The Analytics of Monetary Shocks with Generalized Hazard Functions

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Bank of Finland & CEPR

Summary and Introduction

- Random Menu Cost model:

  Caballero-Engel (1993,1999,2007), Dotsey-King-Wollman (1999)

- Used recently, e.g. Berger-Vavra (2018), Petrella et all (2018), Woodford (2008), Constain-Nakov (2011), others

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  - Flexibility Index $\neq$ summary of IRF.

Firm's Problem

- Price gap: x deviation from ideal markup (no inflation) follows driftless random walk: $dx = \sigma dW$ if price not changed
- Second order approximation to profit function $B\,x^2$, Bx^2 cost of deviation x.
- Firm can always pay fixed cost Ψ and adjust prices
- Each period w/prob. $\kappa\, dt$ firm draws cost $\psi \sim G(\cdot)$ assume (WLOG) that $G : [0, \Psi] \to [0, 1]$.
- Firm minimized expected discounted (at rate r) cost
- Optimal decision rule: sS bands $\bar{x}(\psi)$ increasing in ψ, and $\pm X$. (Caballero and Engel)
- Symmetry: optimal return pint to $x = 0$, and bands $\underline{x}(\psi) = -\bar{x}(\psi)$ and $\pm X$

Firm's Problem: Bellman equation

- $v(x)$ value function of firm w/price gap (markup deviation) x:

$$rv(x) = \min\left\{ Bx^2 + \frac{\sigma^2}{2} v''(x) + \kappa \int_0^{\Psi} \min\{\psi + v(0) - v(x)\,,\, 0\} dG(\psi)\,, \right.$$
$$\left. r\,(v(0) + \Psi) \right\}$$

- $v(0)$ is the minimum, optimal markup adjustment $x = 0$
- Random cost ψ drawn w/prob.: $\kappa\, dG(\psi)$ each period of length dt
- Firm adjust if $\psi \leq v(x) - v(0) =$ reduction in cost
- Firm can always pay (largest) cost Ψ and adjust.

Firm's Problem: Bellman equation (cont.)

- $v(x)$ value function of firm w/price gap (markup deviation) x:

$$rv(x) = \min \Big\{ Bx^2 + \frac{\sigma^2}{2} v''(x) + \kappa \int_0^{\Psi} \min \{\psi + v(0) - v(x)\,,\, 0\} dG(\psi)\,, \\ r\,(v(0) + \Psi) \Big\}$$

- Optimal decision rule X and $\bar{x}(\cdot)$:
  - If $|x|$ reaches $X \to$ adjust with certainty (mass point) .
  - At each $|x| \to$ adjust w/ prob $\Lambda(x) = \kappa\, G\,(v\,(x) - v\,(0))$ per dt
- Generalized Hazard Function $\Lambda(x)$:
  - adjustment probability increases with $|x|$ (Caballero-Engel),
  - at higher deviation $|x|$ more cost are worthwhile to be paid.
- We allow G to have mass points, and also $\Psi = \infty \implies X = \infty$.
- Calvo$^+$: G has a mass point at $\psi = 0$, otherwise it is constant.

Inversion Result

- Caballero Engel: CDF $G \implies \Lambda$ increasing and symmetric GHF.
- **New result:** Λ is lincreasing and symmetric $\implies G$ GHF:

  Fix volatility σ^2, curvature B, and discount rate r.

  Consider an upper bd X & a symmetric, weakly increasing GHF Λ.

  Then there is a unique Ψ and CDF G that rationalize X, Λ.
- **Importance:** can use any weakly increasing symmetric GHF Λ

  to fit data or write models.
- Solving Bellman equation and policy is hard, non-linear o.d.e.
  Solving for G given Λ **easy and constructive**: linear o.d.e.
  (discrete G, linear eqns).

- Sketch of Proof (*skip for time*). Recall for $0 < x < X$:

$$
\begin{aligned}
rv(x) &= Bx^2 + \frac{\sigma^2}{2} v''(x) + \kappa \int_0^{\Psi} \min\{\psi + v(0) - v(x)\,,\, 0\} dG(\psi) \\
&= Bx^2 + \frac{\sigma^2}{2} v''(x) + \kappa \int_0^{v(x)-v(0)} \psi dG(\psi) \\
&+ \kappa \left[v(0) - v(x)\right] G\left(v(x) - v(0)\right)
\end{aligned}
$$

- Differentiate value fcn: $u(x) \equiv v'(x)$, use $\Lambda(x) \equiv \kappa G(v(0) - v(x))$

$$
\left[r + \Lambda(x)\right] u(x) = 2Bx + \frac{\sigma^2}{2} u''(x) \text{ for } x \in [0, X]
$$

- Use symmetry at $x = 0$ and smooth pasting at $x = X$ to motivate boundaries for u in domain $[0, X]$

- Solve for u given Λ. Linear ode (use Sturm-Liouville, Direchlet)
  Show $u \geq 0$ so $u = v' > 0$: $\iff \Lambda'(x) = \kappa G'(v(x) - v(0)) u(x)$.

- Then $v(x) = u'(0) \frac{\sigma^2}{2r} + \int_0^x u(z) dz \quad \text{for } x \in [0, X]$.

Optimal Adjustment Intensity Model

- Alternative model that also produces generalized hazard rateΛ
- Firms chose the intensity of adjustment ℓ at a cost $c(\ell)$.
- Also can pay $\Psi \leq \infty$ if they want to adjust w/certainty

$$rv(x) = \min \left\{ Bx^2 + \frac{\sigma^2}{2} v''(x) + \min_{\ell \geq 0} \ell \left(v(0) - v(x) \right) + c(\ell) \right.$$
$$\left. , \; r \left(\Psi + v(0) \right) \right\}$$

- Similar to Rational Inattention: Woodford (2007) and others.
- Solution: GHF $\Lambda(x) = \ell^*(x)$, symmetric, increasing.
- Also: any increasing $\Lambda(\cdot)$ is rationalized by a cost $c(\cdot)$.
- Technical No restriction on tail of Λ.

Steady State Statistics: Invariant Dist. & Frequency

- Intermediate step: $f(x)$ invariant distribution of price gaps

  Solves the KFE for all $x \in [-X, X]$, $x \neq 0$:

  $$f(x)\Lambda(x) = \frac{\sigma^2}{2} f''(x) \quad \text{and } f \text{ continuous at all } x$$

  with $f(-X) = f(X) = 0$ and $\int_{-X}^{X} f(x)dx = 1$.

- N_a Number of price changes per unit of time:

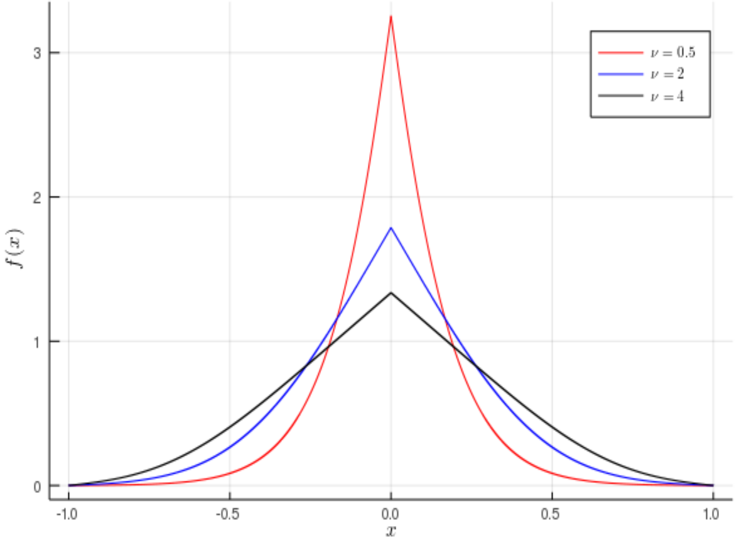
  $$N_a = \underbrace{2}_{symmetry} \left[\underbrace{\int_0^X f(x)\Lambda(x)dx}_{\#w/x \times \Pr \Delta p} \underbrace{- \frac{\sigma^2}{2} f'(X)}_{\Pr \Delta p at \pm X} \right]$$

  uses symmetry of f around $x = 0$.

- Interpretation of $\Lambda(x)f(x)$ and $\frac{\sigma^2}{2}|f'(X)|$.

Figure: Density function for the Invariant distribution of gaps: $f(x)$

Power hazard function: $\Lambda(x) = \kappa \left(\frac{x}{X}\right)^{\nu}$, $X = 1 < \infty$



Steady State Statistics: Dist. of price changes Δp

- Price changes have *mass point(s)* at $\pm X$ if $X < \infty$
- Symmetric *density* of price changes q for all $x \in (-X, X)$

$$\Delta p = \begin{cases} -x & \text{w/ density } q(x) = \frac{\Lambda(x)\, f(x)}{N_a} \text{ provided that } |x| < X, \\ -X & \text{w/ probability mass } \frac{\frac{\sigma^2}{2}|f'(X)|}{N_a} \end{cases}$$

- CDF $Q : [-X, X] \to [0, 1]$ so that $Q' = q$ in the interior.

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- CDF $Q : [-X, X] \to [0, 1]$ so that $Q' = q$ in the interior.
- $k-$ moment of price changes (e.g. variance, kurtosis):

$$\mathbb{E}(\Delta p^k) = \frac{2\left[\int_0^X x^k \Lambda(x) f(x) dx - X^k \frac{\sigma^2}{2} f'(X)\right]}{N_a}$$

Examples: Power Generalized Hazard Function Λ

- A common case used it the literature is $\Lambda(x) \propto x^2$.
- Solution for f are Bessel functions.
- We consider a generalization $\Lambda(x) = \kappa \left(\frac{x}{X}\right)^{\nu}$ for $\nu > 0$.
- Note $\nu = 0$ is Calvo and $\nu \to \infty$ Golosov-Lucas.
- $Kurt(\Delta p)$ depends on shape, controlled by ν and $\frac{\sigma^2}{2\kappa}/X^2$.
- $Kurt(\Delta p)$ of quadratic case ($\nu = 2$) is low, about 1.75.

  More so if X is finite.

Example of density price changes: $q(\Delta p)$

Quadratic Hazard $\Lambda(x) = \kappa x^2$, $X = \infty$, different $\eta \equiv \left(\frac{2\kappa}{\sigma^2}\right)^{\frac{1}{4}}$

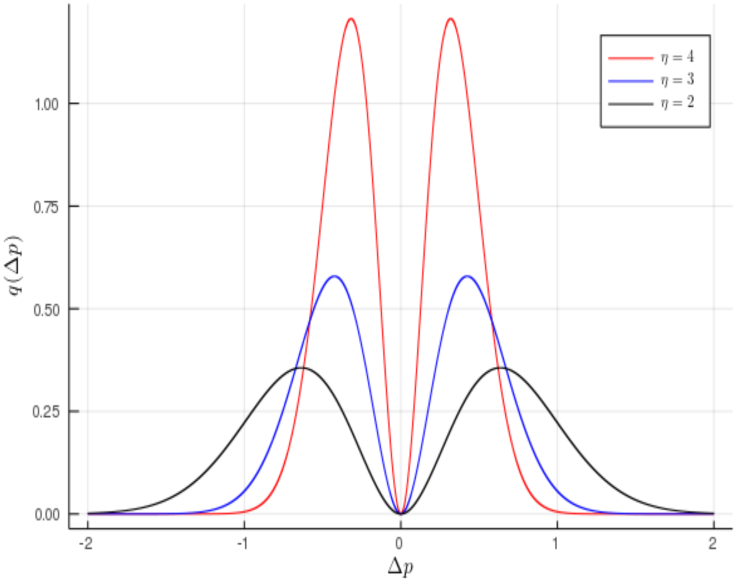


Figure: Kurtosis of power hazard function as power ν varies *(skip)*

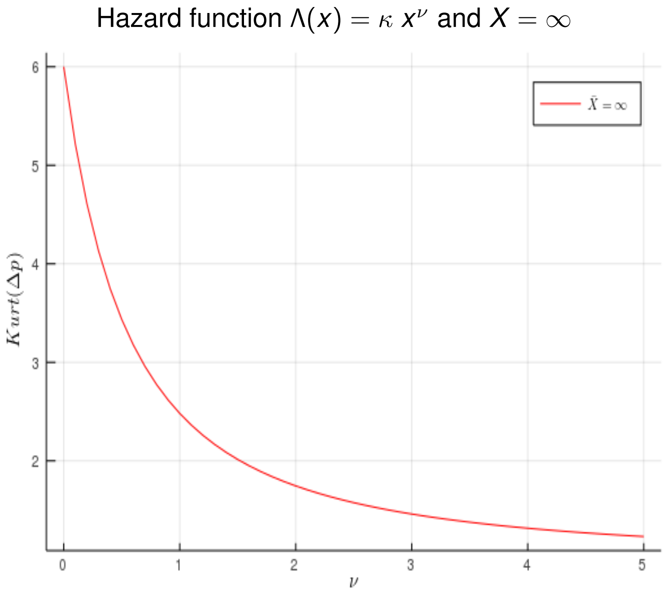
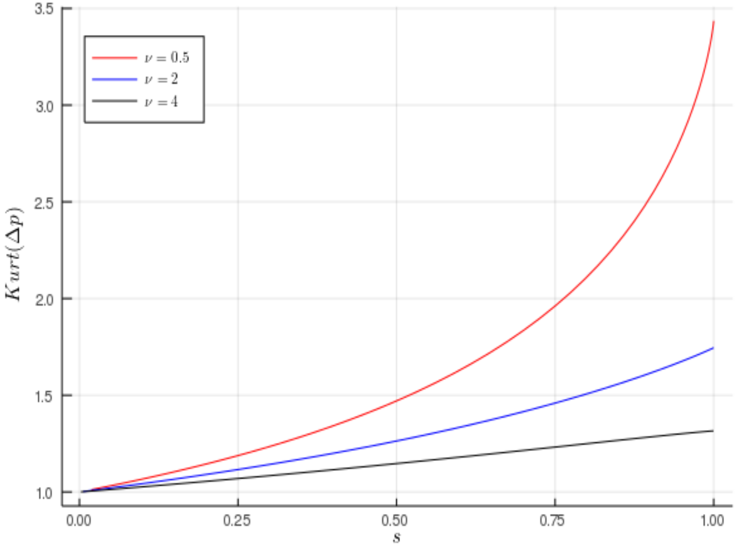


Figure: Kurtosis behavior with a power hazard function as (skip)
s: fraction of price changes away from boundary X varies

Hazard function $\Lambda(x) = \kappa \left(\frac{x}{X}\right)^{\nu}$ and $X < \infty$



Recovering f, Λ and G (or c) from Δp dist. q

- Price changes have density $\Delta p \sim q(\cdot)$ for $x \in (-X, X)$.
- Assume q is symmetric., let Q be its CDF of q, so $q = Q'$.
- Using equations above: $q \implies f \implies \Lambda$:

  Invariant distribution f fo price gaps:

  $$f(x) = \frac{2}{Var(\Delta p)} \left[\int_x^\infty (1 - Q(z))\, dz \right] \text{ for all } x \geq 0$$

  and Generalized Hazard Function Λ:

  $$\Lambda(x) = \frac{N_a\, Var(\Delta p)}{2} \frac{q(x)}{\int_x^\infty (1 - Q(z))\, dz} \text{ for all } x > 0$$

- Given Λ & previous recovery results, we get:

  $q \implies f \implies \Lambda \implies G$ (distribution of menu cost) or

  $q \implies f \implies \Lambda \implies c$ (cost function)

Recovering f, Λ and G from q: trivial proof *(skip)*

- Model gives us:

$$\frac{\Lambda(x)f(x)}{N_a} = q(-x) \;\; \text{all } x \geq 0$$

$$\frac{\sigma^2}{2} f''(x) = \Lambda(x)f(x) \;\; \text{all } x \geq 0$$

$$\sigma^2 = Var(\Delta p)\, N_a$$

- Replace KFE into eqn for q:

$$\frac{\sigma^2}{2} \frac{f''(x)}{N_a} = q(-x) \;\; \text{all } x \geq 0$$

- Integrate w.r.t x twice, used expression for σ^2 gives $f(x)$.
- Use f and definition of q again to get Λ.

Estimating distribution Q

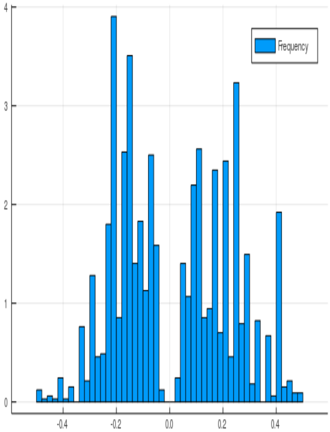
- Cavallo's scraped data $\{\Delta p_{it}\}$: no time agg. + low meas. error.
- Mixture of distribution w/same Kurtosis has higher Kurtosis
- Unobserved heterogeneity b_i across products i within category

$$\Delta p_{it} = b_i \, \Delta \tilde{p}_t \text{ for } i \in I \text{ and } t \in T(i)$$
$$\Delta \tilde{p}_t \sim Q \text{ and } b_i \perp \Delta \tilde{p}_t$$

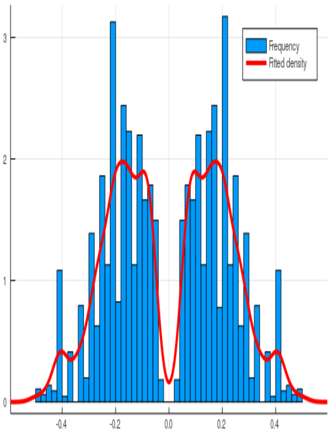
- Theory: price changes iid across t <u>for a given product</u>.
- Non-parametrically identify common distribution Q for a category variation on Kotlarski's lemma (non-parametric random effects).
- Estimate Kurtosis, $Kurt(\Delta p)$ of dist. Q, small due to:
  - correlation of price changes for a given product (induced by b_i),
  - no time aggregation and very low measurement error.

Figure: Distribution of pooled price changes in a narrow category

Histogram for Δp

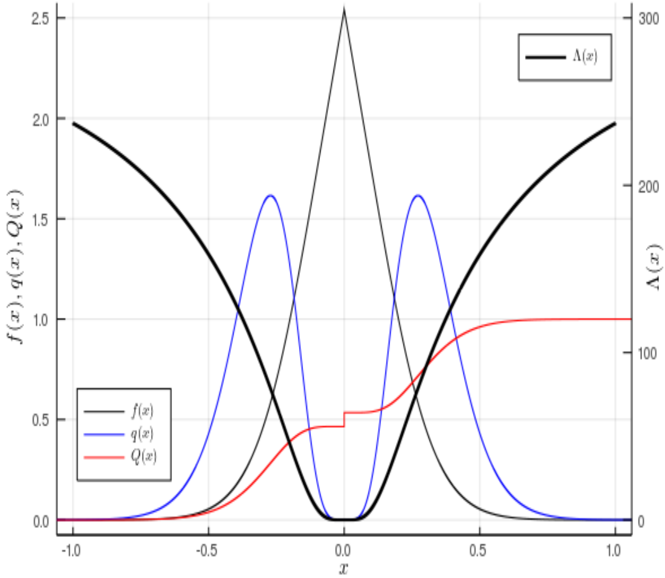


Symmetrized histogram
with a fitted density

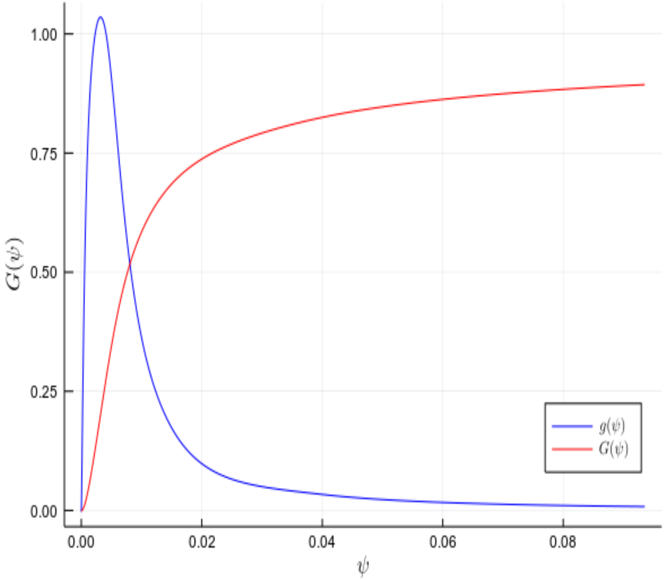


Pooling all products "Non-durable household goods"

Estimated $Q(\cdot)$ and $q(\cdot)$, recovered $f(\cdot)$ and $\Lambda(\cdot)$



Recovered CDF G and density g of menu costs



Estimating Kurtosis of underlying distribution

- Recall:

$$\Delta p_{it} = b_i \, \Delta \tilde{p}_t \text{ for } i \in I \text{ and } t \in T(i)$$
$$\Delta \tilde{p}_t \sim Q \text{ and } b_i \perp \Delta \tilde{p}_t$$

- Then

$$Kurt(\Delta \tilde{p}_t) = \frac{Kurt(\Delta p_{it})}{1 + \text{corr}(\Delta p_{it}^2, \Delta p_{is}^2) CV(\Delta p_{it}^2) CV(\Delta p_{is}^2)}$$

- Autocorrelated square changes of product i inconsistent w/model

- Autocorrelated square changes indicate heterogeneity.

Kurtosis Estimates for categories w/1000+ products

Number Products	Number P. changes	$\hat{\mathbb{E}}(\Delta p_{it})$	$\hat{\sigma}(\Delta p_{it})$	Kurtosis Pooled	Kurtosis w/Unobs. Heterog.
3437	74464	0.002	0.342	3.367 (0.160)	1.640 (0.065)
3225	56527	0.002	0.329	3.807 (0.094)	1.950 (0.047)
2551	30343	-0.001	0.246	3.520 (0.263)	2.049 (0.155)
1401	27321	0.002	0.344	2.923 (0.087)	1.671 (0.051)
1388	30111	0.003	0.309	3.579 (0.236)	2.035 (0.116)
1154	20995	0.007	0.309	3.467 (0.139)	1.983 (0.049)
1032	17724	0.002	0.261	3.314 (0.216)	1.773 (0.128)

Table: Bootstrapped standard errors in parenthesis

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GHF Λ and (duration) Survival Function S

- Let $S(t)$ be survival function of duration of spell:

$$S(t) = \mathbb{E}\left[e^{-\int_0^t \Lambda(x(s))ds} \mid x(0) = 0\right]$$

  assuming, to simplify, that $X = \infty$.

  - Hazard $h(t) = -\frac{S'(t)}{S(t)}$ pr. Δp as function of **duration** t
  - Hazard $h(\cdot)$, as function of duration is observable.

- Identification: Given $S(t)$ (or $h(t)$) $\implies \Lambda(x)$

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  - Hazard $h(\cdot)$, as function of duration is observable.
- Identification: Given $S(t)$ (or $h(t)$) $\implies \Lambda(x)$
- For instance, using hazard rates $h(t)$:
$$h(0) = \Lambda(0) \geq 0\,,\ \frac{\partial h(t)}{\partial t}|_{t=0} = \frac{\sigma^2}{2}\frac{\partial^2 \Lambda(x)}{\partial x^2}|_{x=0}$$
$$\frac{\partial^2 h(t)}{\partial t^2}|_{t=0} = \left(\frac{\sigma^2}{2}\right)^2 \frac{\partial^4 \Lambda(x)}{\partial x^4}|_{x=0}\,,$$
$$\frac{\partial^3 h(t)}{\partial t^3}|_{t=0} = \left(\frac{\sigma^2}{2}\right)^3 \frac{\partial^6 \Lambda(x)}{\partial x^6}|_{x=0} - 4\left(\frac{\sigma^2}{2}\frac{\partial^2 \Lambda(x)}{\partial x^2}|_{x=0}\right)^2$$

GHF Λ and (duration) Survival Function S *(skip)*

- Let $S(t)$ be survival function of duration of spell:

$$S(t) = \mathbb{E}\left[e^{-\int_0^t \Lambda(x(s))ds} \,|\, x(0) = 0 \right]$$

assuming, to simplify, that $X = \infty$.

- All the derivatives $S(t)$ at $t = 0$ identify the level and all the even derivatives of $\Lambda(x)$ at $x = 0$.

  Thus, if Λ is analytical, then it is identified by S.

- Using hazard rates:

$$h(0) = \Lambda(0) \geq 0\,, \; \frac{\partial h(t)}{\partial t}|_{t=0} = \frac{\sigma^2}{2}\frac{\partial^2 \Lambda(x)}{\partial x^2}|_{x=0}$$

$$\frac{\partial^2 h(t)}{\partial t^2}|_{t=0} = \left(\frac{\sigma^2}{2}\right)^2 \frac{\partial^4 \Lambda(x)}{\partial x^4}|_{x=0}\,,$$

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Measure of Price Changes independent of state

- Define *Calvo-ness* index $\mathcal{C} \equiv \frac{\Lambda(0)}{N_a}$:
  fraction of price changes independent of the price gap x **(state).**

$$\mathcal{C} = q(0)\,\frac{Var(\Delta p_{it})}{2\,\mathbb{E}[|\Delta p_{it}|]}$$

  for $t \neq s$.

- Density close to zero $q(0) > 0$ only if state not important.

- Can be adapted (see paper) w/unobserved heterogeneity.
- For Cavallo's data we estimate $\mathcal{C} \approx 6\%$.
  Small values because small density of small price changes $q(0)$.

Measure of Price Changes independent of state (*skip*)

- Define *Calvo-ness* index $\mathcal{C} \equiv \frac{\Lambda(0)}{N_a}$, in words: fraction of price changes independent of the price gap (state).

- Proposition:

$$\mathcal{C} = \mathcal{C}_{pooled} \left(1 + \frac{Cov(b_i^{-1}, b_i^2)}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]} \right) < \mathcal{C}_{pooled}$$

  where the two components are given by

$$\mathcal{C}_{pooled} = \frac{q(0)\, Var(\Delta p_{it})}{2\, \mathbb{E}[|\Delta p_{it}|]} \quad \text{and} \quad 1 + \frac{Cov(b_i^{-1}, b_i^2)}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]} = \frac{\mathbb{E}[|\Delta p_{it}|^{-1}|\Delta p_{is}|^2]}{\mathbb{E}[|\Delta p_{it}|^{-1}]\mathbb{E}[|\Delta p_{it}|^2]}$$

  for $t \neq s$.

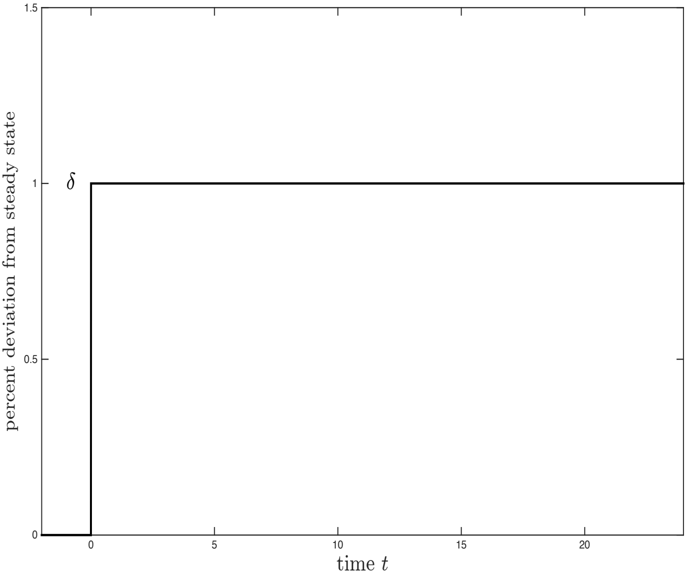
- For Cavallo's data we estimate $\mathcal{C} \approx 6\%$. Small values because small density of small price changes $q(0)$.

Cumulative Response to Aggregate Shock

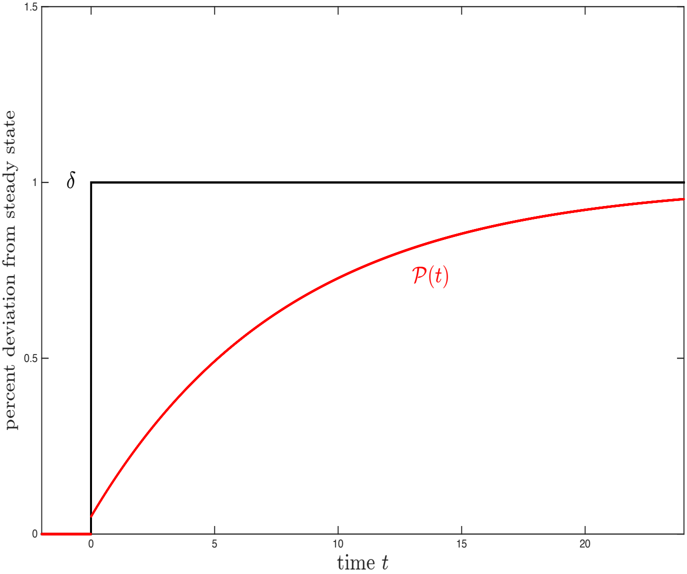
- Once and for all monetary (cost) shock of size δ
- Effect of aggregate price level $\mathcal{P}(t;\delta)$ at horizon t
- Effect on output $Y(t;\delta) = (1/\epsilon)\,[\delta - \mathcal{P}(t,\delta)]$ at horizon t.
- Cumulative IRF (CIRF) $\mathcal{M}(\delta) = \int_0^\infty Y(t,\delta)dt$
- Small shock $\mathcal{M}(\delta) \approx \mathcal{M}'(0)\delta$ since by definition $\mathcal{M}(0) = 0$.
- For models with no first order strategic complementarity.

  Keep same decision rules
- Show that $\mathcal{M}(\delta) = \frac{Kurt(\Delta p)}{6N_a}\delta + o(\delta^2)$

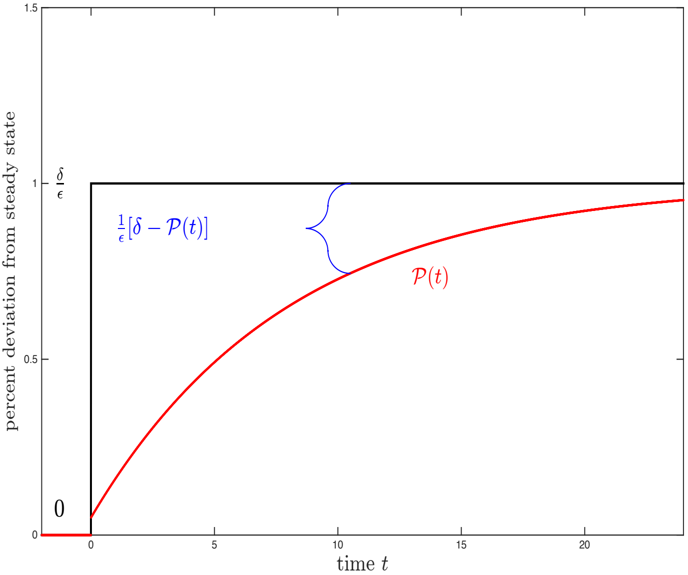
Once and for all monetary shock at $t = 0$.



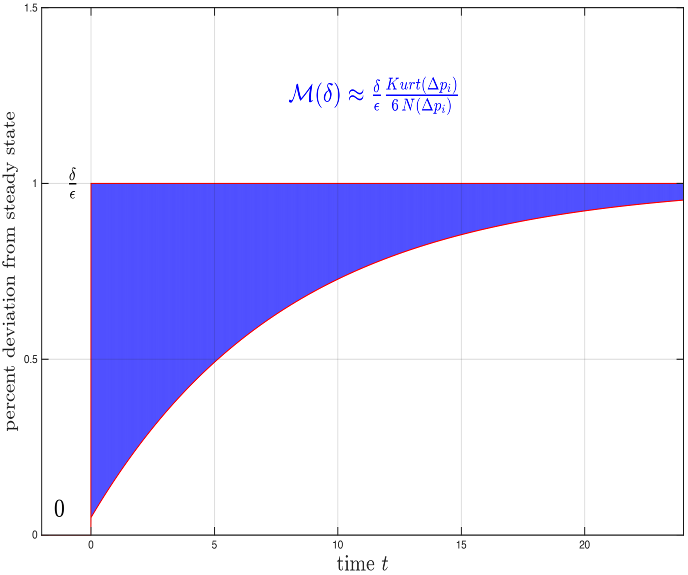
Impulse response of Price Level



Impulse response of output



Area under impulse response of output



Definition of CIR, cumulative response function (skip)

- Symmetry: suffices to consider effect until first Δp after shock δ.
- At t, a firm that has not adjusted contributes $-x$ to output $Y(t)$
- $m(x)$: contribution to CIR of firm starting w/gap x after shock δ.
- Let τ stopping time until $|x| = X$

$$m(x) = -\mathbb{E}\left[\int_0^\tau e^{-\int_0^t \Lambda(x(s))ds} x(t)dt \,|\, x(0) = x\right]$$

- Just before the shock, firms gap x are distributed as $f(\cdot)$.
- Since shock moves each x to $x - \delta$, CIR is given by

$$\mathcal{M}(\delta) = \int_{-X}^{X} m(x)f(x+\delta)dx = \delta\int_{-X}^{X} m(x)f'(x)dx + o(\delta^2)$$

Sufficient Statistic for CIR, first result

- Let Λ be *ANY* Generalized Hazard Function.

  Then the cumulative impulse response (CIR) for a small monetary shock:

  $$\mathcal{M}(\delta) = \mathcal{M}'(0)\delta + o(\delta^2) = \frac{Kurt(\Delta p)}{6\, N_a}\delta + o(\delta^2)$$

  The approximation is accurate, since $\mathcal{M}''(0) = 0$.

- $Kurt(\Delta p)$ depends on shape of $\Lambda(\cdot)$, not its level
  - multiplying $\Lambda(\cdot)$ & σ^2 by constant $\implies$ same dist. q & $Kurt(\Delta p)$.
  - multiplying $\Lambda(\cdot)$ and σ^2 by constant scales N_a by same constant.

Sufficient Statistic for CIR, bounds, second result

- Recall

$$\mathcal{M}(\delta) = \frac{Kurt(\Delta p)}{6N_a}\delta + o(\delta^2)$$

- Only weakly increasing Λ can be rationalized as random menu cost or optimal adjustment intensity model.

- Among the weakly increasing Λ the inverted-L hazard Λ (i.e. Golosov and Lucas) has the smallest $Kurt(\Delta p) = 1$.

- Among the weakly increasing Λ the constant hazard Λ (i.e. Calvo) has the largest $Kurt(\Delta p) = 6$.

- Decreasing Λ have $Kurt(\Delta p) > 6$, and can be arbitrarily large.

- These results imply that $Kurt(\Delta p)$ measure positive selection for price increases after a positive aggregate shock.

Scope & limitations of $Kurt(\Delta p)/(6\,N_a)$ *skip if short in time* as a Cumulative Impulse Response (*CIR*) sufficient statistic

- Also hold on:
  - Calvo$^+$ + Multiproduct $n \geq 1$ products, (Alvarez-Lebihan-Lippi AER 16), common case Calvo$^+$ w/$n = 1$.
  - General Rational inattentiveness model (Reis 06), (Alvarez-Lippi-Paciello, RES 15) common case: pure Calvo.
  - Insensitive, up to first order, to add:
    - steady state inflation
    - asymmetry in objective function
- Does NOT hold:
  - Large inflation (Sheshinsky-Weiss, Blanco-Bailey)
  - Firm do not close gap (Alvarez-Lippi AEJ 19) Temporary price changes.
  - Stochastic volatility.
  - Strategic Interactions (?).

Beyond CIR: Impulse Response $Y(t;\delta) = \delta - \mathcal{P}(t,\delta)$

- Use Eigenvalue-Eigenfunction to solve for $Y(t;\delta)$ and $\mathcal{P}(t,\delta)$.

- Caballero-Engel meet Schrodinger-Dirac:
  - Quadratic Λ has the same eigenfunctions-eigenvalues as famous “Quantum-Oscillator".
  - In general, Schrodinger Eqn for positive symmetric potential.

- Use this result to compare with other proposed sufficient statistic: Flexibility Index.

- Flexibility Index $\mathcal{F} \equiv \lim_{t\to 0,\delta\to 0} \frac{\partial}{\partial_t} Y(t,\delta)$
  - initial slope of the IRF with respect to time, at a small shock.

- Flexibility Index $\mathcal{F}$ does not give same ordering as $\mathcal{M}$ CIR.
  - Construct examples with same $\mathcal{F}$ & are very different CIR $\mathcal{M}$.
  - Even effect for $Y(t;\delta)$ for small t can be quite different.

skip

- IRF of price level, recall $Y(t;\delta) = \delta - \mathcal{P}(t,\delta)$:

$$\mathcal{P}(t,\delta) = \Omega(\delta) + \int_0^t \omega(s,\delta)\, ds$$

- $\omega(t,\delta) = \frac{\partial}{\partial t}\mathcal{P}(t,\delta)$ flow at t, and $\Omega(\delta)$ initial jump at $t = 0$:

$$\omega(t,\delta) = -\int_{-X}^{X} x\Lambda(x) f(x,t) dx + X\sigma^2 \left[f'(-X,t) - f'(X,t)\right]$$

$$\Omega(\delta) = \int_{-X}^{-X+\delta} (-x+\delta)\; f(x,0)\, dx$$

- $f(x,t)$ solves KFE with initial condition $f(x,0) = f(x+\delta)$ does not have to be steady state.

- Flexibility Index $\mathcal{F} \equiv \frac{\partial}{\partial \delta}\omega(0,\delta)|_{\delta=0}$,

- If $X < \infty \implies$ Flexibility Index $\mathcal{F} = \infty$

skip

- Simplify arguments by using $X = \infty$,
  (Flexibility index is always infinite with barriers)
- IRF of price level, recall $Y(t; \delta) = \delta - \mathcal{P}(t, \delta)$:

$$\mathcal{P}(t, \delta) = \int_0^t \omega(s, \delta)\, ds$$

- $\omega(t, \delta) = \frac{\partial}{\partial t}\mathcal{P}(t, \delta)$ flow at t,

$$\omega(t, \delta) = -\int_{-X}^{X} x\Lambda(x) f(x, t) dx$$

- $f(x, t)$ solves KFE with initial condition $f(x, 0) = f(x + \delta)$
- Flexibility Index $\mathcal{F} \equiv \frac{\partial}{\partial \delta}\omega(0, \delta)|_{\delta=0}$,
- $\mathcal{F}$ is easy to compute, only requires $f(x, 0)$ and gives:

$$\mathcal{F} = N_a \left(1 + \int_{-\infty}^{\infty} \frac{x\Lambda'(x)}{\Lambda(x)} q(x) dx\right)$$

skip, if short of time

- Flexibility Index $\mathcal{F} \equiv \frac{\partial}{\partial \delta}\omega(0,\delta)|_{\delta=0}$,
- Advantages of $\mathcal{F}$: is easy to compute, only requires $f(x,0)$:

$$\mathcal{F} = N_a\left(1 + \int_{-\infty}^{\infty} \frac{x\Lambda'(x)}{\Lambda(x)} q(x)dx\right)$$

- Disadvantages of Flexibility Index:
  By design, it only measures effect at very short term.
- Example: $\Lambda(x) = \Lambda(0) + \kappa x^{\nu}$ so that:
  - Same frequency of adjustment $N_a = 1$
  - Same Flexibility Index $\mathcal{F} = 3$
  - Different Cumulative Impulse Response $\mathcal{M}$, i.e. different Kurtosis.
- Literature “misinterpret" idea of sufficient statistic.

Figure: Values of $Kurt(\Delta p)$ or CIR relative to quadratic $\Lambda(x)$
Each case (a dot), different parameters ν for $\Lambda(x) = \Lambda(0) + \kappa x^{\nu}$
All cases have values $\Lambda(0), \kappa$ so that they have the same $\mathcal{F} = 3$ and $N_a = 1$

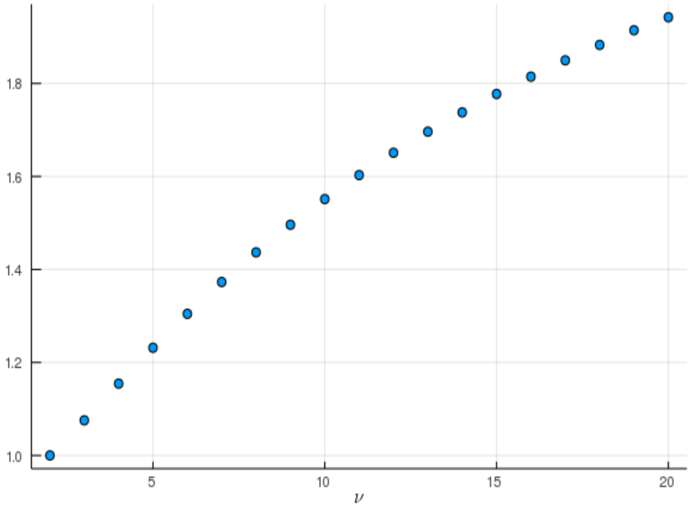
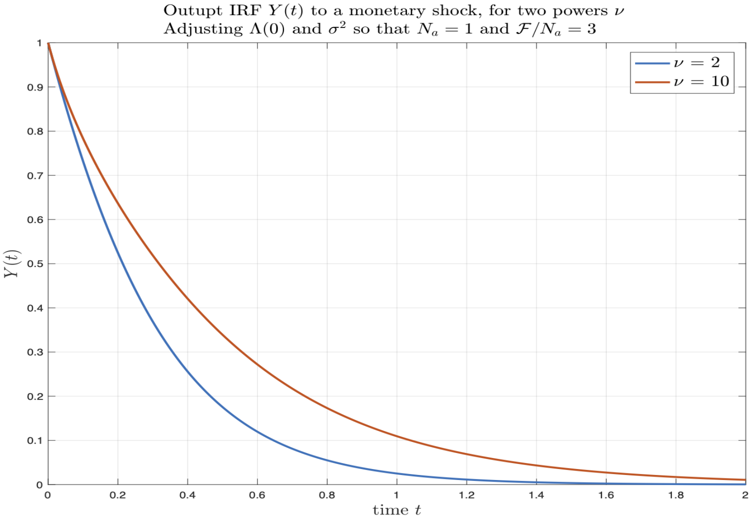


Figure: Impulse Responses for two cases of power plus $\Lambda(x) = \Lambda(0) + \kappa x^{\nu}$
Both cases with same $\mathcal{F}$ and N_a.
Same slope at zero $Y'(0)$, but IRF $Y(t)$ different even in short run,
since expected time to adjustment $N_a = 1$



Compute Entire Impulse Response function (*skip*)

- Consider operator $\frac{\sigma^2}{2}\frac{d^2}{dx^2} - \Lambda(x)$ on functions g on $[-X, X]$, Direchlet boundary $g(-X) = g(X) = 0$.
- If $X = \infty$ assume $\Lambda(x) \to \infty$, so operator is compact.
- Impulse response on shift δ to invariant f for function$g$ is

$$Y(t) = \delta \sum_{j=1}^{\infty} e^{\lambda_j t} \langle \varphi_j \, , \, g \rangle \langle \varphi_j , \, , \, f'(x) \rangle$$

- φ_j eigenfunctions, λ_j eigenvalues.
- A lot is known about eigenfunctions and eigenvalues.

Conclusions

- Analyze popular model in the literature,
  Caballero and Engel, Dotsey-King-Wollman, Woodford, ...
- Models: random-menu cost G or optimal adjustment intensity c
- Any weakly increasing Λ can be rationalized
- Sufficient statistic for CIR of monetary shock $\frac{Kurt(\Delta p)}{6N_a}$
- Span from GL (smallest effect) to Calvo (largest effect)
- Characterization: G (or c) $\iff \Lambda \iff f \iff q$
- Characterization: $S \iff \Lambda$, where S survival function (duration).
- Quadratic Λ, used in literature: small effects, $\approx$ Cavallo's data
- Define & estimate small % of price changes independent of state.
- Eigenvalues-Eigenfunctions used to computing entire IRF
- Examples: Flexibility index is constant, but CIR changes a lot.