta_p^{REE}$, $\beta_{i,t}^k = \beta_i^{REE}$, $\beta_{q,t}^k = \beta_q^{REE}$, $\zeta_{p,t}^k = \zeta_p^{REE}$, $\zeta_{i,t}^k = \zeta_i^{REE}$, $\zeta_{q,t}^k = \zeta_q^{REE}$, for all $k \geq 1$. In particular, the latter imply that $(\widetilde{\Sigma}_t^k)_{1,1} = (\Sigma^{REE})_{1,1}$, for all $k \geq 1$.

We are left to derive $\beta_{Tr,t}^k$, $\beta_{T,t}^k$, and the α^k's, for all $k \geq 1$. For $k = 1$, since we assume that the expectations of level-1 households coincide with the distributions of REE variables before the intervention by the central bank, the sensitivity of transfers and taxes to the current shock coincides with its REE counterpart *in the absence* of asset purchases:

$$
\begin{aligned}
\beta^{1}_{Tr,t} =& 0, \\
\xi^{1}_{Tr,t} =& 0, \\
\beta^{1}_{\overline{T},t} =& \frac{1-\delta}{(1+r)v}\overline{D}, \\
\xi^{1}_{\overline{T},t} =& \frac{1}{v}(\overline{D}+\overline{B}).
\end{aligned}
$$

Proceeding recursively for $k > 1$, we have

$$
\begin{aligned}
\beta^{k}_{Tr,t} =& \frac{1-\delta}{(1+r)v}D_{t+1}, \\
\xi^{k}_{Tr,t} =& 0 \\
\beta^{k}_{\overline{T},t} =& \frac{1-\delta}{(1+r)v}\left(\overline{D}-D_{t+1}\right), \\
\xi^{k}_{\overline{T},t} =& \frac{1}{v}(\overline{D}+\overline{B})
\end{aligned}
$$

From equation (C.14) and $\beta^{k}_{q,t} = \beta^{REE}_{q} = 0$, the intercept of the price level p_t remains

$$
\alpha^{k}_{p,t} = vr,
$$

for any $k \geq 1$. For the intercept of i_t, we get $\alpha^{k}_{i,t} = r$. For the intercept of q_t, from equation (C.12), we have

$$
\alpha^{k}_{q,t} = \begin{cases} \frac{1-vr}{r+\delta}, & k = 1, \\ \frac{1-vr+(1-\delta)\alpha^{k-1}_{q,t+1}}{1+r} - \frac{r\gamma\sigma^2_m}{(1+r)^2}\left[\frac{1-\delta}{(1+r)v}\left(\overline{D}-D_{t+2}\right) - \beta^{k-1}_{\overline{T},t+1} + \beta^{k-1}_{\vartheta,t+1}\right]\frac{1-\delta}{(1+r)v}, & k > 1, \end{cases}
$$

where the expression for $k = 1$ follows from the fact that, following the policy announcement, level-1 thinkers do not change their expectations about the future. Thus, $\alpha^{k}_{q,t}$ can be rewritten as

$$
\alpha^{k}_{q,t} = \begin{cases} \frac{1-vr}{R-1+\delta}, & k = 1, \\ \frac{1-vr}{r+\delta} + \frac{r}{1+r}\cdot\frac{\gamma(1-\delta)^2}{(1+r)^3v^2}D_{t+2}\sigma^2_m, & k = 2, \\ \frac{1-vr+(1-\delta)\alpha^{k-1}_{q,t+1}}{R}, & k \geq 3. \end{cases}
$$

A few simple steps of algebra let us rewrite the line corresponding to $k \geq 3$ as follows:

$$
\begin{aligned}
\alpha^{k}_{q,t} - \frac{1-vr}{r+\delta} &= \frac{1}{1+r}\left((1-\delta)\,\alpha^{k-1}_{q,t+1} + 1 - vr - (1+r)\frac{1-vr}{r+\delta}\right) \\
&= \frac{1-\delta}{1+r}\left(\alpha^{k-1}_{q,t+1} - \frac{1-vr}{r+\delta}\right),
\end{aligned}
$$

which, after applying the "diagonal iteration", becomes

$$\begin{aligned}\alpha^k_{q,t} - \frac{1-vr}{r+\delta} &= \left(\frac{1-\delta}{1+r}\right)^{k-2}\left(\alpha^2_{q,t+k-2} - \frac{1-vr}{r+\delta}\right)\\ &= \left(\frac{1-\delta}{1+r}\right)^{k-2}\frac{r}{1+r}\cdot\frac{\gamma(1-\delta)^2}{(1+r)^3 v^2}D_{t+k}\sigma^2_m,\end{aligned}$$

for $k \geq 3$. As a result, $\alpha^k_{q,t}$ satisfies

$$\alpha^k_{q,t} - \frac{1-vr}{r+\delta} = \begin{cases} 0, & k=1,\\ \left(\frac{1-\delta}{1+r}\right)^{k-2}\frac{r}{1+r}\cdot\frac{\gamma(1-\delta)^2}{(1+r)^3v^2}D_{t+k}\sigma^2_m, & k\geq 2.\end{cases} \tag{C.18}$$

Reflective equilibrium. The market-clearing condition for long-term bonds is

$$\begin{aligned}\sum_{k=1}^{\infty} f(k)(\Sigma^{REE})_{1,1}(\overline{D} - D_{t+1}) =& \frac{1}{\frac{r}{1+r}\gamma}\sum_{k=1}^{\infty} f(k)(1 - \widetilde{\mathbb{E}}^k_t p_{t+1} + (1-\delta)\,\widetilde{\mathbb{E}}^k_t q_{t+1} - Rq_t)\\ &+ \sum_{k=1}^{\infty} f(k)\widetilde{cov}^k_t(\overline{T}_{t+1} - \vartheta_{t+1}, \mathcal{R}_{d,t+1}),\end{aligned}$$

which, after substituting out $\widetilde{cov}^k_t(\overline{T}_{t+1} - \vartheta_{t+1}, \mathcal{R}_{d,t+1}) = [(1-\delta)\,\beta^k_{q,t} - \beta^k_{p,t}]\beta^k_{\overline{T},t}\sigma^2_m$, $\widetilde{\mathbb{E}}^k_t p_{t+1} = \alpha^k_{p,t} + \xi^k_p\epsilon^m_t$, and $\widetilde{\mathbb{E}}^k_t q_{t+1} = \alpha^k_{q,t} + \xi^k_q\epsilon^m_t$, can be rewritten as

$$\begin{aligned}q_t &= \frac{1-vr}{r+\delta} + \frac{1}{(1+r)v}\epsilon^m_t + \frac{r\gamma}{(1+r)^2}\cdot\frac{(1-\delta)^2}{(1+r)^2v^2}\sigma^2_m\sum_{k=1}^{\infty} f(k)\left(\frac{1-\delta}{R}\right)^{k-1}D_{t+k}\\ &= q^{REE}_t + \frac{r\gamma}{(1+r)^2}\cdot\frac{(1-\delta)^2}{(1+r)^2v^2}\sigma^2_m\sum_{k=1}^{\infty} f(k)\left(\frac{1-\delta}{R}\right)^{k-1}D_{t+k}.\end{aligned}$$

We summarize the outcomes of the reflective equilibrium in the following proposition.

Proposition C.2. *Consider a sequence of quantitative easing policies $\{D_{t+1}\}$. In the cashless limit of the reflective equilibrium, the short-term nominal interest rate and the price level coincide with their REE counterparts, while the long-term bond price satisfies*

$$q_t = q^{REE}_t + \frac{r\gamma}{(1+r)^2}\cdot\frac{(1-\delta)^2}{(1+r)^2v^2}\sigma^2_m\sum_{k=1}^{\infty} f(k)\left(\frac{1-\delta}{R}\right)^{k-1}D_{t+k},$$

where q^{REE}_t denotes the *long-term bond price* in the REE defined in Proposition *C.1*.

In particular, q_t is increasing in the amount of long-term bonds purchased by the central bank.

C.2 FX Interventions in a Two-country Version of the Simple Model

We now present the second extension of the simple model that we use to investigate the effects of international balance sheet policies, such as sterilized FX interventions. Relative to the simple model in Section 2, this extension features both a nominal friction in the form of the demand for money and an international dimension, which borrows elements of the open-economy models of Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006).

There are two countries: home and foreign. Foreign-country variables will bear an asterisk. Both countries produce the same good, which is traded freely across borders. As a result, the law of one price necessarily holds in equilibrium and we have $P_t = E_t P_t^*$, where P_t and P_t^* are the nominal price levels in the home and foreign countries, respectively, and E_t is the nominal exchange rate. The fact that the law of one price holds is a necessary no-arbitrage equilibrium condition: without it, the household problem does not have a solution. The exchange rate is defined as the quantity of home currency bought by one unit of foreign currency. Consequently, an increase in E_t corresponds to a depreciation of home currency. For convenience, we let $e_t \equiv \log E_t$, $p_t \equiv \log P_t$, and $p_t^* \equiv \log P_t^*$.

There are several assets in this world. Households in the home country can hold money issued by their own country's central bank, one-period nominal bonds/reserves created by both countries—which pay, respectively, continuously compounded interest rates i_t and i_t^*—and riskless real assets, available in perfectly elastic supply, that pay off a real net return $r > 0$. Similarly, households in the foreign country can hold money issued by their own country, nominal bonds/reserves created by both countries, and the riskless real assets. For simplicity, we do not consider private risky assets and long-term public bonds. Note that we follow the international economics literature and make the simplifying assumption that households in a country can only hold the money of the country they live in. It is easy to extend the analysis to the case where households can hold money of both countries.

There are two sources of risk in the world economy. Both home- and foreign-country money supplies follow processes given by $\log M_{t+1} = \log \overline{M} + \epsilon_t^m$ and $\log M_{t+1}^* = \log \overline{M}^* + \epsilon_t^{m*}$, where the disturbances ϵ_t^m and ϵ_t^{m*} are assumed to be independent from each other, independent over time, and normally distributed with zero mean and standard deviations σ_m, and σ_{m^*}, respectively. Note that these money-supply processes differ from the one assumed in Section C.1, where, for simplicity of exposition, the money-supply process eliminated any risk in one-period nominal bonds. Since, for simplicity, we do not introduce long-term bonds in this section, it is essential to ensure that short-term bonds are risky.

We focus on home-country households, foreign-country households are symmetric. The size of the home and foreign country are, respectively, ω and $1-\omega$, $\omega \in (0,1)$. Households maximize preferences (C.1) by choosing safe assets s_{t+1}, nominal home bonds $b_{H,t+1}$ (expressed in period-t consumption goods), nominal foreign bonds $b_{F,t+1}$ (expressed in period-t consumption goods), home-country real money balances m_{t+1}, and consumption c_{t+1}, subject to the budget constraint

$$\begin{aligned} &P_t c_t + P_t s_{t+1} + P_t b_{H,t+1} + P_t b_{F,t+1} + P_t m_{t+1} \\ \leq &P_t \left(W_t - T_t\right) + P_t(1+r)s_t + e^{i_{t-1}} P_{t-1} b_{H,t} + E_t e^{i_{t-1}^*} P_{t-1}^* b_{F,t} + P_{t-1} m_t. \end{aligned} \tag{C.19}$$

We now specify the behavior of the central bank and the treasury in each country. The home-country government controls real per capita taxes $\{T_{t+1}\}$, the *real* amount of one-period nominal bonds $\{B_{H,t+1}\}$, the amount of one-period real bonds $\{S_{H,t+1}\}$. Without loss of generality we set $B_{H,t+1} = 0$. Notice, however, that the central bank in the home country will still create reserves, which are equivalent to short-term nominal bonds. The home-country central bank controls the nominal money supply $\{M_{t+1}\}$, the *real* amount of one-period interest-paying reserves $\{R_{t+1}\}$, real transfers to the treasury $\{Tr_{t+1}\}$, and the *real* amount of purchases of foreign-currency one-period bonds $\{B_{F,t+1}\}$. As before, we assume that the purchases are fully financed by creating reserves bonds. Using the law of one price, the latter implies $R_{t+1} = B_{F,t+1}$. A policy of foreign-bond purchases financed with the creation of reserves will be referred to as "(sterilized) FX intervention."

The home-country treasury's per-period budget constraint is

$$P_t(1+r)S_{H,t} = \omega P_t T_t + P_t Tr_t + P_t S_{H,t+1}. \tag{C.20}$$

Similarly, the home-country central bank's per-period budget constraint is

$$P_t Tr_t + e^{i_{t-1}} P_{t-1} R_t + M_t + E_t P_t^* B_{F,t+1} = E_t e^{i^*_{t-1}} P^*_{t-1} B_{F,t} + P_t R_{t+1} + M_{t+1}$$

or, using the law of one price and the requirement that FX interventions are financed entirely with reserves,

$$P_t Tr_t + e^{i_{t-1}} P_{t-1} B_{F,t} + M_t = E_t e^{i^*_{t-1}} P^*_{t-1} B_{F,t} + M_{t+1}. \tag{C.21}$$

We will only consider the case in which balance sheet policies are implemented by the central bank in the home country. The analysis for the foreign country is symmetric. Without loss of generality, we assume that the government in the foreign-country sets money supply and taxes so as to keep a constant level of real bonds B^*.

As described in Section 2, households' expectations are captured by the sequence of one-period ahead distributions $\widetilde{\Phi}_t \equiv \{\widetilde{\phi}_s\}_{s\geq t}$, where $\widetilde{\phi}_s$ is a distribution, conditional on information available at time s, of the vector of endogenous variables Z_{s+1}. Here, $Z_t \equiv (p_t, i_t, \overline{T}_t, Tr_t, p^*_t, i^*_t, \overline{T}^*_t, e_t)$, where $\overline{T}_t$ is the expected present value of taxes defined as in Section C.1. Note that, since in the foreign country transfers from central bank to the treasury are zero, we do not need to consider beliefs about them. As in the simple model, level-k thinkers will form expectations in a recursive way, starting from a status-quo distribution $\widetilde{\Phi}^{SQ}_t$ that corresponds to the linear REE before the intervention.

It is straightforward to adapt all the definitions of equilibria to this environment.

Household optimization. As before, we let $a_t \equiv n_t + h_t$ represent total wealth and $h_t \equiv \widetilde{\mathbb{E}}_t \sum_{j=0}^{\infty} \left(W_{t+j} - T_{t+j}\right)/(1+r)^j$ be the expected human capital at time t. The Bellman equation for the household problem is

$$V_t(a_t, m_t) = \max_{c_t, b_{H,t+1}, b_{F,t+1}, m_{t+1}} \left\{ -\frac{1}{\gamma} \exp\left[-\gamma\left(c_t - \frac{m_t\left[\log\left(m_t/\overline{m}\right)-1\right]}{v}\right)\right] + e^{-\rho}\widetilde{\mathbb{E}}_t V_{t+1}(a_{t+1}, m_{t+1})\right\},$$

subject to

$$\begin{aligned} P_{t+1}n_{t+1} = {} & P_{t+1}(1+r)\left(W_t - T_t + n_t - c_t - b_{H,t+1} - b_{F,t+1} - m_{t+1}\right) \\ & + e^{i_t} P_t b_{H,t+1} + E_{t+1} e^{i^*_t} P^*_t b_{F,t+1} + P_t m_{t+1}. \end{aligned} \tag{C.22}$$

We first rewrite the budget constraint (C.22) in real units by dividing both sides by P_{t+1} and then take a first-order Taylor expansion around $\left(i_t, i^*_t, \pi_{t+1}, \pi^*_{t+1}\right) = (r, r, 0, 0)$:

$$n_{t+1} = (1+r)(W_t - T_t + n_t - c_t) + \left(b_{H,t+1}, b_{F,t+1}, m_{t+1}\right)\mathcal{R}_{t+1}.$$

where $\mathcal{R}_{t+1} \equiv \left(i_t - \pi_{t+1} - r, i^*_t - \pi^*_{t+1} - r, -\pi_{t+1} - r\right)'$ a vector of excess returns on $(b_{H,t+1}, b_{F,t+1}, m_{t+1})$. Our strategy of log-linearizing the budget constraint and treating it as exact follows Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006). This approach yields an approximation to the true solution that allows for an analytic characterization.

We conjecture that the distributions $\{\widetilde{\phi}_t\}$ are such that, conditional on information at time t, the vector of endogenous variables Z_{t+1} is linear in the underlying shocks of the economy. This conjecture will be verified in all the equilibria we consider below. Specifically,

$$x_{t+1} = \alpha_{x,t} + \beta_{x,t}\epsilon^{m}_{t+1} + \xi_{x,t}\epsilon^{m*}_{t+1}, \tag{C.23}$$

for some, possibly time-varying, coefficients $\alpha_{x,t}$, $\beta_{x,t}$, and $\xi_{x,t}$, $x \in \{p, i, \overline{T}, Tr, p^*, i^*, \overline{T}^*, e, \vartheta, \vartheta^*\}$.

We guess and verify that

$$V_t(a,m) = -\frac{1}{\gamma}e^{-\gamma\left[A(a+\vartheta_t) - A_m\frac{m[\log(m/\overline{m})-1]}{v}\right]},$$

where ϑ_{t+1} is a deterministic function of time, which summarizes a number of endogenous variables taken as given by the households. Standard properties of Normal distributions imply

$$\begin{aligned}V_t(a_t,m_t) = \max_{c_t, b_{H,t+1}, b_{F,t+1}, m_{t+1}} & -\frac{1}{\gamma}\exp\left(-\gamma\left(c_t - \frac{m_t\left(\log\left(m_t/\overline{m}\right)-1\right)}{v}\right)\right) \\ & -\frac{1}{\gamma}\exp\left(-\rho - \gamma A\widetilde{\mathbb{E}}_t\left(a_{t+1}+\vartheta_{t+1}\right) + \frac{1}{2}\gamma^2A^2\widetilde{\mathbb{V}}_t\left(a_{t+1}\right) + A_m\gamma\frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}\right),\end{aligned}$$

where

$$\begin{aligned}\widetilde{\mathbb{E}}_t\left(a_{t+1}\right) = & (1+r)\left(W_t - T_t + n_t - c_t\right) + \left(i_t - \alpha_{p,t} + p_t - r\right)b_{H,t+1} \\ & + \left(i^*_t - \alpha_{p^*,t} + p^*_t - r\right)b_{F,t+1} + \left(-\alpha_{p,t} + p_t - r\right)m_{t+1} \\ & + \widetilde{\mathbb{E}}_t\left[h_{t+1}\right]\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathbb{V}}_t\left(a_{t+1}\right) = & \sigma^2_m\left(-\beta_{p,t}b_{H,t+1} - \beta_{p^*,t}b_{F,t+1} - \beta_{p,t}m_{t+1} - \beta_{\overline{T},t}\right)^2 \\ & + \left(\sigma^*_m\right)^2\left(-\xi_{p,t}b_{H,t+1} - \xi_{p^*,t}b_{F,t+1} - \xi_{p,t}m_{t+1} - \xi_{\overline{T},t}\right)^2.\end{aligned}$$

The first-order condition with respect to c_t is

$$e^{-\gamma\left(c_t - \frac{m_t(\log(m_t/\overline{m})-1)}{v}\right)} = A(1+r)e^{-\rho - \gamma A\widetilde{\mathbb{E}}_t[a_{t+1}+\vartheta_{t+1}] + \frac{1}{2}\gamma^2A^2\widetilde{\mathbb{V}}_t[a_{t+1}] + A_m\gamma\frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}}. \tag{C.24}$$

Similarly, the first-order conditions with respect to $b_{H,t+1}$, $b_{F,t+1}$, and m_{t+1} are

$$\widetilde{\Sigma}_t \cdot \begin{pmatrix} b_{H,t+1} \\ b_{F,t+1} \\ m_{t+1}\end{pmatrix} = \frac{1}{A\gamma}\widetilde{\mathbb{E}}_t\mathcal{R}_{t+1} + \frac{1}{A\gamma}\begin{pmatrix} 0 \\ 0 \\ -\frac{A_m}{Av}\log\left(\frac{m_{t+1}}{\overline{m}}\right)\end{pmatrix} + \widetilde{cov}_t\left(\overline{T}_{t+1}, \mathcal{R}_{t+1}\right), \tag{C.25}$$

where the variance-covariance matrix $\widetilde{\Sigma}_t \equiv \widetilde{\mathbb{V}}_t(\mathcal{R}_{t+1})$ is such that $(\widetilde{\Sigma}_t)_{1,1} = (\widetilde{\Sigma}_t)_{3,3} = (\widetilde{\Sigma}_t)_{1,3} = (\widetilde{\Sigma}_t)_{3,1} = \left(\beta_{p,t}\right)^2\sigma^2_m + \left(\xi_{p,t}\right)^2\sigma^2_{m^*}$, $(\widetilde{\Sigma}_t)_{2,2} = \left(\beta_{p^*,t}\right)^2\sigma^2_m + \left(\xi_{p^*,t}\right)^2\sigma^2_{m^*}$, $(\widetilde{\Sigma}_t)_{1,2} = (\widetilde{\Sigma}_t)_{2,1} = (\widetilde{\Sigma}_t)_{3,2} = (\widetilde{\Sigma}_t)_{2,3} = \beta_{p,t}\beta_{p^*,t}\sigma^2_m + \xi_{p,t}\xi_{p^*,t}\sigma^2_{m^*}$. We use $(\widetilde{\Sigma}_t)_{m,n}$ to denote the (m,n)'th element of matrix $\widetilde{\Sigma}_t$. Note that the matrix $\widetilde{\Sigma}_t$ is not invertible because the return on money, the return on short-term bonds, and the one-period ahead return on long-term bonds have the same risk profile. Also, the last term in (C.25) equals

$$\widetilde{cov}_t\left(\overline{T}_{t+1}, \mathcal{R}_{t+1}\right) = -\begin{pmatrix} \beta_{p,t} \\ \beta_{p^*,t} \\ \beta_{p,t} \end{pmatrix} \beta_{\overline{T},t}\sigma_m^2 - \begin{pmatrix} \xi_{p,t} \\ \xi_{p^*,t} \\ \xi_{p,t} \end{pmatrix} \xi_{\overline{T},t}\sigma_{m^*}^2.$$

Combining the first and the third raw of equations (C.25) yields

$$m_{t+1} = \overline{m}e^{-\frac{A}{A_m}vi_t}.$$

An analogous equation holds for the foreign country.

We now verify our guess for the value function. To do this, we evaluate the Bellman equation at the optimum and check if it holds for every values of state variables a_t and m_t. At the optimum, the Bellman equation is

$$\begin{aligned}-\frac{1}{\gamma}e^{-\gamma\left[A(a_t+\vartheta_t)-A_m\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right]} =& -\frac{1}{\gamma}e^{-\gamma\left\{c_t-\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right\}} \\ & -\frac{1}{\gamma}e^{-\rho-\gamma A\mathbb{E}_t(a_{t+1}+\vartheta_{t+1})+\frac{1}{2}\gamma^2A^2\mathbb{V}_t(a_{t+1})+A_m\gamma\frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}},\end{aligned}$$

Using the first order condition for consumption, we write

$$-\frac{1}{\gamma}e^{-\gamma\left[A(a_t+\vartheta_t)-(A_m-1)\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right]} = -\frac{1}{\gamma}e^{-\gamma c_t}\frac{1+A(1+r)}{A(1+r)}.$$

Optimal consumption is obtained from equation (C.24):

$$\begin{aligned}[1+A(1+r)]\,c_t =& \frac{m_t[\log(m_t/\overline{m})-1]}{v} - \frac{1}{\gamma}\log\left[A(1+r)\right] + \frac{\rho}{\gamma} + A(1+r)a_t + A\left(b_{H,t+1}, b_{F,t+1}, m_{t+1}\right)\widetilde{\mathbb{E}}_t\mathcal{R}_{t+1} \\ & + A\vartheta_{t+1} - \frac{1}{2}\gamma A^2\mathbb{V}_t(a_{t+1}) - \frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}.\end{aligned}$$

Combining the last two equations, we get

$$\begin{aligned}&[1+A(1+r)]\left\{A\left(a_t+\vartheta_t\right)-\left(A_m-1\right)\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right\} \\ =& \frac{m_t[\log(m_t/\overline{m})-1]}{v} - \frac{1}{\gamma}\log\left[A(1+r)\right] + \frac{\rho}{\gamma} + A(1+r)a_t + A\left(b_{H,t+1}, b_{F,t+1}, m_{t+1}\right)\widetilde{\mathbb{E}}_t\mathcal{R}_{t+1} \\ & + A\vartheta_{t+1} - \frac{1}{2}\gamma A^2\mathbb{V}_t(a_{t+1}) - \frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v} - [1+A(1+r)]\frac{1}{\gamma}\log\frac{1+A(1+r)}{A(1+r)}.\end{aligned}$$

For our conjecture to be true, the coefficients multiplying a_t and m_t should be identical, that is,

$$\begin{aligned} a_t &: [1+A(1+r)]\,A = A(1+r), \\ \frac{m_t[\log(m_t/\overline{m})-1]}{v} &: -[1+A(1+r)]\left(A_m-1\right) = 1.\end{aligned}$$

As a result,

$$A = A_m = \frac{r}{1+r}.$$

Finally, we can express ϑ_t as follows

$$\vartheta_t = \frac{1}{r\gamma}\left[\rho + \log\frac{r^r}{(1+r)^{1+r}}\right] + \frac{\alpha_{\vartheta,t} + (b_{H,t+1}, b_{F,t+1}, m_{t+1})\,\widetilde{\mathbb{E}}_t\mathcal{R}_{t+1}}{1+r} - \frac{\gamma r}{2(1+r)^2}\widetilde{\mathbb{V}}_t(a_{t+1})$$
$$- \frac{1}{r}\cdot\frac{m_{t+1}[\log(m_{t+1}/\overline{m}) - 1]}{v}.$$

Temporary equilibrium. First, the market-clearing conditions in assets markets in period t are

$$\begin{aligned}
B_{F,t+1} &= \omega b_{H,t+1} + (1-\omega)b^*_{H,t+1},\\
B^* - B_{F,t+1} &= \omega b_{F,t+1} + (1-\omega)b^*_{F,t+1},\\
\frac{\overline{M}}{P_t}e^{\epsilon^m_t} &= \overline{m}e^{-vi_t},\\
\frac{\overline{M}^*}{P^*_t}e^{\epsilon^{m*}_t} &= \overline{m}e^{-vi^*_t}.
\end{aligned}$$

The money-market equilibrium condition implies the following relationship between the price level and the short-term interest rate:

$$p_t = \log\left(\overline{M}/\overline{m}\right) + vi_t + \epsilon^m_t, \tag{C.26}$$

with an analogous equation for the foreign country.

As we did in Section C.1, to streamline the analysis, we focus on the "cashless limit" of our model in which the demand and supply of money are negligibly small. Specifically, we let $\overline{m}$, $\overline{M}$, and $\overline{M}^*$ approach zero, so that the ratios $\overline{m}/\overline{M}$ and $\overline{m}/\overline{M}^*$ approaches one.

In the cashless limit, the market-clearing conditions together with optimal choice of bonds imply

$$\left(\widetilde{\Sigma}_t\right)_{1:2,1:2}\cdot\begin{pmatrix} B_{F,t+1} \\ B^* - B_{F,t+1}\end{pmatrix} = \frac{1}{\frac{r}{1+r}\gamma}\widetilde{\mathbb{E}}_t\left(\mathcal{R}_{t+1}\right)_{1:2} + \widetilde{cov}_t\left(\omega\overline{T}_{t+1} + (1-\omega)\,\overline{T}^*_{t+1}, (\mathcal{R}_{t+1})_{1:2}\right), \tag{C.27}$$

where $(\widetilde{\Sigma}_t)_{1:2,1:2}$ is the upper-left sub-matrix of $\widetilde{\Sigma}_t$, and $(\mathcal{R}_{t+1})_{1:2}$ is the vector containing the first two elements of $\mathcal{R}_{t+1}$. The nominal interest rate on domestic bonds is obtained from the first line of (C.27):

$$i_t = r + \alpha_{p,t} - p_t + RP_t, \tag{C.28}$$

where, for convenience, we let

$$RP_t \equiv \frac{r}{1+r}\gamma\left[\left(\left(\beta_{p,t}\right)^2\sigma^2_m + \left(\xi_{p,t}\right)^2\sigma^2_{m^*}\right)B_{F,t+1} + \left(\beta_{p,t}\beta_{p^*,t}\sigma^2_m + \xi_{p,t}\xi_{p^*,t}\sigma^2_{m^*}\right)(B^* - B_{F,t+1})\right.$$
$$\left. + \beta_{p,t}\left(\omega\beta_{\overline{T},t} + (1-\omega)\,\beta_{\overline{T}^*,t}\right)\sigma^2_m + \xi_{p,t}\left(\omega\xi_{\overline{T},t} + (1-\omega)\,\xi_{\overline{T}^*,t}\right)\sigma^2_{m^*}\right].$$

The price level p_t is obtained by combining C.26 and (C.28), taking into account the cashless limit:

$$\begin{aligned}
p_t &= vi_t + \epsilon^m_t\\
&= \frac{v}{1+v}\left(r + \alpha_{p,t} + RP_t\right) + \frac{1}{1+v}\epsilon^m_t.
\end{aligned} \tag{C.29}$$

Next, we consider fiscal policy. The realized transfers from the central bank to the treasury can be computed by log-linearizing equation (C.21) around $\left(i_t, i_t^*, \pi_{t+1}, \pi_{t+1}^*\right) = (r, r, 0, 0)$ and evaluating it at the cashless limit. We obtain

$$Tr_t = \left(i_{t-1}^* - \pi_t^* - i_{t-1} + \pi_t\right) B_{F,t}. \tag{C.30}$$

Also, by iterating forward the treasury's budget constraint (C.20) and taking expectations, we have that the expected discounted sum of taxes satisfies

$$\omega \overline{T}_t = (1+r) S_{H,t} - \widetilde{\mathbb{E}}_t \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s Tr_{t+s}. \tag{C.31}$$

Since only the central bank in the home country trades assets, transfers to the treasury in the foreign country are zero. The treasury in the foreign country targets a constant supply B^* of bonds, therefore, taxes in the foreign country must satisfy

$$(1-\omega)\overline{T}_t^* = (1+r) S_{F,t} + B^* \widetilde{\mathbb{E}}_t \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s \left(i_{t+s-1}^* - \pi_{t+s}^*\right) \tag{C.32}$$

Finally, to verify our initial conjecture, we need to show that, in the cashless equilibrium, ϑ_t is a deterministic function of time. We do this in two steps. First, we compute foreign demand $b_{H,t+1}^*$ and $b_{F,t+1}^*$ using the analogue of (C.25):[27]

$$\begin{aligned}
\left(\widetilde{\Sigma}_t\right)_{1:2,1:2} \cdot \begin{pmatrix} b_{H,t+1}^* \\ b_{F,t+1}^* \end{pmatrix} &= \frac{1}{\frac{r}{1+r}\gamma} \widetilde{\mathbb{E}}_t (\mathcal{R}_{t+1})_{1:2} + \widetilde{cov}_t \left(\overline{T}_{t+1}^*, \mathcal{R}_{t+1}\right) \\
&= \frac{1}{\frac{r}{1+r}\gamma} \begin{pmatrix} RP_t \\ RP_t^* \end{pmatrix} + \widetilde{cov}_t \left(\overline{T}_{t+1}^*, \mathcal{R}_{t+1}\right),
\end{aligned}$$

where the last line uses (C.28). Since RP_t, RP_t^*, and all the conditional moments are independent of shocks, so are asset demands. Second, we use the demands just computed together with market-clearing conditions to obtain

$$\begin{aligned}
\vartheta_t =& \frac{1}{r\gamma}\left[\rho + \log \frac{r^r}{(1+r)^{1+r}}\right] + \frac{\vartheta_{t+1} + \left(\overline{B}_{H,t+1}, \overline{B}_{F,t+1}\right)\left(RP_t, RP_t^*\right)'}{1+r} \\
& - \frac{\gamma r}{2(1+r)^2} \widetilde{\mathbb{V}}_t(a_{t+1}),
\end{aligned} \tag{C.33}$$

where, $\overline{B}_{H,t+1} \equiv (B_{F,t+1} - (1-\omega) b_{H,t+1}^*)/\omega$ and $\overline{B}_{F,t+1} \equiv (B^* - B_{F,t+1} - (1-\omega) b_{F,t+1}^*)/\omega$ are, respectively, the supply of home-country and foreign-country bonds after netting out net demand from the central bank and the foreign country. Importantly, since RP_t, RP_t^*, and all the conditional moments are independent of shocks, (C.33) implies that ϑ_t is a deterministic function of time, therefore, our initial conjecture is vierified.

Rational expectations equilibrium In the REE, the equilibrium distribution of endogenous variables must be equal to the agents' beliefs about these variables. Specifically, for the price level, we need to

[27] With a slight abuse of notation, we use $b_{H,t+1}^*$ and $b_{F,t+1}^*$ to denote the foreign-country household optimal choices.

make sure that the coefficients $\alpha_{p,t}^{REE}$, $\beta_{p,t}^{REE}$, and $\zeta_{p,t}^{REE}$ satisfy

$$\begin{aligned} p_{t+1} =& \frac{v}{1+v}\left(r+\alpha_{p,t+1}^{REE}+RP_{t+1}\right)+\frac{1}{1+v}\epsilon_{t+1}^{m} \\ \overset{REE}{=}& \alpha_{p,t}^{REE}+\beta_{p,t}^{REE}\epsilon_{t+1}^{m}+\zeta_{p,t}^{REE}\epsilon_{t+1}^{m*}, \end{aligned}$$

for all realizations of the shocks. The latter is satisfied if and only if

$$\begin{aligned} \beta_{p,t}^{REE} &= \frac{1}{1+v}, \\ \zeta_{p,t}^{REE} &= 0. \end{aligned}$$

By simmetry, the price level in the foreign country satisfies $\beta_{p^*,t}^{REE}=0$ and $\zeta_{p^*,t}^{REE}=1/(1+v)$. The short-term nominal interest rate must satisfy

$$\begin{aligned} i_{t+1} &= r+\alpha_{p,t+1}^{REE}-p_{t+1}+RP_{t+1} \\ &\overset{REE}{=} \alpha_{i,t}^{REE}+\beta_{i,t}^{REE}\epsilon_{t+1}^{m}+\zeta_{i,t}^{REE}\epsilon_{t+1}^{m*}, \end{aligned}$$

implying

$$\begin{aligned} \beta_{i,t}^{REE} &= -\frac{1}{1+v}, \\ \zeta_{i,t}^{REE} &= 0, \end{aligned}$$

with symmetric expressions for the foreign country. In addition, from (C.30), central bank transfers are such that

$$\begin{aligned} \beta_{Tr,t}^{REE} &= \frac{1}{1+v}B_{F,t+1}, \\ \zeta_{Tr,t}^{REE} &= -\frac{1}{1+v}B_{F,t+1}. \end{aligned}$$

and, from (C.31), the expected sum of taxes are such that

$$\begin{aligned} \omega\beta_{\overline{T},t}^{REE} &= -\frac{1}{1+v}B_{F,t+1}, \\ \omega\zeta_{\overline{T},t}^{REE} &= \frac{1}{1+v}B_{F,t+1}. \end{aligned}$$

Similarly, from (C.32), taxes in the foreign country satisfy

$$\begin{aligned} (1-\omega)\,\beta_{\overline{T}^*,t}^{REE} &= 0, \\ (1-\omega)\,\zeta_{\overline{T}^*,t}^{REE} &= -\frac{1}{1+v}B^*. \end{aligned}$$

To simplify notation, we omit time subscripts from those coefficients that are constant over time. We thus have

$$\left(\beta_p^{REE},\beta_i^{REE},\beta_{p^*}^{REE},\beta_{i^*}^{REE},\beta_{Tr,t}^{REE},\beta_{\overline{T},t}^{REE},\beta_{\overline{T}^*,t}^{REE}\right)=\frac{1}{1+v}\left(1,-1,0,0,B_{F,t+1},-B_{F,t+1}/\omega,0,0\right),$$
$$\left(\zeta_p^{REE},\zeta_i^{REE},\zeta_{p^*}^{REE},\zeta_{i^*}^{REE},\zeta_{Tr,t}^{REE},\zeta_{\overline{T},t}^{REE},\zeta_{\overline{T}^*,t}^{REE}\right)=\frac{1}{1+v}\left(0,0,1,-1,-B_{F,t+1},B_{F,t+1}/\omega,0,-B^*\right).$$

Finally, for the non-random part of the endogenous variables, we have

$$\begin{aligned}
\alpha_{p,t}^{REE} =&\frac{v}{1+v}\left(r+\alpha_{p,t+1}^{REE}+RP_{t+1}\right),\\
\alpha_{i,t}^{REE} =&r+\alpha_{p,t+1}^{REE}-\alpha_{p,t}^{REE}+RP_{t+1},\\
\alpha_{Tr,t}^{REE} =&\left(\alpha_{i^*,t-1}^{REE}-\alpha_{p^*,t}^{REE}+\alpha_{p^*,t-1}^{REE}-\alpha_{i,t-1}^{REE}+\alpha_{p,t}^{REE}-\alpha_{p,t-1}^{REE}\right)B_{F,t+1},\\
\alpha_{\overline{T},t}^{REE} =&(1+r)S_{H,t+1}-\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s\alpha_{Tr,t+s}^{REE}.
\end{aligned}$$

Combining the expressions above, we immediately obtain that $RP_t = RP_t^* = 0$. We can then solve the system of equations above as follows:

$$\begin{aligned}
\alpha_p^{REE} =&vr,\\
\alpha_i^{REE} =&r,\\
\alpha_{Tr,t}^{REE} =&0,\\
\alpha_{\overline{T},t}^{REE} =&(1+r)S_{H,t+1}.
\end{aligned}$$

with analogous expressions for the foreign-country interest rate and price level.

We summarize these results with the following proposition.

Proposition C.3. *In the cashless limit of the rational expectations equilibrium, the price level and the short-term nominal interest rate in the home country are, respectively, given by*

$$i_t = r-\frac{1}{1+v}\epsilon_t^m,\ \ p_t = vr+\frac{1}{1+v}\epsilon_t^m.$$

For the foreign country,

$$i_t^* = r-\frac{1}{1+v}\epsilon_t^{m*},\ \ p_t^* = vr+\frac{1}{1+v}\epsilon_t^{m*}.$$

Finally, the nominal exchange rate satisfies

$$e_t=\frac{1}{1+v}\left(\epsilon_t^m-\epsilon_t^{m*}\right).$$

In particular, they are all independent of balance sheet policies.

As in all the other settings considered in the paper, in the benchmark with rational expectations, the interest rate, the price level, and the exchange rate are independent of balance sheet policies. First, the interest rate is given by the risk-free real rate minus the shock to the money supply. Intuitively, to stimulate agents to hold more money, the opportunity cost of holding money, that is, the nominal interest rate i_t, must go down. Second, in this economy, the nominal interest rate i_t equals the constant real interest rate r plus expected inflation $\mathbb{E}_t p_{t+1} - p_t$. As a result, the expected inflation must decline following a shock to the money supply. Since the economy is stationary and the future expected price level is constant, the drop in expected inflation is achieved through an increase in the current price level p_t. Third, the law of one price requires that nominal exchange rate depreciates (i.e., e_t goes up, after a positive shock to the home-country money supply). Note also that inflation risk reduces the demand for nominal government bonds, but at the same time, this risk makes future real taxes positively correlated with real bond returns. The last effect increases the demand for nominal bonds. In the REE, the two effects cancel each other out and the nominal interest rate equals the real rate plus expected inflation.

Level-k beliefs. Equations (C.29)-(C.33) define a mapping $\Psi(\cdot)$ from beliefs about policy actions and endogenous variables into distributions over the endogenous variables Z_t.

By iterating $\Psi(\cdot)$, starting from $\widetilde{\Phi}_t^{SQ}$, we can compute the beliefs of level-k agents, for any $k \geq 1$. It is easy to see that some β's and ξ's coincide with their REE counterparts: $\beta_x^k = \beta_p^{REE}$, $\xi_x^k = \xi_x^{REE}$, $x \in \{p, i, p^*, i^*, \overline{T}^*, e\}$, for all $k \geq 1$. In particular, the latter imply that the conditional moments of the excess returns coincide with their REE counterparts for all $k \geq 1$.

We are left to derive $\beta_{Tr,t}^k$, $\beta_{\overline{T},t}^k$, $\xi_{Tr,t}^k$, $\xi_{\overline{T},t}^k$, and the α^k's, for all $k \geq 1$. For $k = 1$, since we assume that the expectations of level-1 households coincide with the distributions of REE variables before the intervention by the central bank, the sensitivity of transfers and taxes to the current shock coincides with its REE counterpart *in the absence* of asset purchases:

$$\beta_{Tr,t}^1 = \xi_{Tr,t}^1 = \beta_{\overline{T},t}^1 = \xi_{\overline{T},t}^1 = 0.$$

Proceeding recursively for $k > 1$, we have

$$\begin{aligned}
\beta_{Tr,t}^k &= \frac{1}{1+v} B_{F,t+1}, \\
\xi_{Tr,t}^k &= -\frac{1}{1+v} B_{F,t+1}, \\
\beta_{\overline{T},t}^k &= -\frac{1}{1+v} B_{F,t+1}, \\
\xi_{\overline{T},t}^k &= \frac{1}{1+v} B_{F,t+1}.
\end{aligned}$$

We are left to compute the intercepts. From (C.29),

$$\alpha_{p,t}^k - vr = \begin{cases} 0, & k = 1, \\ \frac{r}{1+r} \gamma \sigma_m^2 \left(\frac{v}{1+v}\right)^k \frac{1}{v(1+v)} B_{F,t+k}, & k > 1. \end{cases} \tag{C.34}$$

Similarly, from equation (C.28),

$$\alpha_{i,t}^k - r = \begin{cases} 0, & k = 1, \\ \frac{r}{1+r} \gamma \sigma_m^2 \left(\frac{v}{1+v}\right)^k \frac{1}{v^2(1+v)} B_{F,t+k}, & k > 1. \end{cases} \tag{C.35}$$

Note that the expressions for $k = 1$ follow from the fact that, upon observing the policy announcement, level-1 thinkers do not change their expectations about the future.

Similarly, for the foreign country,

$$\alpha_{p^*,t}^k - vr = \begin{cases} 0, & k=1, \\ -\frac{r}{1+r}\gamma\left(\sigma_{m^*}\right)^2\left(\frac{v}{1+v}\right)^k \frac{1}{v(1+v)} B_{F,t+k}, & k>1, \end{cases} \tag{C.36}$$

and

$$\alpha_{i^*,t}^k - r = \begin{cases} 0, & k=1, \\ -\frac{r}{1+r}\gamma\left(\sigma_{m^*}\right)^2\left(\frac{v}{1+v}\right)^k \frac{1}{v^2(1+v)} B_{F,t+k}, & k>1. \end{cases} \tag{C.37}$$

Reflective equilibrium. The market-clearing conditions for the nominal bonds in both countries are

$$\sum_{k=1}^{\infty} f(k)(\Sigma_t^{REE})_{1:2,1:2}\begin{pmatrix} B_{F,t+1} \\ B^* - B_{F,t+1}\end{pmatrix} = \frac{1}{\frac{r}{1+r}\gamma}\sum_{k=1}^{\infty} f(k)\widetilde{\mathbb{E}}_t^k\left(\mathcal{R}_{t+1}\right)_{1:2} + \sum_{k=1}^{\infty} f(k)\widetilde{cov}_t^k\left(\omega\overline{T}_{t+1} + (1-\omega)\overline{T}_{t+1}^*, (\mathcal{R}_{t+1})_{1:2}\right).$$

After substituting out for the expressions of the conditional moments, we can rewrite the last equation as

$$\left(\frac{1}{1+v}\right)^2\begin{pmatrix} \sigma_m^2 B_{F,t+1} \\ \sigma_{m^*}^2\left(B^* - B_{F,t+1}\right)\end{pmatrix} = \frac{1}{\frac{r}{1+r}\gamma}\sum_{k=1}^{\infty} f(k)\widetilde{\mathbb{E}}_t^k\left(\mathcal{R}_{t+1}\right)_{1:2} + f(1)\left(\frac{1}{1+v}\right)^2\begin{pmatrix} 0 \\ 1\end{pmatrix} B^*\sigma_{m^*}^2 + (1-f(1))\left(\frac{1}{1+v}\right)^2\left(\begin{pmatrix} 1 \\ 0\end{pmatrix}\sigma_m^2 B_{F,t+1} + \begin{pmatrix} 0 \\ 1\end{pmatrix}\left(B^* - B_{F,t+1}\right)\sigma_{m^*}^2\right).$$

Finally, using (C.26) together with the intercepts (C.34)-(C.37), we can solve the equation above for the nominal interest rate, both in the home and in the foreign country:

$$\begin{pmatrix} i_t - i_t^{REE} \\ i_t^* - i_t^{*REE}\end{pmatrix} = \frac{r}{1+r}\gamma\begin{pmatrix} \sigma_m^2 \\ -\sigma_{m^*}^2\end{pmatrix}\frac{1}{v(1+v)^2}\sum_{k=1}^{\infty} f(k)\left(\frac{v}{1+v}\right)^k B_{F,t+k}.$$

Analogous steps lead to a solution for p_t, p_t^*, and the nominal exchange rate e_t. We summarize the reflective equilibrium outcomes in the following proposition.

Proposition C.4. *Consider a sequence of (sterilized) FX intervention* $\{B_{F,t+1}\}$. *Let* p_t^{REE}, i_t^{REE}, p_t^{*REE}, i_t^{*REE}, and e_t^{REE} denote the price levels, the nominal interest rates, and the nominal exchange rate in the rational expectations equilibrium defined in Proposition C.3. *Also, let*

$$\Gamma_t \equiv \frac{r}{1+r}\gamma\frac{1}{v(1+v)^2}\sum_{k=1}^{\infty} f(k)\left(\frac{v}{1+v}\right)^k B_{F,t+k}.$$

In the cashless limit of the reflective equilibrium, the price level and the short-term nominal interest rate in the home country are, respectively, given by

$$i_t = i_t^{REE} + \sigma_m^2\Gamma_t, \quad p_t = p_t^{REE} + v\sigma_m^2\Gamma_t.$$

For the foreign country,

$$i_t^* = i_t^{*REE} - \sigma_{m^*}^2\Gamma_t, \quad p_t^* = p_t^{*REE} - v\sigma_{m^*}^2\Gamma_t.$$

Finally, the nominal exchange rate satisfies

$$e_t = e_t^{REE} + v\left(\sigma_m^2 + \sigma_{m^*}^2\right)\Gamma_t.$$

Proposition C.4 contains the main result of this extension. It shows that, in the reflective equilibrium, balance sheet policies affect asset prices, including the exchange rate. In particular, the nominal interest rate and the price level—and, therefore, the exchange rate—are now functions of the entire path of bond purchases. In the reflective equilibrium, some households fail to anticipate that future taxes will now be risky in real terms since bonds promise a risk-free *nominal* payment. In particular, since the FX intervention is sterilized, the tax risk in the home country will be a combination of the shocks to the money supplies in the two countries. As before, if households do not hold rational expectations and, hence, fail to hedge the tax risk, asset prices will be affected.

C.3 A Model with Rational Expectation Agents

To investigate whether the presence of households with rational expectations can undo the non-neutrality result of balance sheet policies, we add a mass of rational-expectations agents to the simple model. Specifically, we assume that fraction $\tau \in [0,1]$ of agents form their expectations rationally, while the remaining fraction $1-\tau$ uses the level-k thinking process. Moreover, the fraction $1-\tau$ is split into groups with different levels k, where groups have mass given by the distribution function $f(k)$.

In the reflective equilibrium augmented with rational-expectations agents, market-clearing in the risky asset market requires

$$\tau\left(\frac{r^x + q_{t+1} - q_t(1+r)}{\gamma\frac{r}{1+r}\sigma_x^2} - X_{t+1}\right) + (1-\tau)\sum_{k=1}^{\infty} f(k)\left(\frac{r^x + \alpha_{q,t}^k - q_t(1+r)}{\gamma\frac{r}{1+r}\sigma_x^2} - \beta_{Tr,t}^k\right) = \overline{X} - X_{t+1}.$$

As in the simple model, the price q_t is deterministic. The first term on the left-hand side is the demand for risky assets by the rational-expectations agents, while the second term represents the demand by level-k thinkers. Conditional on the beliefs of level-k thinkers that that we computed in Section 2, we solve the equation above for price q_t:

$$q_t - q^{REE} = \tau\frac{q_{t+1} - q^{REE}}{1+r} + (1-\tau)\frac{r}{1+r}\gamma\sigma_x^2\sum_{k=1}^{\infty} f(k)\frac{X_{t+k}}{(1+r)^k}. \tag{C.38}$$

To solve for price q_t, we introduce the following new variables:

$$G_t \equiv \frac{r}{1+r}\gamma\sigma_x^2\sum_{k=1}^{\infty} f(k)\frac{X_{t+k}}{(1+r)^k},$$
$$\widehat{q}_t \equiv q_t - q^{REE}.$$

We can then rewrite equation (C.38) as follows:

$$\widehat{q}_t = \tau\frac{\widehat{q}_{t+1}}{1+r} + (1-\tau)G_t,$$

and iterate it forward (imposing a non-bubble condition) to get

$$q_t = q^{REE} + (1-\tau) \sum_{s=0}^{\infty} \left(\frac{\tau}{1+r} \right)^s G_{t+s},$$

or, using the definitions of G_t and $\widehat{q}_t$,

$$q_t = q^{REE} + (1-\tau) \frac{r}{1+r} \gamma \sigma_x^2 \sum_{k=1}^{\infty} \frac{f(k)}{(1+r)^k} \sum_{s=0}^{\infty} \left(\frac{\tau}{1+r} \right)^s X_{t+s+k}. \tag{C.39}$$

Comparing (C.39) to the price of risky assets in the reflective equilibrium in the simple model, equation (17), we note the following. First, the presence of the term $(1-\tau)$ implies that a higher fraction of rational-expectations agents must reduce the effectiveness of balance sheet policies. At the same time, however, the term $(\tau/(1+r))^s$ implies that a higher fraction of rational-expectations agents leads to a lower discounting of future policies and, thus, to a stronger effect of future purchases. These two opposing effects echo the discussion on the implications for balance sheet policies of a higher average level of sophistication in Section 2.5. To make this more explicit, we compute the price q_t in two special cases.

Example 1. Consider an example in which $f(k) = (1-\lambda)\lambda^{k-1}$ and $X_{t+k} = X_{t+1}\mu^{k-1}$. In this case,

$$\begin{aligned} G_t &= \frac{r}{1+r} \gamma \sigma_x^2 X_{t+1} \sum_{k=1}^{\infty} (1-\lambda) \lambda^{k-1} \frac{\mu^{k-1}}{(1+r)^k} \\ &= \frac{r}{1+r} \gamma \sigma_x^2 X_{t+1} \frac{1-\lambda}{1+r-\lambda\mu}. \end{aligned}$$

Thus, the asset price is

$$q_t = q^{REE} + \frac{r}{1+r} \gamma \sigma_x^2 \frac{1-\lambda}{1+r-\lambda\mu} X_{t+1} \frac{(1+r)(1-\tau)}{1+r-\tau\mu}.$$

The last expression has its maximum at $\tau = 0$ and monotonically declines to zero at $\tau = 1$. In this example, the first effect dominates and a higher fraction of rational-expectations agents makes balance sheet policies weaker.

Example 2. Consider now the following path of risky assets purchases $\{X_{t+1}\} = \{0, X_{t+2}, 0, 0, \ldots\}$. In this case,

$$\begin{aligned} G_t &= \frac{r}{1+r} \gamma \sigma_x^2 f(2) \frac{X_{t+2}}{(1+r)^2}, \\ G_{t+1} &= \frac{r}{1+r} \gamma \sigma_x^2 f(1) \frac{X_{t+2}}{(1+r)^1}, \\ G_{t+2} &= 0, \end{aligned}$$

and the asset price is

$$q_t = q^{REE} + \frac{r}{1+r} \frac{\gamma \sigma_x^2}{(1+r)^2} (1-\tau)(\lambda+\tau)(1-\lambda) X_{t+2}.$$

The price is now a non-monotonic function of τ. The derivative of the price with respect to τ is:

$$\frac{dq_t}{d\tau} = \frac{r}{1+r}\frac{\gamma\sigma_x^2}{(1+r)^2}(1-\lambda)\, X_{t+2}\left[1-\lambda-2\tau\right],$$

we have that, when the fraction of rational-expectations agents is low enough, $\tau < (1-\lambda)/2$, then $dq_t/d\tau > 0$. That is, the second effect dominates and balance sheet policies become stronger as τ grows and as long as $\tau < (1-\lambda)/2$.

C.4 A Model with a Learning Mechanism

In this section, we modify our simple model and allow for a simple version of "learning" or "dynamic equilibrium unraveling." In particular, we assume that the level of sophistication of agents changes over time. We introduce this assumption in the simplest possible way to highlight a number of qualitative results. Specifically, we assume that a current level-k thinker becomes level-$(k+h)$ in the subsequent period, where h is a constant non-negative integer number. One interpretation of this assumption is as follows. At the time a new policy is announced, an agent can compute k deductive iterations to form the expectations about future endogenous variables. In every subsequent period, the agent uses the already computed beliefs and performs h additional deductive iterations.

Formally, we assume that the distribution of levels of sophistication changes over time according to

$$f_t(k) = \begin{cases} f(k-ht), & k \geq 1+ht, \\ 0, & k < 1+ht, \end{cases} \tag{C.40}$$

where $f(k)$ is the distribution at the time the policy is announced.

We can compute the price effect of the intervention by evaluating the results in Proposition 2 with the distribution (C.40). Specifically, if we focus, for simplicity, on a permanent intervention of the size X, we obtain

$$q_t = q^{REE} + \frac{r}{1+r}\gamma\sigma_x^2 X \sum_{k=1}^{\infty} f_t(k)\frac{1}{(1+r)^k}.$$

Now the distribution of level-k agents changes over time. Using (C.40) and assuming that the initial distribution is exponential, we get

$$q_t = q^{REE} + \frac{r}{1+r}\gamma\sigma_x^2 \frac{X}{\bar{k}r+1}(1+r)^{-ht}. \tag{C.41}$$

Equation (C.41) shows that, over time, the price approaches q^{REE} at rate $1/(1+r)^h$. Thus, h determines the speed of convergence. The key implication of equation (C.41) is that a central bank cannot stimulate the economy forever by keeping the size of its balance sheet at a constant level.

To counteract the dampening forces coming from equilibrium unraveling, it is crucial that the size of the intervention increases over time. In the specific example, to keep the price q_t at a constant higher level, the central bank needs to increase asset purchases exponentially at the rate $\mu = (1+r)^{h/(1+h)} > 1$. Formally, with $X_{t+1} = \mu^t X_1$, where $\mu \leq ((1+r)/\mu)^h$ (to keep the price bounded), we get

$$q_t - q^{REE} = \frac{r}{1+r}\gamma\sigma_x^2 \cdot \frac{1}{\bar{k}(1+r-\mu)+\mu} X_1 \left(\frac{\mu^{1+h}}{(1+r)^h}\right)^t ,$$

so that, if $\mu^{1+h}/(1+r)^h = 1$, then q_t does not depend on time.

An empirical prediction of equilibrium unraveling is that new rounds of balance sheet policies tend to be weaker than initial rounds. For example, after controlling for the size of the intervention, the first round of quantitative easing implemented by the Federal Reserve in 2009 should have had stronger effects than the second round implemented in 2010.