Parallel Currencies in a New Keynesian Framework

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Abstract

We construct a simple New Keynesian framework with multiple parallel currencies as pricing units, and analyse the macroeconomic dynamics under exchange-rate shocks. In the baseline setup with homogeneous price rigidity, we find that a one-off exchange-rate shock leads to persistent distributional effects between the currency sectors. With heterogeneous price rigidity, we find that the effect of an exchange-rate shock is not neutral as long as it originates from a currency sector with sticky prices. Our simulations of endogenous currency choices show that the non-dollar sector may increase in size when prices in the dollar sector become less rigid, causing greater impacts from exchange-rate shocks.

Keywords: private money, cryptocurrency, currency choice, monetary policy

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1 Introduction

The proliferation of digital payment platforms has facilitated transactions in various currencies. Some currencies are pegged to the fiat, such as bank deposits and stablecoins, while others do not, such as reward points and cryptocurrencies. In this paper, we study, through the lens of a basic New Keynesian (NK) framework, the macroeconomic outcomes of exchange rate shocks in an economy with multiple parallel currencies. Our NK framework features two departures from Galí (2015): Households derive liquidity services from multiple currencies, and firms price their products in currencies of their choices.

Our primary finding is that exchange-rate shocks among parallel currencies lead to macroeconomic volatility when there are firms pricing in a non-dollar currency, and when prices are sticky in the non-dollar sector. This is seen from four particular aspects. Firstly, the nominal exchange rate between any pair of parallel currencies is a random-walk process, given that all currencies are perfect substitutes in providing liquidity services. Secondly, the relative price in each currency sector to the general price level is a state variable in an NK economy. This is a consequence of price rigidity which delays the process of prices converging to their desired level after an exchange-rate shock. The relative price also explains the persistent distributional output effects, and the sectoral inflation dynamics. Thirdly, the responses of aggregate macroeconomic variables vary with the size of the non-dollar sector. Fourthly, an exchange-rate shock is not neutral if it arises from a currency sector in which price adjustments are infrequent. With a discrete choice model, we further find that a change in price rigidity leads to varying sizes of the currency sectors which may alter the impact of exchange-rate shocks.

There are three strands of literature to which this paper contributes. The most closely related is the emerging literature on the economics of private monies and cryptocurrencies. Following the rise of bitcoin and blockchain technology, recent literature such as Schilling and Uhlig (2019) and Fernández-Villaverde and Sanches (2019) has analysed cryptocurrencies as a medium of exchange which is from the monetarists' perspective. We, on the contrary, provide an analysis from the producers' perspective. We model producers who decides the currencies that they wish to price in. This feature of our framework relates us to the literature on currency choices. Gopinath et al. (2010) examine endogenous choice between local currency pricing and producer currency pricing. Gopinath et al. (2019) further explores dominant currency pricing. These currency choice frameworks pertain to open economies. We adopt them and apply to our closed-economy framework with multiple currencies. The third strand of literature that we are related to is the multi-sector NK framework such as Cienfuegos (2019), Barsky et al. (2007) and Sterk (2010), in which heterogeneity among sectors are involved. Our discussion on exchange-rate shocks relates the multi-sector NK literature to the current issues of multi-currency co-existence.

The remainder of the paper are orgainsed as follows. Section 2 describes the model setup for our analyses. Section 3 presents the key equations consisting the linearised NK framework. Section 4 discusses two baseline cases, including dynamics under homogeneous price regidity and with one flexible-price sector. Section 5 analyses equilibrium dynamics under three alternative Taylor rules in details. Section 6 relaxes the assumption of homogeneous price rigidity. Section 7 allows endogenous currency choice and examines transition dynamics as price rigidity changes. Section 8 concludes.

2 Model

The model extends from the basic NK framework presented in Chapter 3 of Galí (2015). We introduce two departures. First, multiple types of money provide liquidity services to the households. This is modelled with a money-in-utility setup. Particularly, all monies are perfect substitutes in providing liquidity services. Second, each firm has a choice of currency in which it sets the price of its goods. The subtle difference in firms' price-setting behaviour, as compared to a conventional NK setup, is that firms have to consider exchange-rate dynamics when setting the optimal price.

2.1 Currencies and price indices

There is a total of K parallel currencies circulating in the economy. Each currency j has money supply $M_{j,t}$ in period t. Among them, there is one centralised currency, of which the money supply is managed by the central bank, and $K-1$ currencies which are created by private entities. Without loss of generality, the centralised currency is indexed by $j = 1$, and is named *dollar*. The price of currency j in terms of dollar in period t is denoted by $\mathcal{E}_{j,t}$, which is also known as the nominal exchange rate between currency j and the dollar. The price of currency j in terms of a different currency j' is then calculated as the ratio between $\mathcal{E}_{j,t}$ and $\mathcal{E}_{j',t}$. The nominal exchange rate of dollar equals to 1 for all periods, in other words,

$$\mathcal{E}_{1,t} = 1. \tag{1}$$

Each currency j can be chosen as the pricing unit by any monopolistic competitive firm i located in unit interval. The firm may only choose one currency as the pricing unit. The price set by firm i in currency j is denoted by $P_{j,t}(i)$. Despite the choice of pricing currency, the firm is always willing to accept any other parallel currency at the prevailing nominal exchange rate. All firms pricing in the same currency form a sector with the same index as the currency's. The sectoral price index for sector j is given by

$$P_{j,t} \equiv \left[\frac{1}{\upsilon_j}\int_{\upsilon_j(t)} P_{j,t}(i)^{1-\epsilon} di\right]^{\frac{1}{1-\epsilon}} \tag{2}$$

where $\upsilon_j(t)$ is the set of firms in sector j in period t, and ϵ is the intra-temporal elasticity of substitution among the differentiated goods. We assume that $\upsilon_j(t)$ is of a constant size υ_j, until we allow for endogenous currency choice in Section 7. It follows that the general price index, expressed in terms of dollar, is a composite of the sectoral price indices

$$P_t = \left[\sum_{j=1}^{K} \upsilon_j \left(\mathcal{E}_{j,t} P_{j,t}\right)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}. \tag{3}$$

Although the general price index and the dollar-sector price index are both expressed in terms of dollar, their values are not necessarily the same. Only in the limiting case when all firms price in dollar, $\upsilon_1 \to 1$, the general price index is identical to the sectoral one, $P_t = P_{1,t}$. We also define the relative price between the price in sector j and the general price level to be

$$\hat{P}_{j,t} = \frac{\mathcal{E}_{j,t} P_{j,t}}{P_t}, \tag{4}$$

which is an indicator of respective currency's purchasing power, with a higher value corresponding to a weaker purchasing power. From the equation above, a weak purchasing power of currency j is associated with a depreciation of itself, higher sectoral price, or an appreciation and higher sectoral price of a different currency. This can also be interpreted as the real effective exchange rate of currency j.

2.2 Households

A representative household's life-time utility is a discounted flow of period utility function of consumption bundle, C_t, real money balances or liquidity, L_t, and labour supply N_t, subject to an exogenous preference shock Z_t:

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbb{U}\left(C_t, L_t, N_t\right) Z_t \tag{5}$$

where E_t is the expectation operator and $\beta < 1$ is the discount factor. The period utility function is

$$\mathbb{U}\left(C_t, L_t, N_t\right) = \begin{cases} \frac{C_t^{1-\sigma}-1}{1-\sigma} + \frac{L_t^{1-\xi}-1}{1-\xi} - \frac{N_t^{1+\varphi}}{1+\varphi} & \text{if} \quad \sigma \neq 1 \\ \log C_t + \frac{L_t^{1-\xi}-1}{1-\xi} - \frac{N_t^{1+\varphi}}{1+\varphi} & \text{if} \quad \sigma = 1 \end{cases} \tag{6}$$

The consumption bundle is an aggregation of consumption goods priced in all currencies given by $C_t \equiv \left[\sum_{j=1}^{K} \int_{\upsilon_j(t)} C_{j,t}(i)^{1-\frac{1}{\epsilon}} di\right]^{\frac{\epsilon}{\epsilon-1}}$, where $C_{j,t}(i)$ is variety i priced in currency j of which the demand function is given by

$$C_{j,t}(i) = \left[\frac{\mathcal{E}_{j,t}P_{j,t}(i)}{P_t}\right]^{-\epsilon} C_t \quad \text{for} \quad i \in \upsilon_j(t) \tag{7}$$

The liquidity is the sum of real money balances in a total of K parallel currencies which are perfect substitutes in providing liquidity services:

$$L_t \equiv \sum_{j=1}^{K} L_{j,t} \quad \text{where} \quad L_{j,t} \equiv \frac{\mathcal{E}_{j,t}M_{j,t}}{P_t} \tag{8}$$

The lifetime utility in Eq. (5) is maximised subject to the period budget constraint

$$C_t + B_t + \sum_{j=1}^{K} L_{j,t} = \frac{\exp\left(i_{t-1}\right)}{\Pi_t} B_{t-1} + \sum_{j=1}^{K} \frac{L_{j,t-1}}{\Pi_t}\frac{\mathcal{E}_{j,t}}{\mathcal{E}_{j,t-1}} + W_t N_t + \Gamma_t \tag{9}$$

where B_t is the holding of real government bonds at the end of period t, i_t is the nominal return from bond holdings in terms of dollar, W_t is the real wage rate, and Γ_t represents the real dividends from the firms. $\Pi_t \equiv \frac{P_t}{P_{t-1}}$ is the general price inflation.

2.3 Firms

Each firm i in the unit interval produces a differentiated good. The production function of a firm that prices in currency j has the following form

$$Y_{j,t}(i) = A_t N_{j,t}(i)^{1-\alpha} \tag{10}$$

where A_t is an exogenous level of technology common all firms.

The price-setting process follows Calvo (1983). In each period, each firm in set $\upsilon_j(t)$ resets its price in currency j with probability $1-\theta_j$. Given an opportunity to reset its price, a firm sets a new optimal price in currency j denoted by $P^*_{j,t}$. This optimal price solves the following profit maximising problem:

$$\max_{P^*_{j,t}} \sum_{\ell=0}^{\infty} \theta_j^{\ell} \mathrm{E}_t \left[Q_{t,t+\ell} \left[\frac{\mathcal{E}_{j,t+\ell}P^*_{j,t}}{P_{t+k}} Y_{j,t+\ell|t} - \Psi_{t+\ell}\left(Y_{j,t+\ell|t}\right) \right] \right] \tag{11}$$

subject to the demand function

$$Y_{j,t+\ell|t} = \left(\frac{\mathcal{E}_{j,t+\ell}P^*_{j,t}}{P_{t+\ell}}\right)^{-\epsilon} Y_{t+\ell} \tag{12}$$

where $Q_{t,t+\ell}$ is a stochastic discount factor, and $\Psi_{t+\ell}\left(\cdot\right)$ is the real total cost of production.

2.4 Equilibrium

In equilibrium, both the goods and labour markets clear. All goods produced by firm i are consumed by the households:

$$Y_{j,t}\left(i\right) = C_{j,t}(i) \tag{13}$$

Define the aggregate output $Y_t = \left(\sum_{j=1}^{K} \int_{\upsilon_j(t)} Y_{j,t}(i)^{1-\frac{1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}$. Then the aggregate output and consumption are equal:

$$Y_t = C_t. \tag{14}$$

The labour market clearing condition leads to the following

$$N_t = \int_0^1 N_t(i) di = \left(\frac{Y_t\, D_t}{A_t}\right)^{\frac{1}{1-\alpha}} \tag{15}$$

where $D_t \equiv \left[\sum_{j=1}^{K} \int_{\upsilon_j(t)} \left(\frac{\mathcal{E}_{j,t} P_{j,t}(i)}{P_t}\right)^{-\frac{\epsilon}{1-\alpha}} di\right]^{1-\alpha}$ is known as the price dispersion. The monetary policy is the conventional Taylor rule:

$$\exp(i_t) = \exp(i)\Pi_t^{\phi_\pi}\left(\frac{Y_t}{Y_t^n}\right)^{\phi_y} \tag{16}$$

where Y_t^n is the natural level of aggregate output.

3 Linearised NK framework

The model presented in Section 2 is log-linearised at the first order around the zero-inflation steady state. In this section, we present the key equations that consist the linear NK framework with K parallel currencies. Unless otherwise stated, lower cases are used to denote the deviations from the logarithmic steady states of the upper-cased variables. Detailed derivations are shown in Appendix A.

3.1 Exchange-rate dynamics

Proposition 1. *The nominal exchange rate between any pair of parallel currencies j and j' follows a random-walk process:*

$$e_{j,t} - e_{j',t} = \mathrm{E}_t\left[e_{j,t+1} - e_{j',t+1}\right]. \tag{17}$$

Proof. See Appendix B.1. □

Proposition 1 generalises the fundamental equation presented in Schilling and Uhlig (2019) to a context with any number of currencies. In a particular case between any currency j and dollar, the price of currency j in terms of dollar is given by:

$$e_{j,t} = \mathrm{E}_t \left[e_{j,t+1} \right]. \tag{18}$$

3.2 Sectoral price dynamics and NKPC's

In Appendix A, we show that, when prices are flexible in sector j, $\theta_j \to 0$, firms set prices to the desired levels given by:

$$\tilde{p}_{j,t} = \Theta \, \mathrm{mc}_t + p_t - e_{j,t}, \tag{19}$$

where $\Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\epsilon}$. Use $\tilde{p}_t$ to represent the components independent of currency choice, $\tilde{p}_t \equiv \Theta \, \mathrm{mc}_t + p_t$. We learn that the desired prices for all currency sectors are the same, when they are denominated in dollar:

$$\tilde{p}_{j,t} + e_{j,t} = \tilde{p}_t \quad \text{for all} \quad j = 1, ..., K \tag{20}$$

When prices are rigid, from the first-order condition of the firm's profit-maximising problem, the optimal price set by a firm pricing in currency j is given by the following forward-looking function of future desired price:

$$p^*_{j,t} = (1 - \beta\theta_j) \sum_{\ell=0}^{\infty} (\beta\theta_j)^{\ell} \, \mathrm{E}_t \left[\tilde{p}_{j,t+\ell} \right] \tag{21}$$

Using the fact that the sectoral price level is the weighted average between its past value and the optimal price, $p_{j,t} = \theta \, p_{j,t-1} + (1-\theta) \, p^*_{j,t}$, one can express the sectoral inflation in terms of its expected value one period ahead, and a markup over the desired price:

$$\pi_{j,t} = \beta \, \mathrm{E}_t \left[\pi_{j,t+1} \right] - \lambda_j \, (p_{j,t} - \tilde{p}_{j,t}) \tag{22}$$

where $\lambda_j \equiv \frac{(1-\theta_j)(1-\beta\theta_j)}{\theta_j}$. Substitute Eq. (19), the market clearing condition $y_t = c_t$, and households' optimality condition into the markup in Eq. (22). One can derive that

$$p_{j,t} - \tilde{p}_{j,t} = \hat{p}_{j,t} - \Theta \left[\left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right) y_t - \frac{\varphi + 1}{1 - \alpha} a_t \right] \tag{23}$$

where the relative price of sector j, defined as $\hat{p}_{j,t} \equiv p_{j,t} + e_{j,t} - p_t$, has the following law of motion:

$$\hat{p}_{j,t} = \hat{p}_{j,t-1} + \pi_{j,t} + \Delta e_{j,t} - \pi_t \tag{24}$$

From Eq. (23), the price markup can be expressed in terms of output gap and relative price. When prices are flexible in all sectors, all firms price at the same desired level. In this case, $p_{j,t} = \tilde{p}_{j,t}$, $\hat{p}_{j,t} = 0$, and the aggregate output is at its natural level y_t^n. The price markup equation becomes:

$$0 = -\Theta \left[\left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right) y_t^n - \frac{\varphi + 1}{1 - \alpha} a_t \right] \tag{25}$$

Taking the difference between Eqs. (23) and (25) provides an expression for the markup in terms of the sector's relative price and the output gap:

$$p_{j,t} - \tilde{p}_{j,t} = \hat{p}_{j,t} - \Theta \left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right) \tilde{y}_t \tag{26}$$

The NKPC for sector j is then an equation of its expected value for the next period, the output gap, and the sectoral relative price.

$$\pi_{j,t} = \beta \, \mathrm{E}_t \left[\pi_{j,t+1} \right] + \kappa_j \tilde{y}_t - \lambda_j \, \hat{p}_{j,t} \tag{27}$$

where $\kappa_j \equiv \lambda_j \, \Theta \left(\sigma + \frac{\varphi + \alpha}{1 - \alpha} \right)$. Two distinctive features are worth to be noticed. Firstly, the sectoral inflation is influenced by the aggregate output gap, not just the sectoral one. Secondly, there is an additional term on the relative price in the sectoral NKPC. Both features are signs of network effects across currency sectors.

3.3 Output-gap dynamics and the dynamic IS curve

The dynamic IS equation is the same as one in a standard NK framework which is derived from households' Euler equation and the market clearing condition:

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] - \frac{1}{\sigma} \left[\hat{i}_t - \mathrm{E}_t[\pi_{t+1}] - \hat{r}_t^n \right], \tag{28}$$

where $\hat{i}_t$ is the deviation of nominal interest rate from its steady state, and $\hat{r}^n_t$ is the natural rate of interest which is a linear combination of productivity and preference shocks

$$\hat{r}^n_t = -\sigma\left(1-\rho_a\right)\psi_{ya}a_t + \left(1-\rho_z\right)z_t. \tag{29}$$

The real interest rate is defined as the nominal interest rate adjusted for expected inflation:

$$\hat{r}_t = \hat{i}_t - \mathrm{E}_t[\pi_{t+1}]. \tag{30}$$

From the demand equations for sectoral consumption goods, the sectoral output is given by $y_{j,t} = -\epsilon\,\hat{p}_{j,t} + y_t$. With flexible prices in all sectors, $y^n_{j,t} = y^n_t$. When prices are rigid, the sectoral output gap is given by

$$\tilde{y}_{j,t} = -\epsilon\,\hat{p}_{j,t} + \tilde{y}_t \tag{31}$$

3.4 Key equations

We use bold fonts to symbolise $K \times 1$ vectors of sectoral parameters and variables. In particular,

$$\begin{aligned}
\boldsymbol{\upsilon} &\equiv \left[\upsilon_1, \upsilon_2, ..., \upsilon_K\right]' \\
\boldsymbol{\lambda} &\equiv \left[\lambda_1, \lambda_2, ..., \lambda_K\right]' \\
\boldsymbol{\kappa} &\equiv \left[\kappa_1, \kappa_2, ..., \kappa_K\right]' \\
\boldsymbol{\pi}_t &\equiv \left[\pi_{1,t}, \pi_{2,t}, ..., \pi_{K,t}\right]' \\
\hat{\boldsymbol{p}}_t &\equiv \left[\hat{p}_{1,t}, \hat{p}_{2,t}, ..., \hat{p}_{K,t}\right]' \\
\boldsymbol{\Delta e}_t &\equiv \left[\Delta e_{1,t}, \Delta e_{2,t}, ..., \Delta e_{K,t}\right]'
\end{aligned}$$

From its definition, the general price inflation can be expressed as a vector product between the sizes of the currency sectors and the exchange-rate-adjusted sectoral inflation:

$$\pi_t = \boldsymbol{\upsilon}'\left(\boldsymbol{\pi}_t + \boldsymbol{\Delta e}_t\right). \tag{32}$$

The random-walk process of the nominal exchange rates Eq. (18) implies that the vector of expected value in the next period is $\mathrm{E}_t\left[\boldsymbol{\Delta e}_{t+1}\right] = \mathbf{0}$, so that the one-period-ahead expected inflation is

$$\mathrm{E}_t\left[\pi_{t+1}\right] = \boldsymbol{\upsilon}'\mathrm{E}_t\left[\boldsymbol{\pi}_{t+1}\right]. \tag{33}$$

We can then express the NK framework with K parallel currencies using the following $(2K+2)$-equation system, which includes a standard Taylor rule

$$\tilde{y}_t = \mathrm{E}_t\left[\tilde{y}_{t+1}\right] - \sigma^{-1}\left(\hat{i}_t - \boldsymbol{\upsilon}'\,\mathrm{E}_t\left[\boldsymbol{\pi}_{t+1}\right] - r_t^n\right) \tag{34}$$

$$\boldsymbol{\pi}_t = \beta\,\mathrm{E}_t\left[\boldsymbol{\pi}_{t+1}\right] + \boldsymbol{\kappa}\,\tilde{y}_t - \boldsymbol{\lambda}\circ\hat{\boldsymbol{p}}_t \tag{35}$$

$$\hat{\boldsymbol{p}}_t = \hat{\boldsymbol{p}}_{t-1} + \left(\mathbf{I} - \mathbf{1}\,\boldsymbol{\upsilon}'\right)\left(\boldsymbol{\pi}_t + \Delta\boldsymbol{e}_t\right) \tag{36}$$

$$\hat{i}_t = \phi_\pi\,\boldsymbol{\upsilon}'\boldsymbol{\pi}_t + \phi_y\,\tilde{y}_t \tag{37}$$

where $\circ$ is an operator for element-wise multiplication.

The main difference of this NK framework with K parallel currencies from the one in Galí (2015) lies in that an exchange-rate shock in any sector spills over to the aggregate economy. An unexpected one-off appreciation of currency j leads to a higher price in sector j and a higher general price level. The relative price is higher in sector j, but those in all the other sectors are lower as seen from Eq. (36). Demand for goods sold by sector j is now relatively lower, leading to lower inflation in sector j but higher inflation elsewhere.

In addition, Eq. (36) depicts that despite the exchange-rate shock being one-off, its effects can persist. Due to the infrequent price adjustments, in each period, only a fraction of the firms are able to optimise their prices in response to the exchange-rate shock. The relative prices therefore persist until all firms reset their prices.

These key equations are similar to the NK framework with production network presented in Cienfuegos (2019). In particular, the relative prices are state variables of the economy. Although we do not discuss the production network in this paper, an exchange-rate shock arising from any non-dollar currency also leads to a network effect on the relative prices of all other sectors.

From Eqs. (32) and (35), we derive the generalised aggregate inflation:

$$\pi_t = \beta\,\mathrm{E}_t\left[\pi_{t+1}\right] + \boldsymbol{\upsilon}'\boldsymbol{\kappa}\,\tilde{y}_t - \boldsymbol{\upsilon}'\left(\boldsymbol{\lambda}\circ\hat{\boldsymbol{p}}_t\right) + \boldsymbol{\upsilon}'\Delta\boldsymbol{e}_t \tag{38}$$

Both the sectoral relative prices and the nominal exchange rates influence the aggregate inflation. The extents of such influences are contingent on the sizes of the respective currency sectors.

4 Baseline cases

Two baseline cases are worth discussing. The first one is when price rigidity is homogeneous across all currency sectors. The second one is when prices are flexible in one currency sector while being rigid in the others.

4.1 Homogeneous price rigidity

We summarise the bilateral associations in an economy with homogeneous price rigidity with the following proposition.

Proposition 2. *Between any two sectors j and j' with homogeneous price rigidity θ,*

1. *the optimal prices in both sectors are equivalent, $p^*_{j,t} + e_{j,t} = p^*_{j',t} + e_{j',t}$;*
2. *the bilateral relative price is an autoregressive process, $s_{jj',t} = \theta\,(s_{jj',t-1} + \Delta e_{j,t} - \Delta e_{j',t})$;*
3. *the inflation differential is linear in bilateral relative price, $\pi_{j,t} - \pi_{j',t} = -\frac{1-\theta}{\theta}\, s_{jj',t}$;*
4. *the output-gap differential is linear in bilateral relative price, $\tilde{y}_{j,t} - \tilde{y}_{j',t} = -\epsilon\, s_{jj',t}$.*

Proof. See Appendix B.2. □

The first result is parallel to Proposition 1 of Gopinath et al. (2010), which states that local currency pricing and producer currency pricing are equivalent. We argue here that while different choices of pricing currencies end up with equivalent price in the baseline case of homogeneous price rigidity, this result may not hold when price rigidities differ across the currency sectors, except when the expected desired price level is constant.

The second to the fourth results states that the bilateral relative price is a state variable for inter-sector differentials in inflation and output gaps. These differentials are regardless of the choice of monetary policy, as no assumption on the monetary policy is needed to arrive at these conclusions. Instead, price rigidity is the only characteristic of the economy that causes the different inflation dynamics between the currency sectors. The output gap differential is influenced by elasticity of substitution, in addition to price rigidity. From the negative signs, the sector that experiences a currency appreciation always produces less output and has lower inflation, as compared to a sector with no currency appreciation.

Two scenarios are relevant to Proposition 2. The first scenario is when currency j experiences an appreciation, while currency j' does not. In the period of an unexpected appreciation of currency j, the dollar-denominated prices of sector-j products deviate above their desired levels. The price in sector j, relative to that in sector j', becomes higher. Demands for the sectoral goods change as they are sensitive to the relative prices. The second scenario is when both currencies j and j' do not experience an appreciation. Suppose that the exchange-rate shock does not arise from either of the two currencies, but from a third currency. The inflation and output-gap dynamics are identical between sectors j and j'.

Extending the assumption of homogeneous price rigidity to the aggregate economy, we arrive at an NKPC that is similar to one in Galí (2015). We present this NKPC in the following proposition.

Proposition 3. *The new Keynesian Philips curve for aggregate inflation is independent of the relative price dynamics if price rigidity is homogeneous across all currency sectors:*

$$\pi_t = \beta\,\mathrm{E}_t\left[\pi_{t+1}\right] + \kappa\,\tilde{y}_t + \boldsymbol{v}'\Delta\boldsymbol{e}_t \tag{39}$$

Proof. See Appendix B.3. □

Proposition 3 posits that when price rigidity is homogeneous, the net effect of sectoral prices on the aggregate inflation is zero.

4.2 Dynamics in a flexible sector

The second baseline case we analyse considers a currency sector with flexible prices among other currency sectors with the same degree of price rigidity. It differs from the case when prices are flexible across all currency sectors.

Proposition 4. *An exchange-rate shock to any non-dollar currency j does not spillover to the other currency sectors if prices are flexible in sector j.*

Proof. See Appendix B.4. □

In other words, Proposition 4 states that an exchange-rate shock leads to economy-wide responses as long as it arises from a sector with sticky prices. This proposition considers the effects of an exchange-rate shock on a sector with flexible prices according to the sources of the shock. When the shock originates from the sector with flexible prices, the price tends to deviate from the steady state, which is the desired price. Firms in this sector adjust their prices so that the effect of exchange-rate shock is offset. As a result, the dollar-denominated price remains the same as before the exchange-rate shock. Hence, the relative price is unchanged. There is no change in macroeconomic dynamics. The exchange-rate shock is therefore neutral.

However, the flexible sector does not remain unchanged when there is an exchange-rate shock from another sector with price rigidity. Due to the infrequent adjustment of prices, there are changes in aggregate price level and output gap as firms adjust their prices towards the desired level. As a result, the desired price varies. Since firms in this sector always price at the desired level, the inflation and output gap vary, following the changes in desired price.

5 Equilibrium dynamics under Taylor rules

With the linear NK framework laid out in the previous section, we can analyse the economic dynamics. Our analyses involve the responses of output gaps and inflation to an unexpected exchange-rate shock under three variants of the well-known Taylor rule. The first variant, which is also the baseline rule, entails a nominal interest rate responding to Aggregate Inflation and Aggregate Output gap (AIAO). We find that this NK framework differs from one in standard literature only in the existence of an exchange-rate disturbance in the NKPC, under the assumption of homogeneous price rigidity. In the second variant, the nominal interest rate responds to the Dollar-sector Inflation and Aggregate Output gap (DIAO), assuming that the central bank may only be able to observe the inflation in the dollar sector. The last variant features a nominal interest rate responding to the Dollar-sector Inflation and Dollar-sector Output (DIDO). This assumes that the central bank only responds to economic dynamics in the dollar sector.

From the central bank's perspective, the macroeconomic consequences of its monetary policy consist of reactions in both the dollar sector and a collection of the non-dollar sectors. For the convenience of policy evaluation, we consider a two-sector economy, $K = 2$. As in the previous section, firms in sector 1 price in dollar, and firms in sector 2 price in a non-dollar currency. Regardless of the number of currencies, the currency in sector 2 is interpreted as a numeraire currency of all non-dollar currencies. This two-sector setup allows us to examine how the dollar and non-dollar sectors interact with alternative monetary policy rules.

The notations in a two-sector environment can be simplified as follows. Let the size of the non-dollar sector be $\upsilon_2 = \upsilon$, then the size of the dollar sector is $\upsilon_1 = 1 - \upsilon$. The nominal exchange rate of dollar is normalised to $e_{1,t} = 0$, and we let the nominal exchange rate of the alternative currency be $e_{2,t} = e_t$. It is also convenient to drop the currency indices in the bilateral relative price of the two sectors so that $s_t \equiv \hat{p}_{2,t} - \hat{p}_{1,t}$. From Proposition 2, the law of motion of the bilateral relative price resembles an autoregressive process

$$s_t = \theta \left(s_{t-1} + \Delta e_t \right) \tag{40}$$

where $\Delta e_t \sim N(0, \sigma^2_{\Delta e})$ is the residual of the random-walk process in Proposition 1. It follows from the definition of general price index, $(1 - \upsilon)\, \hat{p}_{1,t} + \upsilon\, \hat{p}_{2,t} = 0$, that the relative prices in the respective sectors can be expressed in terms of s_t, so $\hat{p}_{1,t} = -\upsilon s_t$ and $\hat{p}_{2,t} = (1 - \upsilon)\, s_t$. With homogeneous price rigidity across the two sectors, $\theta_1 = \theta_2 = \theta$, the subscripts of the parameters κ_1, κ_2, λ_1, and λ_2 can be dropped.

The sectoral NKPC's are expressed in terms of the output gap and the bilateral relative price as:

$$\pi_{1,t} = \beta \, \mathrm{E}_t[\pi_{1,t+1}] + \kappa \, \tilde{y}_t + \lambda \, \upsilon \, s_t \tag{41}$$

$$\pi_{2,t} = \beta \, \mathrm{E}_t[\pi_{2,t+1}] + \kappa \, \tilde{y}_t - \lambda \, (1 - \upsilon) \, s_t \tag{42}$$

The model parameters are summarised in Table 1. Most of the parameters follow Galí (2015). The size of the non-dollar sector, υ, and the standard deviation of the exchange-rate shock, $\sigma_{\Delta e}$, are new. We let the non-dollar sector to take 20% of the market. The size of the exchange-rate shock is comparable to a monetary policy shock in Galí (2015) at 0.25%.

5.1 Aggregate inflation, aggregate output gap

Using Proposition 3, the linearised NK framework describing the dynamics of aggregate output gap and inflation, including the baseline Taylor rule, condenses to a three-equation system as follows

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] - \sigma^{-1} \left(\hat{i}_t - \mathrm{E}_t[\pi_{t+1}] - \hat{r}^n_t \right) \tag{43}$$

$$\pi_t = \beta \, \mathrm{E}_t[\pi_{t+1}] + \kappa \tilde{y}_t + \upsilon \, \Delta e_t \tag{44}$$

$$\hat{i}_t = \phi_\pi \, \pi_t + \phi_y \, \tilde{y}_t \tag{AIAO}$$

The above equations differ from a standard NK framework only in the exchange-rate shock in the NKPC. The additional term in the NKPC here implies that the path of aggregate inflation is influenced by both the size of the non-dollar sector, and the standard deviation of the exchange-rate shock. Since the nominal interest rate responds to the aggregate inflation and the aggregate output gap, the exchange-rate shock therefore has a direct impact on the nominal interest rate.

As all exogenous shocks are assumed to be uncorrelated, the economic dynamics under a productivity shock and a preference shock are identical to those in Galí (2015). We therefore focus only on the exchange-rate shock. The natural rate of interest in the dynamic IS curve vanishes as productivity and preference shocks are not in place. In Fig. 1, the impulse responses to a *25 basis-point nominal appreciation in the non-dollar currency* are presented. We discuss the impulse responses in details next.

5.1.1 Aggregate dynamics

The three plots in the first row of Fig. 1 depict that the aggregate output gap, the aggregate inflation, and the nominal interest rate return to their steady states immediately after the period of the exchange-rate shock. To see the analytical solution, substitute the Taylor rule Eq. (AIAO) into the dynamic IS curve, and use the method of undetermined coefficients to solve for the paths of the aggregate inflation and the aggregate output gap. The impulse responses as functions of the exchange-rate shock are:

$$\tilde{y}_t = -\upsilon\, \phi_\pi\, \Omega\, \Delta e_t \tag{45}$$

$$\pi_t = \upsilon\, (\sigma + \phi_y)\, \Omega\, \Delta e_t, \tag{46}$$

where $\Omega \equiv \frac{1}{\sigma+\phi_y+\kappa\phi_\pi} > 0$. As the equations put forward, an unexpected appreciation of the non-dollar currency, represented by a positive exchange-rate shock, unambiguously induces a negative response in the output gap, and a positive response in the aggregate inflation. This is because the exchange-rate shock leads to a higher consumer price and hence lower aggregate demand. The resulting effect on the monetary policy is a contractionary one, as seen from the positive coefficient in the nominal interest rate:

$$\hat{i}_t = \upsilon\, \sigma\, \phi_\pi\, \Omega\, \Delta e_t. \tag{47}$$

The random-walk process of the nominal exchange rate means that $\mathrm{E}_t[\pi_{t+1}] = 0$, and hence the real interest rate coincides with the nominal interest rate.

Note that the responses of the aggregate variables are proportional to the size of the non-linear sector. When few firms opt to price in the non-dollar currency, $\upsilon \to 0$, an unexpected exchange-rate shock has negligible influence on the aggregate economic dynamics. In addition, the fact that the coefficient Ω increases in θ means that higher responsiveness of the aggregate variables to an exchange-rate shock can be associated with higher price rigidity in the economy.

5.1.2 Sectoral dynamics

We now zoom in to sectoral dynamics so as to analyse the interactions between the dollar and the non-dollar sectors, as shown in the second and third rows of Fig. 1. From the impulse responses, there are distributional effects between the two sectors. The output in the dollar sector is above the natural level with higher inflation, while the output in the non-dollar sector is below the natural level with lower inflation.

Proposition 2 has shown that the bilateral differentials in sectoral prices, inflation, and output gaps are results of price rigidity and imperfect substitution among the goods. The inflation and output-gap differentials are both linear in the bilateral relative price:

$$\pi_{2,t} - \pi_{1,t} = -\frac{1-\theta}{\theta} s_t. \tag{48}$$

$$\tilde{y}_{2,t} - \tilde{y}_{1,t} = -\epsilon\, s_t. \tag{49}$$

The negative signs in Eqs. (48) and (49) imply that both the output gap and inflation are higher in the dollar sector compared to those in the non-dollar sector, given an appreciation of the non-dollar currency.

Substituting the aggregate output gap into the sectoral output gap functions Eq. (31) gives the following dynamics of sectoral output gaps:

$$\tilde{y}_{1,t} = \upsilon\, \epsilon\, s_t - \upsilon\, \phi_\pi\, \Omega\, \Delta e_t \tag{50}$$

$$\tilde{y}_{2,t} = -\,(1-\upsilon)\; \epsilon\, s_t - \upsilon\, \phi_\pi\, \Omega\, \Delta e_t \tag{51}$$

The first components of the sectoral output gaps are the substitution effects arising from a change in the relative sectoral price. The second components are the income effects due to the lower aggregate output, seen in Eq. (45). The contemporaneous responses to an unexpected appreciation in the non-dollar currency are net outcomes of both the income effect and the substitution effect. The income effect drives the sectoral outputs below their natural level. The substitution effect, on the other hand, causes the households to consume relatively more goods from the dollar sector and less from the non-dollar sector. Both effects result in lower output in the non-dollar sector, but in the dollar sector, the resulting output depends on which effect is dominant. Substitute Eq. (40) into Eq. (50) and combine the coefficients of nominal exchange-rate shock. The dollar-sector firms produce above the natural level of output during the period of exchange-rate shock if:

$$\frac{1}{\epsilon\,\theta} - \kappa < \frac{\sigma + \phi_y}{\phi_\pi} \tag{52}$$

in which case the substitution effect dominates the income effect. The right-hand side is an indicator of the nominal interest rate's responsiveness to aggregate inflation and output gap. A greater value on the right-hand side is due to a greater response to aggregate output gap and / or a smaller response to aggregate inflation. Given that the output gap response is negative and the inflation response is positive to an exchange-rate shock from Eqs. (45) and (46), the interest rate declines more when the right-hand-side value is larger, ceteris paribus.

The inequality also implies that an unexpected exchange-rate shock may cause the dollar-sector firms to produce below the natural level when either the elasticity of substitution among the consumer goods is sufficiently low (ϵ is sufficiently small), or the interest rate is sufficiently responsive to aggregate inflation (ϕ_π is sufficiently large), and the interest rate is not responsive enough to the output gap (ϕ_y is sufficiently small). In the first instance the small substitution elasticity limits households' willingness to consume more dollar-sector goods and less non-dollar-sector goods. In the second instance, the decline in nominal interest rate is small, limiting the income effect of the exchange-rate shock. The degree of price rigidity, however, has no clear influence on the direction of response.

In the periods following the exchange-rate shock, the dollar-sector output is persistently above the natural level, while the non-dollar-sector output is persistently below it. This is because the exchange-rate shock no longer causes a change in the aggregate inflation. The income effect is not present after the period of shock. The national income is back at its natural level. Due to price rigidity, the substitution effect, however, remains until the bilateral relative price returns to its steady state. Because the income effect is only one-off, it is expected that there are kinks in the impulse responses in the second period.

Using the method of undetermined coefficients, one can solve the sectoral inflation in terms of the exchange-rate shock and the relative price as:

$$\pi_{1,t} = \frac{\upsilon\,(1-\theta)}{\theta} s_t - \upsilon\,\kappa\,\phi_\pi\,\Omega\,\Delta e_t \tag{53}$$

$$\pi_{2,t} = -\frac{(1-\upsilon)\,(1-\theta)}{\theta} s_t - \upsilon\,\kappa\,\phi_\pi\,\Omega\,\Delta e_t \tag{54}$$

As in the case of output responses, the exchange-rate shock causes aggregate demand to be lower, and hence a downward pressure on the inflation. The substitution effect results in a demand-pulled inflation in the dollar sector. Firms adjust their price when opportunities arise. The rate at which the inflation changes is subject to the degree of price rigidity. Substitute Eq. (40) into Eq. (53) and combine the coefficients of the exchange-rate shock. The inflation in the dollar sector during the period of exchange-rate shock is higher if

$$(1-\beta\,\theta)\;\Theta\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) < \frac{\sigma+\phi_y}{\phi_\pi}. \tag{55}$$

This inequality also implies that, in the period of an exchange-rate shock, the dollar sector may respond with lower inflation when either the degree of price rigidity is sufficiently low (θ is sufficiently small), or the interest rate is sufficiently responsive to aggregate inflation (ϕ_π is sufficiently large), and less responsive to aggregate output gap (ϕ_y is sufficiently small). In the non-dollar sector, the inflation is always below the steady state, as firms need to offset the price hike due to the exchange-rate shock. In the periods after the exchange-rate shock, the inflation dynamics are linear in the bilateral relative price.

It is of policy interest to find from Eqs. (50) and (53) that the dollar-sector variables are proportional to the size of the non-dollar sector. This is similar to the aggregate variables. In the case when no firm prices in the non-dollar currency, the dollar-sector variables do not respond to an exchange-rate shock.

5.2 Alternative simple rules

In the next two variants of Taylor rule, the nominal interest rate responds to the dollar-sector inflation instead of the aggregate one. Under such monetary policy rules, the exchange-rate shock does not influence the nominal interest rate directly, but via the bilateral relative price which causes changes in the dollar-sector inflation. The NK framework can be rearranged to a three-equation system with the aggregate output gap, the dollar-sector inflation, the nominal interest rate, and the bilateral relative price as the endogenous variables. The non-policy block consists of the following dynamic IS curve and dollar-sector NKPC, in addition to Eq. (40):

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] - \sigma^{-1} \left[\hat{i}_t - \mathrm{E}_t[\pi_{1,t+1}] + \upsilon\,(1-\theta)\,s_t - \hat{r}^n_t\right] \tag{56}$$

$$\pi_{1,t} = \beta\,\mathrm{E}_t[\pi_{1,t+1}] + \kappa\,\tilde{y}_t + \lambda\,\upsilon\,s_t. \tag{57}$$

We consider the following two variants of the Taylor rule:

$$\hat{i}_t = \phi_\pi\,\pi_{1,t} + \phi_y\,\tilde{y}_t \tag{DIAO}$$

$$\hat{i}_t = \phi_\pi\,\pi_{1,t} + \phi_y\,(\tilde{y}_t + \epsilon\,\upsilon\,s_t) \tag{DIDO}$$

Under both regimes, the nominal interest rate responds to the dollar-sector inflation. The difference lies in the output gaps that enter the policy rule. In DIDO, the nominal interest rate responds to the dollar-sector output gap.

In Fig. 2, we show the impulse responses of the macroeconomic variables to a 25 basis-point nominal appreciation in the non-dollar currency under the DIAO and DIDO regimes. Both regimes behave similarly when responding to an exchange rate shock, with the impulse response curves of DIDO below those of DIAO, except for the real interest rate. To keep the paper concise, we discuss the impulse responses of DIAO in the main body of the paper, and provide the equations of DIDO impulse response functions in Appendix C.3.

The equation system formed from Eqs. (40), (56), (57) and (DIAO) can be interpreted as one with $\tilde{y}_t$, $\pi_{1,t}$, and $\hat{i}_t$ being the endogenous variables, and s_t being an autoregressive exogenous process. It is then straightforward to express the endogenous variables in terms of the bilateral relative price:

$$\tilde{y}_t = -\lambda\,\upsilon\,\phi_\pi\,\Lambda\,s_t \tag{58}$$

$$\pi_{1,t} = \frac{\upsilon\,(1-\theta)}{\theta}\,(1-\kappa\,\phi_\pi\,\Lambda)\,s_t \tag{59}$$

$$\hat{i}_t = -\upsilon\,(\kappa - \lambda\,\sigma)\,(1-\theta)\,\phi_\pi\,\Lambda\,s_t \tag{60}$$

where $\Lambda \equiv \frac{1}{(1-\beta\,\theta)[\sigma\,(1-\theta)+\phi_y]+\kappa\,(\phi_\pi-\theta)} > 0$. Contrary to the baseline case, the one-off exchange-rate shock translates into a persistent shock to bilateral relative price. The variables now behave differently.

The aggregate output in Fig. 2 responds with a level below the natural one, and returns to its steady state gradually. The direction of response is the same as in the baseline case shown in Fig. 1. The gradual decay of the impulse is because the nominal interest rate is a function of the sectoral inflation which returns to its steady state only when the bilateral relative price does. The aggregate output gap, which is sensitive to nominal interest rate changes, also follows the behaviour of the bilateral relative price.

The directions of responses of the dollar-sector inflation and the nominal interest rate are no longer unambiguous. They depend on the parameters. In particular, the nominal interest rate decreases when $\kappa\,\sigma^{-1} > \lambda$, also elaborated as

$$\sigma^{-1} > \frac{1-\Theta}{\Theta}\frac{1-\alpha}{\alpha+\varphi} \tag{61}$$

which refers to a sufficiently large elasticity of output gap to real interest rate (sufficiently small σ). This condition always holds true in the particular case of a constant return to scale, $\alpha = 0$. The dollar-sector inflation is lower if

$$\frac{(\kappa-\lambda\,\sigma)\,\theta}{1-\beta\,\theta} > \phi_y \tag{62}$$

which holds true when $\kappa\,\sigma^{-1} > \lambda$, and when either prices are sufficiently rigid (sufficiently large θ), or ϕ_y is sufficiently small. Note that when $\kappa\,\sigma^{-1} < \lambda$, both the nominal interest rate and the dollar-sector inflation increase. From the dynamic IS curve, the real interest rate is found to increase unambiguously, despite the uncertain response of the nominal interest rate:

$$\hat{r}_t = \upsilon\,\lambda\,\sigma\,\left(1-\theta\right)\,\phi_\pi\,\Lambda\,s_t > 0. \tag{63}$$

Inflation in the non-dollar sector is derived by adding the inflation differential Eq. (48) to the dollar-sector inflation

$$\pi_{2,t} = -\frac{1-\theta}{\theta}\left(1-\upsilon+\upsilon\,\kappa\,\phi_\pi\,\Lambda\right)\,s_t \tag{64}$$

which is always below the steady state. The sectoral output gaps are derived from the demand functions

$$\tilde{y}_{1,t} = -\upsilon\,(\lambda\,\phi_\pi\,\Lambda - \epsilon)\;s_t \tag{65}$$

$$\tilde{y}_{2,t} = -\left[\lambda\,\upsilon\,\phi_\pi\,\Lambda + \epsilon\;(1-\upsilon)\right]\,s_t \tag{66}$$

As in the dollar-sector inflation, the response of the dollar-sector output depends on the parameters. For sufficiently large value of ϵ, firms in the dollar sector produce above the natural level, as households find it easier to substitute one consumption good for another. Firms in the non-dollar sector, instead, always produce below the natural level, as the dollar-denominated price is higher.

Thus far, under the DIAO regime, all the variables presented are only linear in the bilateral relative price. This is due to the choice of the monetary policy where the nominal interest rate responds to variables which are associated with the bilateral relative price. However, an exception is the aggregate inflation which is additionally influenced by the contemporaneous exchange-rate shock:

$$\pi_t = -\frac{1-\theta}{\theta}\,\upsilon\,\kappa\,\phi_\pi\,\Lambda\,s_t + \upsilon\,\Delta e_t \tag{67}$$

The exchange-rate shock offsets some effects of the bilateral relative price. The contemporaneous response of the aggregate inflation is always positive as shown in the appendix. From the second period on, the aggregate inflation is below its steady state, and is linear only in the bilateral relative price. As such, a kink is observed in the response of the aggregate inflation.

In summary, there is one similarity and two differences between economic dynamics under the AIAO regime and the DIAO (or DIDO) regime. The similarity is that economic dynamics in the aggregate economy and the dollar sector are proportional to the size of the non-dollar sector. For DIAO, this is seen from Eqs. (58), (60) and (67) for the aggregate variables, and from Eqs. (59) and (65) for the dollar-sector variables. In the limiting case when no firm prices in the non-dollar currency, it is expected that the economy is not affected by the exchange-rate shock.

The two differences between the regimes are as follows. Firstly, the aggregate output gap and inflation experience one-period volatility under the AIAO regime while exhibiting persistent movements in the DIAO regime. Secondly, the responses of the nominal interest rate and dollar-sector inflation are unambiguous under the AIAO regime, while being dependent on the parameter values under the DIAO regime. Dynamics in the non-dollar sector, on the other hand, have shown similar patterns between the two sectors.

6 Heterogeneous price rigidity

The reality may be more complicated than the case of homogeneous price rigidity, in that the price rigidity may differ across various pricing currencies. For example, goods listed online may be subjected to more frequent price changes as it is less costly for merchants to do so online than at physical boutiques. In this section, we analyse the economic dynamics when the dollar and the non-dollar sectors differ in the extents of price rigidity.

The NK framework with different price rigidities, and with the baseline AIAO monetary policy are summarised by the following five equations:

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] - \sigma^{-1}\left[\hat{i}_t - \mathrm{E}_t\left[(1-\upsilon)\,\pi_{1,t+1} + \upsilon\,\pi_{2,t+1}\right] - \hat{r}_t^n\right] \tag{68}$$

$$\pi_{1,t} = \beta\,\mathrm{E}_t[\pi_{1,t+1}] + \kappa_1\,\tilde{y}_t + \lambda_1\,\upsilon\,s_t \tag{69}$$

$$\pi_{2,t} = \beta\,\mathrm{E}_t[\pi_{2,t+1}] + \kappa_2\,\tilde{y}_t - \lambda_2\,(1-\upsilon)\,s_t \tag{70}$$

$$s_t = s_{t-1} + \pi_{2,t} - \pi_{1,t} + \Delta e_t \tag{71}$$

$$\hat{i}_t = \phi_\pi\,\left[(1-\upsilon)\,\pi_{1,t} + \upsilon\,(\pi_{2,t} + \Delta e_t)\right] + \phi_y\,\tilde{y}_t \tag{72}$$

Notice that the parameters κ_1, κ_2, λ_1, and λ_2 are now with subscripts as the price rigidities differ. This equation system contains the aggregate output gap, the two sectoral inflation, and the nominal interest rate as the endogenous variables, the bilateral relative price as the state variable, and the nominal appreciation of the non-dollar currency as the exogenous variable. Since the price rigidities are different between the two sectors, Proposition 2 no longer holds, meaning the aggregate inflation is not necessarily independent of the bilateral relative price as in our earlier simulations of the AIAO regime.

We first consider a scenario in which the non-dollar sector has a lower price rigidity than the dollar sector. This is done by adjusting the parameter θ_2, so that it corresponds to two price changes per year ($\theta_2 = 0.5$) and four price changes per year ($\theta_2 = 0$), while holding the frequency of price changes in the dollar sector at once a year ($\theta_1 = 0.75$). The impulse responses are shown in Fig. 3. The circle-marked lines are the impulse responses in Fig. 1, when the frequency of price change in the non-dollar sector is the same as that in the dollar sector. As price rigidity reduces in the non-dollar sector, more firms respond to the exchange-rate shock by adjusting their non-dollar prices, offsetting the increase in their dollar-denominated prices. As a result, we see a smaller impact of the exchange-rate shock on the bilateral relative price. This leads to a smaller substitution effect between the consumption goods, and hence smaller responses in most of the variables. Also note that the aggregate output, the aggregate inflation, and the nominal interest rate no longer experience one-off volatility. They take time to return to the steady states. In the limiting case when all firms in the non-dollar sector are able to adjust prices in response to the exchange-rate shock ($\theta_2 = 0$), the change in inflation in the non-dollar sector fully offsets the impact of the exchange-rate shock, leading to no change in their dollar-denominated prices. Therefore, when prices are fully flexible in the non-dollar sector, the exchange-rate shock is neutral, and has no influence on economic dynamics. This result is inline with Proposition 4.

We then examine the economic dynamics when the dollar sector becomes more flexible, as compared to the non-dollar sector. Similar to the above simulation, we vary the price rigidity in the dollar sector while holding the frequency of price change in the non-dollar sector at once a year. Fig. 4 shows the impulse responses when the price rigidity in the dollar sector changes from one change a year to two and four changes a year. When the non-dollar sector is hit by an exchange-rate shock, the rigidity in the non-dollar sector causes the dollar-denominated price of its goods to be higher. The re-distributed demand from the non-dollar sector to the dollar sector motivates the firms in the dollar sector to price higher. When the price rigidity in the dollar sector is lower, more firms increase their prices to meet the increased demand. We therefore observe that the response of the dollar-sector inflation first shows a dip, caused by the one-off exchange-rate shock, then a value in the second period that increases with price flexibility.

It is important to note that even when prices are fully flexible in the dollar sector, the exchange-rate shock is not neutral as in the case of flexible non-dollar price. Again, this is inline with Proposition 4. It is because the dollar-denominated price of the non-dollar sector goods, resulted from the exchange-rate shock, does not coincide with the desired price level. As such, when the dollar-sector firms price their goods at desired levels, there is a price differential between the goods from the two sectors. While firms in the non-dollar sector take time to reset the prices to their desired level, firms in the dollar sector optimise their prices every period. Therefore, we see the non-negligible impulse responses even when prices are fully flexible in the dollar sector.

To obtain a more general picture of economic dynamics at different extents of price rigidity, in Fig. 5, we compute the cumulative impulse responses for aggregate output gap and aggregate inflation over two years (eight quarters). As expected, when prices are flexible in the non-dollar sector ($\theta_2 = 0$), an exchange-rate shock does not cause any movements in the aggregate output gap and the aggregate inflation. However, when prices are also flexible in the dollar sector ($\theta_1 = 0$), the exchange-rate shock is not neutral, unless prices are flexible in the non-dollar sector. Further more, throughout the range of dollar-sector price rigidity, the cumulative impulse responses are generally greater when price rigidity in the non-dollar sector is higher.

In this the previous section, we have shown that the impulse responses to an exchange rate shock for the overall economy and the dollar sector are proportional to the size of the non-dollar sector. To see if this result also holds in an environment of heterogeneous price rigidities, we simulate for different values of υ. The impulse responses are shown in Fig. 6. The price rigidities are set to $\theta_1 = 0.75$ for the dollar sector, and $\theta_2 = 0.5$ for the non-dollar sector. When the size of the non-dollar sector increases from 0.2 to 0.5, the responses of the economy increase accordingly. Whereas, when the size of the non-dollar sector diminishes to 0, there are no responses to the exchange-rate shock. Therefore, we infer that, with heterogeneous price rigidities, the responses of the overall economy and the dollar sector to an exchange-rate shock also vary with the size of the non-dollar sector.

7 Currency choice

We now depart from the assumption that a firm sticks to the same pricing currency all the time. We allow the firms to choose a different pricing currency when they are given the opportunity to reset prices. As a result, a rational firm chooses the currency that provides it with the best utility outcome.

We use a discrete choice model to specify the conditions under which firms choose to price in one currency versus the other. Let $\mathcal{U}_{j,t|t}$ denote the utility of a producer, who resets price in period t.

$$\mathcal{U}_{j,t|t} = \log\left(\gamma_j\, \mathcal{V}_{j,t|t}\right) + \varepsilon_{j,t} \tag{73}$$

where $\mathcal{V}_{j,t|t}$ is the value to the firm upon resetting the price, γ_j is a weighing parameter dependent on the currency choice, and $\varepsilon_{j,t}$ is an idiosyncratic preference shock. The shocks are independent across the parallel currencies with type I extreme value distribution $F\left(\varepsilon_{j,t}\right) = \exp\left(-\exp(-\varepsilon_{j,t})\right)$. Let $\mathcal{V}^*_{j,t|t}$ and $\mathcal{U}^*_{j,t|t}$ be the value and utility from the optimised price in period t. A firm chooses to price in currency j if the utility from the optimised value is at least as high as all the other alternatives:

$$\mathcal{U}^*_{j,t|t} \geq \mathcal{U}^*_{j',t|t} \forall j' = 0, ..., K. \tag{74}$$

which holds when

$$\varepsilon_{j',t} \leq \log\left(\frac{\gamma_j\, \mathcal{V}^*_{j,t|t}}{\gamma_{j'}\, \mathcal{V}^*_{j',t|t}}\right) + \varepsilon_{j,t} \forall j' = 0, ..., K. \tag{75}$$

The joint probability from K currencies gives the probability of a firm pricing in currency j:

$$\Pr_{j,t} = \frac{\gamma_j\, \mathcal{V}^*_{j,t|t}}{\sum_{j'=1}^{K} \gamma_{j'}\, \mathcal{V}^*_{j',t|t}} \tag{76}$$

Following Gopinath et al. (2010), we write the value to the firm in the following recursive form

$$\mathcal{V}_{j,t|t} = \Xi\left(p\,|\,x_t\right) + \theta_j \mathrm{E}_t Q_{t,t+1} \mathcal{V}_{j,t+1|t} + (1-\theta_j)\,\mathrm{E}_t Q_{t,t+1} \mathcal{V}_{t+1} \tag{77}$$

where $\Xi\left(p\,|\,x_t\right)$ is the profit function given the state of the economy x_t. In the period following the price adjustment, the firm continues using the current optimal price with probability θ_j. Otherwise, with probability $(1-\theta_j)$, it gets the opportunity to reset its price with a different currency, and the value to the firm is $\mathcal{V}_{t+1}$, which is the average of optimised values from all currencies:

$$\mathcal{V}_t = \sum_{j=1}^{K} \mathrm{Pr}_{j,t} \mathcal{V}^*_{j,t|t} \tag{78}$$

Iterating the value function gives the following infinite sum:

$$\mathcal{V}^*_{j,t|t} = \mathrm{E}_t \left[\sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell} \Xi\left(p^*_{j,t} + e_{j,t+\ell}\,|\,x_{t+\ell}\right)\right] + \theta_j\left(1-\theta_j\right)\mathrm{E}_t\left[\sum_{\ell'=1}^{\infty} \theta_j^{\ell'} Q_{t,t+\ell'} \mathcal{V}_{t+\ell'}\right] \tag{79}$$

A proposition from the baseline case of homogeneous price rigidity can be summarised as follows:

Proposition 5. *Assume homogeneous price rigidity in a two-sector economy. Let $\gamma_1 = 1$, $\gamma_2 = \gamma$. The second order approximation to the probability of pricing in dollar is*

$$\mathrm{Pr}_{1,t} \approx \frac{1}{1+\gamma} + \frac{\gamma\, K\left(x_t\right)}{\left(1+\gamma\right)^2 \tilde{\mathcal{V}}_t} \sum_{\ell=0}^{\infty} \left(\beta\theta\right)^{\ell} \mathrm{Var}_t\left(e_{t+\ell}\right)\left[\frac{1}{2} - \frac{\mathrm{Cov}_t\left(\tilde{p}_{t+\ell}, e_{t+\ell}\right)}{\mathrm{Var}_t\left(e_{t+\ell}\right)}\right] \tag{80}$$

where $K(x_t) \equiv -\tilde{\Xi}_{pp}(x_t)$, and $\tilde{\mathcal{V}}_t$ is the value function evaluated at the desired price.

Proof. See Appendix B.5. □

The endogenous currency choice leads to a time-varying sizes of the currency sectors. With a time subscript to the parameter υ_j, the law of motion for $\upsilon_{j,t}$ is:

$$\upsilon_{j,t} = \theta_j\, \upsilon_{j,t-1} + \mathrm{Pr}_{j,t} \sum_{j'=1}^{K} \left(1-\theta_{j'}\right)\, \upsilon_{j',t-1} \tag{81}$$

The first term is the proportion of firms that did not change prices from the previous period. The second term is the sum of all firms choosing to price in currency j from all the flexible firms.

In Fig. 7, we show the transition dynamics from the baseline specification to different extents of price rigidities in a two-sector economy. The non-dollar sector increases in size when price rigidity in the dollar sector is lower. This is because with lower price rigidity, firms in the dollar sector switch to pricing in the non-dollar currency more easily. It is the same reason for the non-dollar sector to decrease in size when the price rigidity in the non-dollar sector is lower. In general, when transiting from the baseline specification to one with lower price rigidity, steady states of the key macroeconomic variables are unchanged. However, the variables deviate from the steady states temporarily.

The evolving size of the currency sector is related to our earlier finding of proportional macroeconomic responses to the size of the non-dollar sector under an exchange rate shock. In an environment with endogenous currency choice, the central bank may see changes in the impact of exchange-rate shocks when price rigidity in the dollar or the non-dollar sector changes. In particular, when prices in the dollar sector become less rigid, exchange-rate shocks may pose increasing risk to economic stability when the non-dollar sector grows in size.

8 Conclusion

Extending the basic NK framework in Galí (2015), we examine the macroeconomic dynamics when multiple parallel currencies co-exist in an economy. Our baseline case with homogeneous price rigidity finds an NKPC that is similar to a conventional one, with an additional disturbance term from nominal exchange rate. Upon relaxing the assumption of homogeneous price rigidity, we find that the exchange-rate shock is neutral only when the shock originates from a currency sector with flexible prices. We have also discussed a scenario when firms are able to change the pricing currency. We find that the non-dollar sector may increase in size when prices in the dollar sector are less rigid, posing higher risk to economic stability.

Our analyses are with limitations. The vision that a considerable proportion of firms price in a non-dollar currency may seem futuristic. Existing regulatory frameworks are mostly based on dollar, resulting in majority of firms using dollar as the pricing unit. However, private monies such as Libra and stablecoins have made to the headlines. With increasing globalisation, and the vast initiatives to provide the unbanked and underbanked with affordable payment means, the jury is still out on the demise of private monies. The future that we envision here may have arrived.

The framework presented in this paper is at most a stereotype model. A benefit of this simple model is that it is easy to build on it to analyse more complicated issues, for example, a non-random path of exchange rate, which we shall leave for future research.

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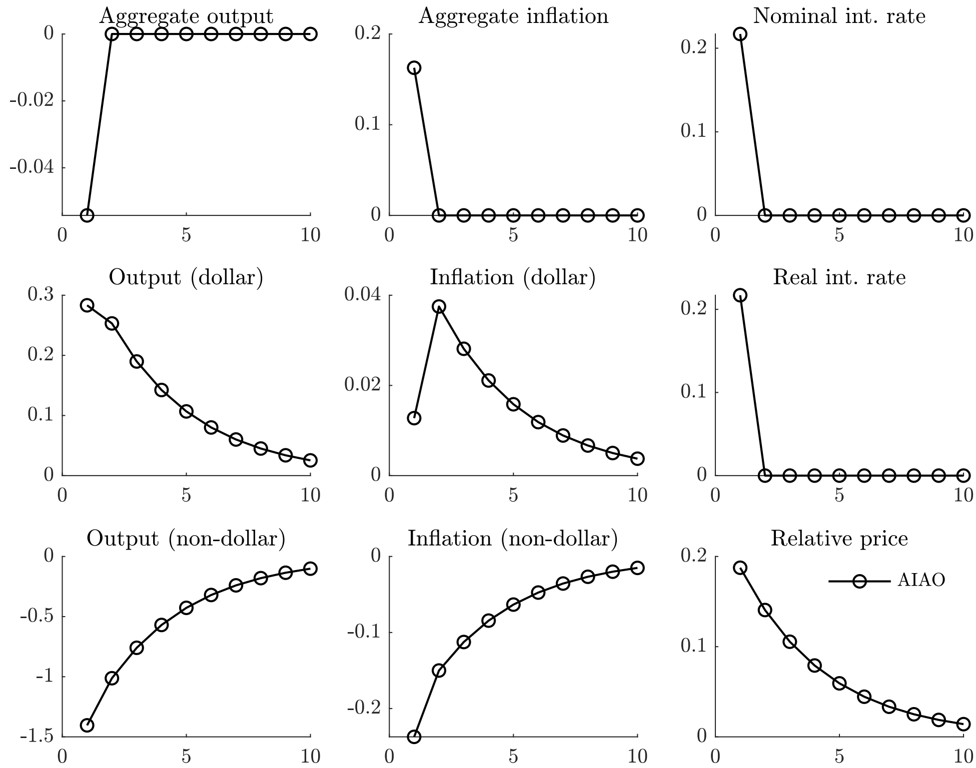

Figure 1: Impulse responses to a 0.25% shock in non-dollar exchange rate. Under the AIAO regime, the nominal interest rate responds to the aggregate inflation and the aggregate output gap. Vertical axes indicate percentage deviations from the steady states. Horizontal axes indicate quarters after the exchange-rate shock.

Table 1: Parameter values in benchmark model.

Parameter	Value	Description
α	0.250	Share of labour input in production function
σ	1.000	Coefficient of risk aversion
φ	5.000	Inverse Frisch elasticity of labour supply
β	0.990	Discount factor
θ_1	0.750	Probability of not adjusting prices in dollar sector
θ_2	0.750	Probability of not adjusting prices in non-dollar sector
ϵ	9.000	Elasticity of substitution among consumption goods
ϕ_π	1.500	Interest-rate reaction to inflation
ϕ_y	0.125	Interest-rate reaction to output gap
υ	0.200	Size of non-dollar sector
$\sigma_{\Delta e}$	0.250	Standard deviation of exchange-rate shock

Note: All parameters, except υ and $\sigma_{\Delta e}$, are obtained from Galí (2015).

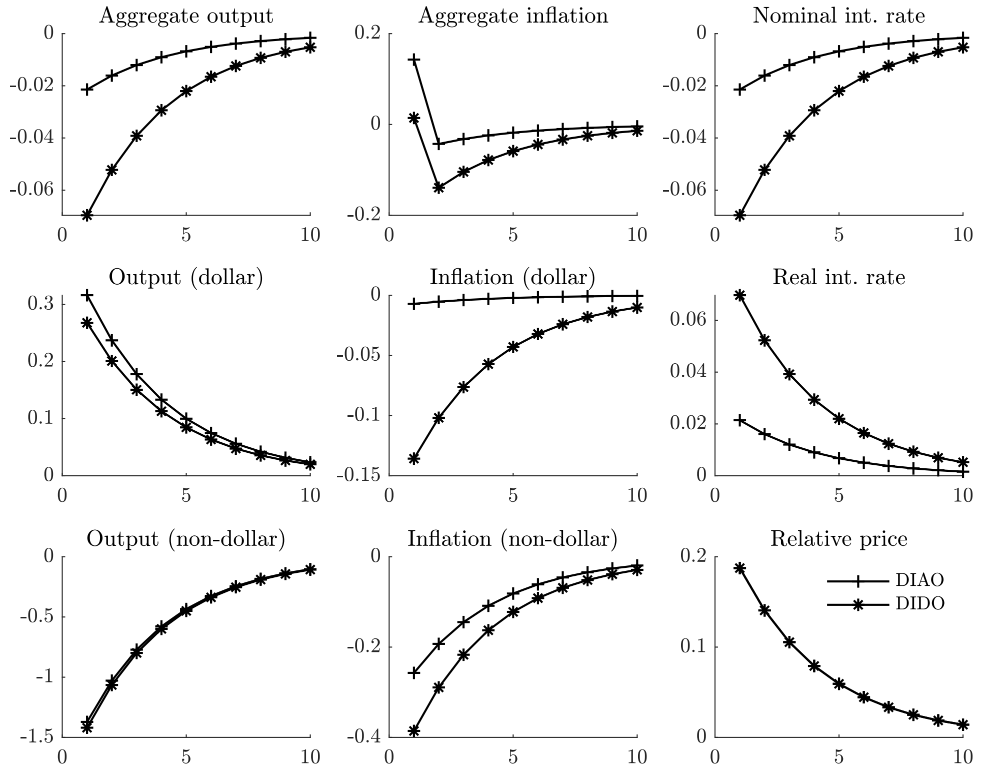

Figure 2: Impulse responses to a 0.25% shock in non-dollar exchange rate. Under the DIAO regime, the nominal interest rate responds to the dollar-sector inflation and the aggregate output gap. Under the DIDO regime, the nominal interest rate responds to the dollar-sector inflation and the dollar-sector output gap. Vertical axes indicate percentage deviations from the steady states. Horizontal axes indicate quarters after the exchange-rate shock.

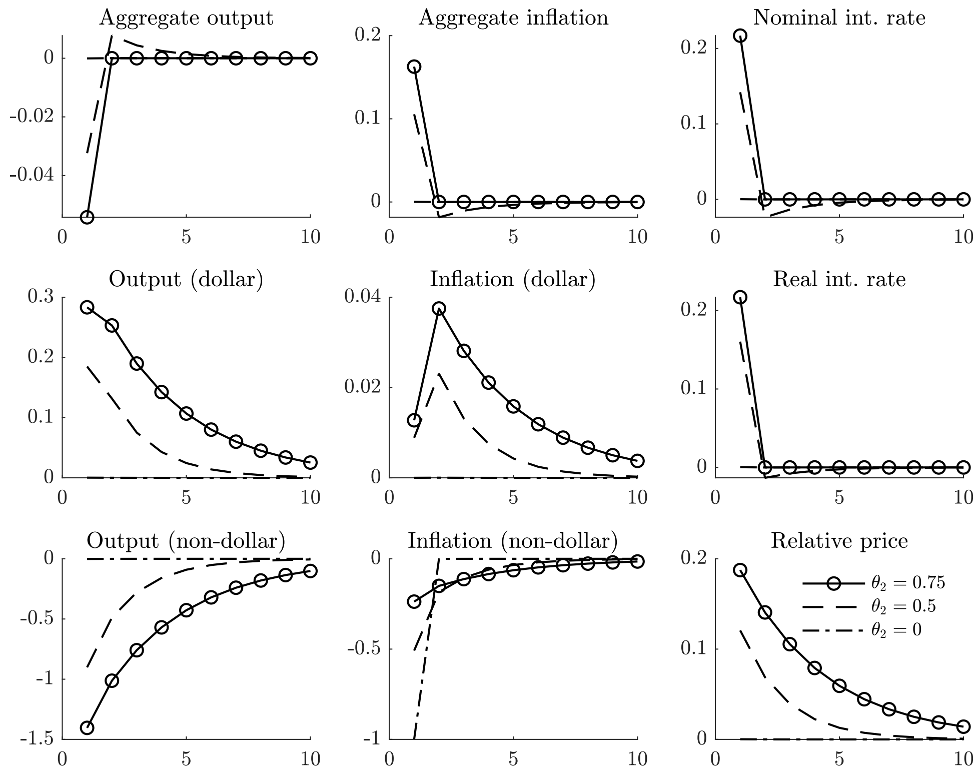

Figure 3: Impulse responses to a 0.25% shock in non-dollar exchange rate. Price rigidity in the dollar sector is kept at $\theta_1 = 0.75$, corresponding to a frequency of one price change per year. Price rigidity in the non-dollar sector varies among 0.75, 0.5, and 0.25, corresponding to one, two and four price changes per year, respectively. The monetary policy regime is AIAO. Vertical axes indicate percentage deviations from the steady states. Horizontal axes indicate quarters after the exchange-rate shock.

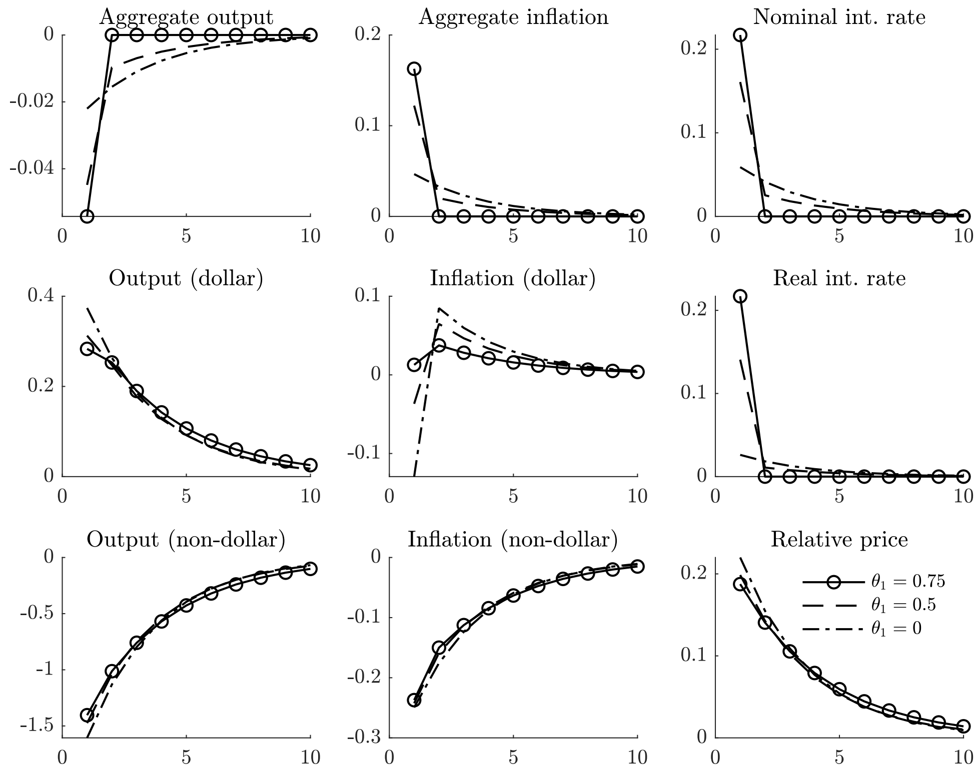

Figure 4: Impulse responses to a 0.25% shock in non-dollar exchange rate. Price rigidity in the non-dollar sector is kept at $\theta_2 = 0.75$, corresponding to a frequency of one price change per year. Price rigidity in the dollar sector varies among 0.75, 0.5, and 0.25, corresponding to one, two and four price changes per year, respectively. The monetary policy regime is AIAO. Vertical axes indicate percentage deviations from the steady states. Horizontal axes indicate quarters after the exchange-rate shock.

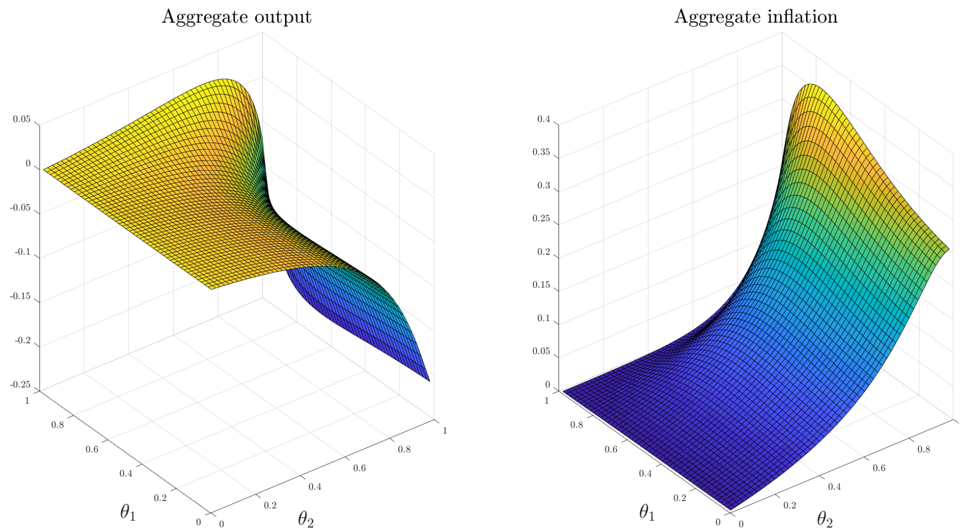

Figure 5: Two-year cumulative responses of aggregate output gap and aggregate inflation to a 0.25% shock in non-dollar exchange rate. The vertical axes indicate cumulative percentage deviations from the steady states.

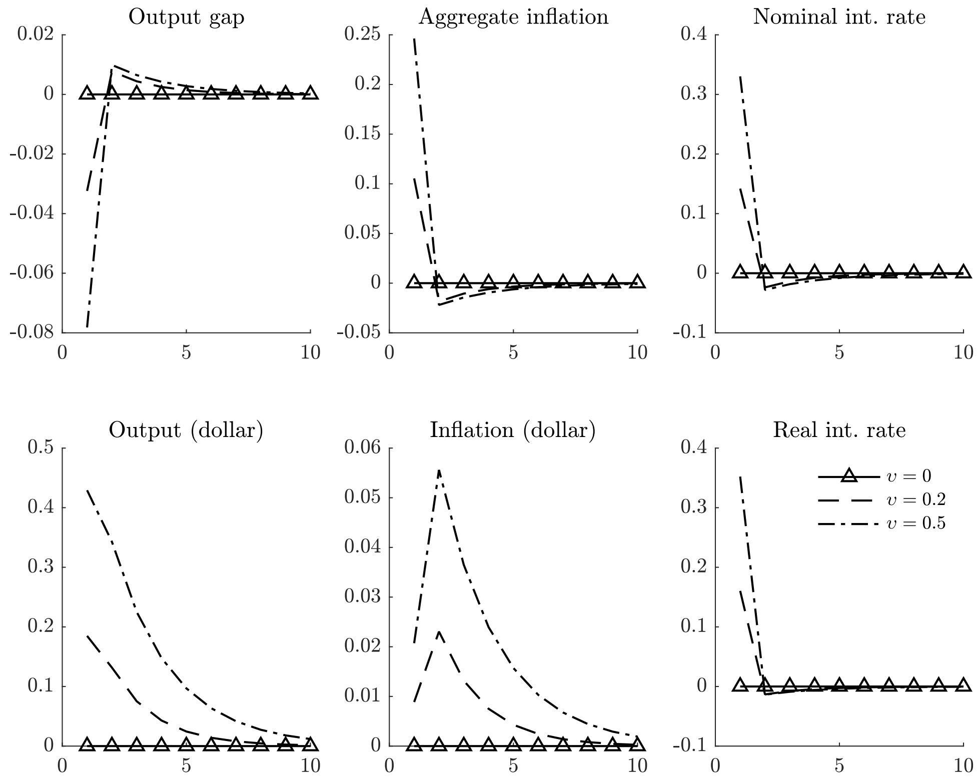

Figure 6: Impulse responses to a 0.25% shock in non-dollar exchange rate. Price rigidity prarameters are set to $\theta_1 = 0.75$ and $\theta_2 = 0.5$. The monetary policy regime is AIAO. Vertical axes indicate percentage deviations from the steady states. Horizontal axes indicate quarters after the exchange-rate shock.

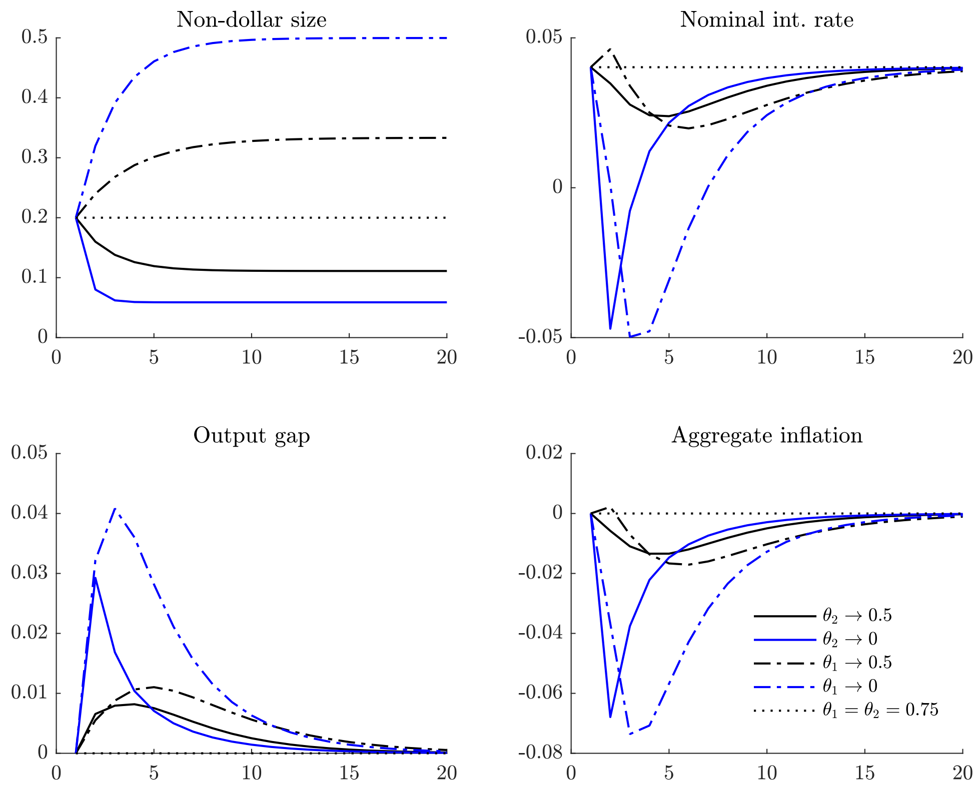

Figure 7: Transition dynamics to more flexible prices.

A Mathematical derivations for the linearised NK model

A.1 Households

Lifetime utility of a representative household is given by:

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbb{U}\left(C_t, L_t, N_t\right) Z_t \tag{A.1}$$

$$\mathbb{U}\left(C_t, L_t, N_t\right) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} + \frac{L_t^{1-\xi} - 1}{1-\xi} - \frac{N_t^{1+\varphi}}{1+\varphi} \tag{A.2}$$

The marginal utility of consumption, liquidity, and labour:

$$\mathbb{U}_{C,t} = C_t^{-\sigma} \tag{A.3}$$

$$\mathbb{U}_{L,t} = L_t^{-\xi} \tag{A.4}$$

$$\mathbb{U}_{N,t} = -N_t^{\varphi}. \tag{A.5}$$

The household's budget constraint is

$$C_t + B_t + \sum_{j=1}^{K} L_{j,t} = \frac{\exp\left(i_{t-1}\right) B_{t-1}}{\Pi_t} + \sum_{j=1}^{K} \frac{L_{j,t-1}}{\Pi_t} \frac{\mathcal{E}_{j,t}}{\mathcal{E}_{j,t-1}} + W_t N_t + \Gamma_t \tag{A.6}$$

The first-order conditions with respect to N_t, B_t, and $L_{j,t}$, are

$$N_t: \quad W_t = \frac{N_t^{\varphi}}{C_t^{-\sigma}} \tag{A.7}$$

$$B_t: \quad C_t^{-\sigma} = \beta \exp\left(i_t\right) \mathrm{E}_t\left[C_{t+1}^{-\sigma} \frac{1}{\Pi_{t+1}} \frac{Z_{t+1}}{Z_t}\right] \tag{A.8}$$

$$L_{j,t}: \quad \frac{L_t^{-\xi}}{C_t^{-\sigma}} = 1 - \beta\, \mathrm{E}_t\left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{1}{\Pi_{t+1}} \frac{\mathcal{E}_{j,t+1}}{\mathcal{E}_{j,t}} \frac{Z_{t+1}}{Z_t}\right] \tag{A.9}$$

Upon log-linearisation at the first order around the zero-inflation steady state, the optimality conditions are expressed as:

$$w_t = \varphi\, n_t + \sigma\, c_t \tag{A.10}$$

$$c_t = \mathrm{E}_t[c_{t+1}] - \frac{1}{\sigma}\left(\hat{i}_t - \mathrm{E}_t[\pi_{t+1}]\right) + \frac{1}{\sigma}\left(1-\rho_z\right) z_t \tag{A.11}$$

$$\xi\, l_t - \sigma\, c_t = \frac{\beta}{1-\beta}\left[\sigma\left(c_t - \mathrm{E}_t[c_{t+1}]\right) - \mathrm{E}_t[\pi_{t+1}] + \mathrm{E}_t[\Delta e_{j,t+1}] - \left(1-\rho_z\right) z_t\right] \tag{A.12}$$

A.2 Firms

The production function of firm i that prices in currency j is

$$Y_{j,t}(i) = A_t N_{j,t}(i)^{1-\alpha} \tag{A.13}$$

Cost minisation requires the marginal product of labour to equate the unit labour cost

$$\mathrm{MPN}_{j,t}(i) = \frac{W_t}{\mathrm{MC}_{j,t}(i)} \tag{A.14}$$

Substitute the expression for marginal product of labour. The real marginal cost of a firm in sector j is then given by

$$\begin{aligned}
\mathrm{MC}_{j,t}(i) &= \frac{W_t}{(1-\alpha)\, A_t N_{j,t}(i)^{-\alpha}} && \text{(A.15)}\\
&= \frac{W_t}{(1-\alpha)\, A_t} \left(\frac{Y_{j,t}(i)}{A_t}\right)^{\frac{\alpha}{1-\alpha}} && \text{(A.16)}\\
&= \frac{W_t}{(1-\alpha)\, A_t^{\frac{1}{1-\alpha}}} Y_t^{\frac{\alpha}{1-\alpha}} \left(\frac{\mathcal{E}_{j,t} P_{j,t}(i)}{P_t}\right)^{-\frac{\alpha\,\epsilon}{1-\alpha}} && \text{(A.17)}
\end{aligned}$$

The average real marginal cost of the economy

$$\begin{aligned}
\mathrm{MC}_t &= \sum_{j=1}^{K} \int_{\upsilon_j(t)} \mathrm{MC}_{j,t}(i) di && \text{(A.18)}\\
&= \frac{W_t}{(1-\alpha)\, A_t^{\frac{1}{1-\alpha}}} Y_t^{\frac{\alpha}{1-\alpha}} \left[\sum_{j=1}^{K} \int_{\upsilon_j(t)} \left(\frac{\mathcal{E}_{j,t} P_{j,t}(i)}{P_t}\right)^{-\frac{\alpha\,\epsilon}{1-\alpha}} di\right] && \text{(A.19)}
\end{aligned}$$

The first-order approximation for the summation in the bracket is 0. The log-linearised real marginal cost is

$$\mathrm{mc}_t = w_t + \frac{\alpha}{1-\alpha}\, y_t - \frac{1}{1-\alpha}\, a_t \tag{A.20}$$

It follows that an individual firm's real marginal cost and the average real marginal cost, after log-linearisation, are associated by the following equation:

$$\mathrm{mc}_{j,t}(i) = \mathrm{mc}_t - \frac{\alpha\,\epsilon}{1-\alpha}\left[p_{j,t}(i) + e_{j,t} - p_t\right] \tag{A.21}$$

The log-linearised desired price is

$$\tilde{p}_{j,t} = \mathrm{mc}_{j,t|t} + p_t - e_{j,t} \tag{A.22}$$

where (from Eq. (A.21))

$$\mathrm{mc}_{j,t|t} = \mathrm{mc}_t - \frac{\alpha\,\epsilon}{1-\alpha}\left[\tilde{p}_{j,t} + e_{j,t} - p_t\right] \tag{A.23}$$

The desired price can be solved as

$$\tilde{p}_{j,t} = \Theta\,\mathrm{mc}_t + p_t - e_{j,t} \tag{A.24}$$

Note that the dollar-denominated desired price, $\tilde{p}_{j,t} + e_{j,t}$, is independent of currency choice. When prices are sticky, the log-linearised optimal price is

$$p^*_{j,t} = (1-\beta\,\theta_j)\sum_{\ell=0}^{\infty}(\beta\,\theta_j)^{\ell}\,\mathrm{E}_t\left[\mathrm{mc}_{t+\ell|t} + p_{t+\ell} - e_{j,t+\ell}\right] \tag{A.25}$$

To simplify the equation, use Eq. (A.21) to establish the relationship between the optimising firm's real marginal cost with the economy's average marginal cost:

$$\mathrm{mc}_{t+\ell|t} = \mathrm{mc}_{t+\ell} - \frac{\alpha\,\epsilon}{1-\alpha}\left[p^*_{j,t} + e_{j,t+\ell} - p_{t+\ell}\right] \tag{A.26}$$

Substitute into Eq. (A.25) and simplify:

$$\begin{aligned}
p^*_{j,t} &= (1-\beta\,\theta_j)\sum_{\ell=0}^{\infty}(\beta\,\theta_j)^{\ell}\,\mathrm{E}_t\left[\mathrm{mc}_{t+\ell} - \frac{\alpha\,\epsilon}{1-\alpha}\left[p^*_{j,t} + e_{j,t+\ell} - p_{t+\ell}\right] + p_{t+\ell} - e_{j,t+\ell}\right] && \text{(A.27)}\\
&= \Theta^{-1}(1-\beta\,\theta_j)\sum_{\ell=0}^{\infty}(\beta\,\theta_j)^{\ell}\,\mathrm{E}_t\left[\Theta\,\mathrm{mc}_{t+\ell} + p_{t+\ell} - e_{j,t+\ell} - (1-\Theta)\,p^*_{j,t}\right] && \text{(A.28)}\\
&= -\left(\Theta^{-1}-1\right)p^*_{j,t} + \Theta^{-1}(1-\beta\,\theta_j)\sum_{\ell=0}^{\infty}(\beta\,\theta_j)^{\ell}\,\mathrm{E}_t\left[\Theta\,\mathrm{mc}_{t+\ell} + p_{t+\ell} - e_{j,t+\ell}\right] && \text{(A.29)}\\
&= (1-\beta\,\theta_j)\sum_{\ell=0}^{\infty}(\beta\,\theta_j)^{\ell}\,\mathrm{E}_t\left[\tilde{p}_{j,t+\ell}\right] && \text{(A.30)}
\end{aligned}$$

which can be written in a recursive form

$$p^*_{j,t} = \beta\,\theta_j\,\mathrm{E}_t\left[p^*_{j,t+1}\right] + (1-\beta\,\theta_j)\,\tilde{p}_{j,t} \tag{A.31}$$

The law of motion for the inflation is derived using the sectoral price index

$$p^*_{j,t} - p_{j,t-1} = \beta\,\theta_j \mathrm{E}_t\left[p^*_{j,t+1} - p_{j,t}\right] + (1-\beta\,\theta_j)\;\tilde{p}_{j,t} - p_{j,t-1} + \beta\,\theta_j\,p_{j,t} \tag{A.32}$$

$$= \beta\,\theta_j \mathrm{E}_t\left[p^*_{j,t+1} - p_{j,t}\right] - (1-\beta\,\theta_j)\;(p_{j,t} - \tilde{p}_{j,t}) + \pi_{j,t} \tag{A.33}$$

From the identity $p_{j,t} = \theta\,p_{j,t-1} + (1-\theta)\;p^*_{j,t}$, we have $\pi_{j,t} = (1-\theta)\left(p^*_{j,t} - p_{j,t-1}\right)$. The above equation becomes

$$(1-\theta)^{-1}\,\pi_{j,t} = (1-\theta)^{-1}\,\beta\,\theta_j \mathrm{E}_t\left[\pi_{j,t+1}\right] - (1-\beta\,\theta_j)\;(p_{j,t} - \tilde{p}_{j,t}) + \pi_{j,t} \tag{A.34}$$

Rearrange terms to obtain the law of motion for sectoral inflation

$$\pi_{j,t} = \beta\,\mathrm{E}_t\left[\pi_{j,t+1}\right] - \lambda_j\;(p_{j,t} - \tilde{p}_{j,t}) \tag{A.35}$$

where $p_{j,t} - \tilde{p}_{j,t}$ is interpreted as the price markup.

A.3 Equilbrium

The market clearing conditions for the goods and labour markets are

$$Y_t = C_t \tag{A.36}$$

where the aggregate output is defined as

$$Y_t \equiv \left(\sum_{j=1}^{K} \int_{\upsilon_j} Y_{j,t}(i)^{1-\frac{1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}} \tag{A.37}$$

The labour market clears when

$$N_t = \sum_{j=1}^{K} \int_{\upsilon_j} N_{j,t}(i) di \tag{A.38}$$

$$= \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} D_t \tag{A.39}$$

where $D_t \equiv \left[\sum_{j=1}^{K} \int_{\upsilon_j(t)} \left(\frac{\mathcal{E}_{j,t} P_{j,t}(i)}{P_t}\right)^{-\frac{\epsilon}{1-\alpha}} di\right]$ is a version of price dispersion for a multi-sector economy. The price dispersion is elaborated as

$$D_t = \sum_{j=1}^{K} \left(\frac{\mathcal{E}_{j,t} P_{j,t}}{P_t}\right)^{-\frac{\epsilon}{1-\alpha}} \int_{\upsilon_j(t)} \left(\frac{P_{j,t}(i)}{P_{j,t}}\right)^{-\frac{\epsilon}{1-\alpha}} di \tag{A.40}$$

$$= \sum_{j=1}^{K} \hat{P}_{j,t}^{-\frac{\epsilon}{1-\alpha}} \int_{\upsilon_j(t)} \left(\frac{P_{j,t}(i)}{P_{j,t}}\right)^{-\frac{\epsilon}{1-\alpha}} di \tag{A.41}$$

$$= \sum_{j=1}^{K} \hat{P}_{j,t}^{-\frac{\epsilon}{1-\alpha}} D_{j,t} \tag{A.42}$$

where $D_{j,t}$ is the sectoral price dispersion

$$D_{j,t} = \int_{\upsilon_j(t)} \left(\frac{P_{j,t}(i)}{P_{j,t}}\right)^{-\frac{\epsilon}{1-\alpha}} di \tag{A.43}$$

$$= \upsilon_j (1-\theta_j) \left(\frac{P_{j,t}^*}{P_{j,t}}\right)^{-\frac{\epsilon}{1-\alpha}} + \int_{\upsilon_j(t)\cap S(i)} \left(\frac{P_{j,t-1}(i)}{P_{j,t}}\right)^{-\frac{\epsilon}{1-\alpha}} di \tag{A.44}$$

$$= \upsilon_j (1-\theta_j) \left(\frac{P_{j,t}^*}{P_{j,t}}\right)^{-\frac{\epsilon}{1-\alpha}} + \int_{\upsilon_j(t)\cap S(i)} \left(\frac{P_{j,t-1}}{P_{j,t}} \frac{P_{j,t-1}(i)}{P_{j,t-1}}\right)^{-\frac{\epsilon}{1-\alpha}} di \tag{A.45}$$

$$= \upsilon_j (1-\theta_j) \left(\frac{P_{j,t}^*}{P_{j,t}}\right)^{-\frac{\epsilon}{1-\alpha}} + \Pi_{j,t}^{\frac{\epsilon}{1-\alpha}} \int_{\upsilon_j(t)\cap S(i)} \left(\frac{P_{j,t-1}(i)}{P_{j,t-1}}\right)^{-\frac{\epsilon}{1-\alpha}} di \tag{A.46}$$

$$= \upsilon_j (1-\theta_j) \left(\frac{P_{j,t}^*}{P_{j,t}}\right)^{-\frac{\epsilon}{1-\alpha}} + \theta_j \Pi_{j,t}^{\frac{\epsilon}{1-\alpha}} D_{j,t-1} \tag{A.47}$$

The labour market condition is linearised as

$$n_t = \frac{y_t - a_t}{1-\alpha} \tag{A.48}$$

Note that the price dispersion vanishes at the first order.

A.4 Deriving the sectoral NKPC

Substitute the desired price into the price markup:

$$p_{j,t} - \tilde{p}_{j,t} = \hat{p}_{j,t} - \Theta\, \mathrm{mc}_t \tag{A.49}$$

$$= \hat{p}_{j,t} - \Theta \left(w_t - \frac{1}{1-\alpha} a_t + \frac{\alpha}{1-\alpha} y_t\right) \tag{A.50}$$

$$= \hat{p}_{j,t} - \Theta \left[\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right) y_t - \frac{\varphi+1}{1-\alpha} a_t\right] \tag{A.51}$$

The last equation eliminates the real wage using the household's optimality condition for labour supply. Under flexible prices, $p_{j,t} = \tilde{p}_{j,t}$, $\hat{p}_{j,t} = 0$, and we have

$$0 = -\Theta \left[\left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right) y_t^n - \frac{\varphi + 1}{1 - \alpha} a_t \right] \tag{A.52}$$

Solve Eq. (A.52) for the natural level of output

$$y_t^n = \psi_{ya} \, a_t \tag{A.53}$$

Take the difference between Eqs. (A.51) and (A.52). The price markup is expressed in terms of the output gap defined as $\tilde{y}_t \equiv y_t - y_t^n$:

$$p_{j,t} - \tilde{p}_{j,t} = \hat{p}_{j,t} - \Theta \left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right) \tilde{y}_t \tag{A.54}$$

Substitute the price markup back into the law of motion for sectoral inflation. The sectoral NKPC is derived as:

$$\pi_{j,t} = \beta \, \mathrm{E}_t \left[\pi_{j,t+1} \right] + \kappa_j \, \tilde{y}_t - \lambda_j \, \hat{p}_{j,t} \tag{A.55}$$

A.5 Dynamic IS curve

Using the market clearing condition $y_t = c_t$, one can rewrite the optimality condition for government bonds as

$$y_t = \mathrm{E}_t[y_{t+1}] - \frac{1}{\sigma} \left(i_t - \mathrm{E}_t[\pi_{t+1}] \right) + \frac{1}{\sigma} \left(1 - \rho_z \right) z_t \tag{A.56}$$

Substracting the flexible counterpart from A.56 gives

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] - \frac{1}{\sigma} \left(i_t - \mathrm{E}_t[\pi_{t+1}] - r_t^n \right) \tag{A.57}$$

where the natural rate of interest is a linear combination of exogenous shocks

$$r_t^n \equiv -\sigma \left(1 - \rho_a \right) \psi_{ya} a_t + \left(1 - \rho_z \right) z_t \tag{A.58}$$

A.6 NKPC in nonlinear form

The firm's profit maximisation problem is expressed as

$$\max_{P^*_{j,t}} \sum_{\ell=0}^{\infty} \theta_j^\ell \mathrm{E}_t \left[Q_{t,t+\ell} \left[\frac{\mathcal{E}_{j,t+\ell}\, P^*_{j,t}}{P_{t+\ell}}\, Y_{j,t+\ell|t} - \Psi_{t+\ell}\left(Y_{j,t+\ell|t}\right)\right]\right] \tag{A.59}$$

subject to the demand function

$$Y_{j,t+\ell|t} = \left(\frac{\mathcal{E}_{j,t+\ell}\, P^*_{j,t}}{P_{t+\ell}}\right)^{-\epsilon} Y_{t+\ell} \tag{A.60}$$

where $\Psi_{t+\ell}(\cdot)$ is the real total cost of production. The first-order condition for price setting is

$$\sum_{\ell=0}^{\infty} {\theta_j}^\ell \mathrm{E}_t \left[Q_{t,t+\ell}\, \mathcal{E}_{j,t+\ell}\, (1/P_{t+\ell}) \left(Y_{j,t+\ell|t} + P^*_{j,t} \frac{\partial Y_{j,t+\ell|t}}{\partial P^*_{j,t}} - \frac{P_{t+\ell}\, \mathrm{MC}_{j,t+\ell|t}}{\mathcal{E}_{j,t+\ell}} \frac{\partial Y_{j,t+\ell|t}}{\partial P^*_{j,t}} \right)\right] = 0 \tag{A.61}$$

which can be simplified to the following condition

$$\sum_{\ell=0}^{\infty} \theta_j^\ell \mathrm{E}_t \left[\beta^\ell C_{t+\ell}^{-\sigma} \left(\Pi^*_{j,t} \hat{P}_{j,t}\right)^{-\epsilon} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\epsilon} Y_{t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \left(\Pi^*_{j,t} \hat{P}_{j,t+\ell} - \frac{\epsilon}{\epsilon-1} \frac{P_{j,t+\ell}}{P_{j,t}}\, \mathrm{MC}_{j,t+\ell|t} \right)\right] = 0 \tag{A.62}$$

Multiply both sides by $\hat{P}^\epsilon_{j,t}$

$$\sum_{\ell=0}^{\infty} \theta_j^\ell \mathrm{E}_t \left[\beta^\ell C_{t+\ell}^{-\sigma} {\Pi^*_{j,t}}^{-\epsilon} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\epsilon} Y_{t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \left(\Pi^*_{j,t} \hat{P}_{j,t+\ell} - \frac{\epsilon}{\epsilon-1} \frac{P_{j,t+\ell}}{P_{j,t}}\, \mathrm{MC}_{j,t+\ell|t} \right)\right] = 0 \tag{A.63}$$

where $\Pi^*_{j,t} \equiv \frac{\mathcal{E}_t P^*_{j,t}}{P_{j,t}}$. Rearrange terms,

$$\sum_{\ell=0}^{\infty} \theta_j^\ell \mathrm{E}_t \left[\beta^\ell C_{t+\ell}^{-\sigma} {\Pi^*_{j,t}}^{1-\epsilon} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\epsilon} Y_{t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \hat{P}_{j,t+\ell} \right] \tag{A.64}$$

$$= \sum_{\ell=0}^{\infty} \theta_j^\ell \mathrm{E}_t \left[\beta^\ell C_{t+\ell}^{-\sigma} {\Pi^*_{j,t}}^{-\epsilon} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\epsilon} Y_{t+\ell} \frac{\epsilon}{\epsilon-1}\, \mathrm{MC}_{j,t+\ell|t} \right] \tag{A.65}$$

Substitute the expression for idiosyncratic marginal cost:

$${\Pi^*_{j,t}}^{1+\frac{\alpha\epsilon}{1-\alpha}} \sum_{\ell=0}^{\infty} \theta_j^\ell \mathrm{E}_t \left[\beta^\ell C_{t+\ell}^{-\sigma} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\epsilon} \frac{P_{j,t}}{P_{j,t+\ell}} \hat{P}_{j,t+\ell} Y_{t+\ell} \right] \tag{A.66}$$

$$= \frac{\epsilon}{\epsilon-1} \sum_{\ell=0}^{\infty} \theta_j^\ell \mathrm{E}_t \left[\beta^\ell C_{t+\ell}^{-\sigma} \left(\frac{\mathcal{E}_{j,t+\ell}}{\mathcal{E}_{j,t}} \frac{P_t}{P_{t+\ell}} \right)^{-\epsilon} \left(\frac{P_{j,t}}{P_{j,t+\ell}} \hat{P}_{j,t+\ell} \right)^{-\frac{\alpha\epsilon}{1-\alpha}} Y_{t+\ell}\, \mathrm{MC}_{t+\ell} \right] \tag{A.67}$$

which can be rewritten as

$$\Pi_{j,t}^{*\,1+\frac{\alpha\epsilon}{1-\alpha}} x_{2,t} = \frac{\epsilon}{\epsilon-1} x_{1,t} \tag{A.68}$$

where

$$x_{2,t} = C_t^{-\sigma} Y_t \hat{P}_{j,t} + \beta\theta_j \mathrm{E}_t \left[\Pi_{j,t+1}^{\epsilon-1} \left(\frac{\hat{P}_{j,t+1}}{\hat{P}_{j,t}} \right)^{-\epsilon} x_{2,t+1} \right] \tag{A.69}$$

and

$$x_{1,t} = C_t^{-\sigma} Y_t \hat{P}_{j,t}^{-\frac{\alpha\epsilon}{1-\alpha}} \mathrm{MC}_t + \beta\theta_j \mathrm{E}_t \left[\Pi_{j,t+1}^{\frac{\epsilon}{1-\alpha}} \left(\frac{\hat{P}_{j,t+1}}{\hat{P}_{j,t}} \right)^{-\epsilon} x_{1,t+1} \right] \tag{A.70}$$

The relative price follows

$$\hat{P}_{j,t} = \hat{P}_{j,t-1} \frac{\Pi_t^e \Pi_{j,t}}{\Pi_t} \tag{A.71}$$

In the case of flexible price, the desired price is given by

$$\tilde{P}_{j,t} = \frac{P_t}{\mathcal{E}_{j,t}} \left(\frac{\epsilon}{\epsilon-1} \mathrm{MC}_t \right)^{\Theta} \tag{A.72}$$

A.7 Value function in nonlinear form

Iterate the value functifon:

$$\begin{aligned}
\mathcal{V}_{j,t|t} &= \Xi_{j,t|t} + \theta_j \mathrm{E}_t Q_{t,t+1} \mathcal{V}_{j,t+1|t} + (1-\theta_j)\, \mathrm{E}_t Q_{t,t+1} \mathcal{V}_{t+1} && \text{(A.73)}\\
&= \Xi_{j,t|t} + (1-\theta_j)\, \mathrm{E}_t Q_{t,t+1} \mathcal{V}_{t+1} + \theta_j \mathrm{E}_t Q_{t,t+1} \left[\Xi_{j,t+1|t} + \theta_j Q_{t+1,t+2} \mathcal{V}_{j,t+2|t} + (1-\theta_j)\, Q_{t+1,t+2} \mathcal{V}_{t+2} \right] && \text{(A.74)}\\
&= \mathrm{E}_t \left[\sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell} \Xi_{j,t+\ell|t} \right] + (1-\theta_j)\, \mathrm{E}_t \left[\sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell+1} V_{t+\ell+1} \right] && \text{(A.75)}\\
&= \mathrm{E}_t \left[\sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell} \frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} Y_{j,t+\ell|t} \right] - \mathrm{E}_t \left[\sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell} \mathrm{MC}_{j,t+\ell|t} Y_{j,t+\ell|t} \right] + (1-\theta_j)\, \mathrm{E}_t \left[\sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell+1} V_{t+\ell+1} \right] && \text{(A.76)}
\end{aligned}$$

Value function to the firm

$$\mathcal{V}_{j,t} = \mathcal{R}_{j,t} - \mathcal{C}_{j,t} + \mathcal{X}_{j,t} \tag{A.77}$$

The revenue is given by

$$\mathcal{R}_{j,t} = \sum_{\ell=0}^{\infty} \theta_j^\ell Q_{t,t+\ell} \frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} Y_{j,t+\ell|t} \tag{A.78}$$

$$= \sum_{\ell=0}^{\infty} (\beta\theta_j)^\ell \left(\frac{C_{t+\ell}}{C_t}\right)^{-\sigma} \frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} Y_{j,t+\ell|t} \tag{A.79}$$

$$= \sum_{\ell=0}^{\infty} (\beta\theta_j)^\ell \left(\frac{C_{t+\ell}}{C_t}\right)^{-\sigma} \frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}} \left(\frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}}\right)^{-\epsilon} Y_{t+\ell} \tag{A.80}$$

$$= \sum_{\ell=0}^{\infty} (\beta\theta_j)^\ell \left(\frac{C_{t+\ell}}{C_t}\right)^{-\sigma} \left(\frac{\mathcal{E}_{j,t+\ell} P_{j,t}^*}{P_{t+\ell}}\right)^{1-\epsilon} Y_{t+\ell} \tag{A.81}$$

$$= \sum_{\ell=0}^{\infty} (\beta\theta_j)^\ell \left(\frac{C_{t+\ell}}{C_t}\right)^{-\sigma} \left(\Pi_{j,t}^* \hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{1-\epsilon} Y_{t+\ell} \tag{A.82}$$

$$= \frac{1}{C_t^{-\sigma}} {\Pi_{j,t}^*}^{1-\epsilon} \sum_{\ell=0}^{\infty} (\beta\theta_j)^\ell C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{1-\epsilon} Y_{t+\ell} \tag{A.83}$$

Let $\tilde{\mathcal{R}}_{j,t} \equiv \mathcal{R}_{j,t} {\Pi_{j,t}^*}^{\epsilon-1} C_t^{-\sigma}$, then

$$\tilde{\mathcal{R}}_{j,t} = \sum_{\ell=0}^{\infty} (\beta\theta_j)^\ell \mathrm{E}_t \left[C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{1-\epsilon} Y_{t+\ell}\right] \tag{A.84}$$

$$= C_t^{-\sigma} \hat{P}_{j,t}^{1-\epsilon} Y_t + \sum_{\ell=1}^{\infty} (\beta\theta_j)^\ell \mathrm{E}_t \left[C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{1-\epsilon} Y_{t+\ell}\right] \tag{A.85}$$

$$= C_t^{-\sigma} \hat{P}_{j,t}^{1-\epsilon} Y_t + \mathrm{E}_t \left[\left(\frac{P_{j,t}}{P_{j,t+1}}\right)^{1-\epsilon} \sum_{\ell=1}^{\infty} (\beta\theta_j)^\ell C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t+1}}{P_{j,t+\ell}}\right)^{1-\epsilon} Y_{t+\ell}\right] \tag{A.86}$$

$$= C_t^{-\sigma} \hat{P}_{j,t}^{1-\epsilon} Y_t + \beta\theta_j \mathrm{E}_t \left[\left(\frac{P_{j,t}}{P_{j,t+1}}\right)^{1-\epsilon} \tilde{\mathcal{R}}_{j,t+1}\right] \tag{A.87}$$

$$= C_t^{-\sigma} \hat{P}_{j,t}^{1-\epsilon} Y_t + \beta\theta_j \mathrm{E}_t \left[\Pi_{j,t+1}^{\epsilon-1} \tilde{\mathcal{R}}_{j,t+1}\right] \tag{A.88}$$

The discounted cost is

$$\mathcal{C}_{j,t} = \sum_{\ell=0}^{\infty} \theta_j^\ell Q_{t,t+\ell} Y_{j,t+\ell|t} \mathrm{MC}_{j,t+\ell|t} \tag{A.89}$$

$$= \sum_{\ell=0}^{\infty} \theta_j^\ell \left(\frac{C_{t+\ell}}{C_t}\right)^{-\sigma} \left(\Pi_{j,t}^* \hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{-\epsilon} Y_{t+\ell} \left(\Pi_{j,t}^* \hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{-\frac{\alpha\epsilon}{1-\alpha}} \mathrm{MC}_{j,t+\ell} \tag{A.90}$$

$$= \sum_{\ell=0}^{\infty} \theta_j^\ell \left(\frac{C_{t+\ell}}{C_t}\right)^{-\sigma} \left(\Pi_{j,t}^* \hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{-\frac{\epsilon}{1-\alpha}} Y_{t+\ell} \mathrm{MC}_{j,t+\ell} \tag{A.91}$$

$$= \frac{1}{C_t^{-\sigma}} {\Pi_{j,t}^*}^{-\frac{\epsilon}{1-\alpha}} \sum_{\ell=0}^{\infty} (\beta\theta_j)^\ell C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}}\right)^{-\frac{\epsilon}{1-\alpha}} Y_{t+\ell} \mathrm{MC}_{t+\ell} \tag{A.92}$$

Let $\tilde{\mathcal{C}}_{j,t} \equiv \mathcal{C}_{j,t} C_t^{-\sigma} \Pi_{j,t}^{*\,\frac{\epsilon}{1-\alpha}}$, then

$$\tilde{\mathcal{C}}_{j,t} = \sum_{\ell=0}^{\infty} (\beta\theta_j)^{\ell} C_{t+\ell}^{-\sigma} \left(\hat{P}_{j,t+\ell} \frac{P_{j,t}}{P_{j,t+\ell}} \right)^{-\frac{\epsilon}{1-\alpha}} Y_{t+\ell} \mathrm{MC}_{t+\ell} \tag{A.93}$$

$$= C_t^{-\sigma} \hat{P}_{j,t}^{-\frac{\epsilon}{1-\alpha}} Y_t \mathrm{MC}_t + \beta\theta_j \mathrm{E}_t \left(\frac{P_{j,t}}{P_{j,t+1}} \right)^{-\frac{\epsilon}{1-\alpha}} \tilde{\mathcal{C}}_{j,t+1} \tag{A.94}$$

$$= C_t^{-\sigma} \hat{P}_{j,t}^{-\frac{\epsilon}{1-\alpha}} Y_t \mathrm{MC}_t + \beta\theta_j \mathrm{E}_t \left[\Pi_{j,t+1}^{\frac{\epsilon}{1-\alpha}} \tilde{\mathcal{C}}_{j,t+1} \right] \tag{A.95}$$

The infinite sum for the weighted value

$$\mathcal{X}_{j,t} = (1-\theta_j) \sum_{\ell=0}^{\infty} \theta_j^{\ell} Q_{t,t+\ell+1} V_{t+\ell+1} \tag{A.96}$$

Let $\tilde{\mathcal{X}}_{j,t} \equiv \frac{\mathcal{X}_{j,t} C_t^{-\sigma}}{\beta(1-\theta_j)}$, then

$$\tilde{\mathcal{X}}_{j,t} = \mathrm{E}_t \left[C_{t+1}^{-\sigma} V_{t+1} \right] + \beta\theta_j \mathrm{E}_t \left[\tilde{\mathcal{X}}_{j,t+1} \right] \tag{A.97}$$

The steady states are

$$\mathcal{V}_j = \mathcal{V} = \frac{Y\,(1-\mathrm{MC})}{1-\beta} \tag{A.98}$$

B Proofs of propositions

B.1 Proof of Proposition 1

The optimality conditions for real holdings of currencies j and j' are

$$\xi\, l_t - \sigma\, c_t = \frac{\beta}{1-\beta}\left[\sigma\,(c_t - \mathrm{E}_t[c_{t+1}]) - \mathrm{E}_t[\pi_{t+1}] + \mathrm{E}_t[\Delta e_{j,t+1}] - (1-\rho_z)\, z_t\right] \tag{B.1}$$

$$\xi\, l_t - \sigma\, c_t = \frac{\beta}{1-\beta}\left[\sigma\,(c_t - \mathrm{E}_t[c_{t+1}]) - \mathrm{E}_t[\pi_{t+1}] + \mathrm{E}_t[\Delta e_{j',t+1}] - (1-\rho_z)\, z_t\right] \tag{B.2}$$

Taking the difference between the two equations proves the proposition

$$e_{j,t} - e_{j',t} = \mathrm{E}_t\left[e_{j,t+1} - e_{j',t+1}\right]. \tag{B.3}$$

B.2 Proof of Proposition 2

B.2.1 First result

With the exchange rate being a random-walk process, the optimal price is rewritten as:

$$p^*_{j,t} + e_{j,t} = (1-\beta\,\theta_j)\sum_{k=0}^{\infty}(\beta\theta_j)^k\,\mathrm{E}_t\left[\tilde{p}_{t+k}\right] \tag{B.4}$$

Let the price rigidity be the same for an arbitrary pair of currencies j and j', $\theta_j = \theta_{j'} = \theta$. Eq. (B.4) can be written as:

$$p^*_{j,t} + e_{j,t} = (1-\beta\,\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\,\mathrm{E}_t\left[\tilde{p}_{t+k}\right],$$

$$p^*_{j',t} + e_{j',t} = (1-\beta\,\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\,\mathrm{E}_t\left[\tilde{p}_{t+k}\right].$$

The right-hand sides are identical, and independent of the currency choice. Hence, prices set in currencies j and j' are equivalent. Mathematically, $p^*_{j,t} + e_{j,t} = p^*_{j',t} + e_{j',t}$.

B.2.2 Second result

Price levels in sectors j and j' are weighted sums of optimal prices and the price levels from the previous period:

$$p_{j,t} = (1-\theta)\, p^*_{j,t} + \theta\, p_{j,t-1}$$

$$p_{j',t} = (1-\theta)\, p^*_{j',t} + \theta\, p_{j',t-1}$$

From the definition of $s_{jj',t}$:

$$
\begin{aligned}
s_{jj',t} &= p_{j,t} + e_{j,t} - (p_{j',t} + e_{j',t}) \\
&= (1-\theta)\left(p^*_{j,t} - p^*_{j',t}\right) + \theta\,(p_{j,t-1} - p_{j',t-1}) + e_{j,t} - e_{j',t} \\
&= (1-\theta)\left(p^*_{j,t} + e_{j,t} - p^*_{j',t} - e_{j',t}\right) + \theta\,(p_{j,t-1} + e_{j,t} - p_{j',t-1} - e_{j',t}) \\
&= \theta\, s_{jj',t-1} + \theta\,(\Delta e_{j,t} - \Delta e_{j',t})
\end{aligned}
$$

The last equation makes use of the first result.

B.2.3 Third result

The expression for $s_{jj',t}$ from the second result can be rearranged as an expression for s_{t-1}:

$$
s_{jj',t-1} = \frac{1}{\theta}\, s_{jj',t} - \Delta e_{j,t} + \Delta e_{j',t}
$$

The law of motion of $s_{jj',t} = s_{jj',t-1} + \pi_{j,t} + \Delta e_{j,t} - \pi_{j',t} - \Delta e_{j',t}$ is rearranged as

$$
\begin{aligned}
\pi_{j,t} - \pi_{j',t} &= s_{jj',t} - s_{jj',t-1} - \Delta e_{j,t} + \Delta e_{j',t} \\
&= -\frac{1-\theta}{\theta}\, s_{jj',t}
\end{aligned}
$$

B.2.4 Fourth result

The output-gap differential is obtained by taking the difference between the demand functions of the sectoral goods:

$$
\begin{aligned}
\tilde{y}_{j,t} - \tilde{y}_{j',t} &= -\epsilon\, \hat{p}_{j,t} + \tilde{y}_t - (-\epsilon\, \hat{p}_{j',t} + \tilde{y}_t) \\
&= -\epsilon\, s_{jj',t}
\end{aligned}
$$

B.3 Proof of Proposition 3

Let $\theta_j = \theta$ for all j. The second and third terms on the right-hand side of Eq. (38) are simplified as $\boldsymbol{\upsilon}'\boldsymbol{\kappa}\,\tilde{y}_t = \kappa\,\tilde{y}_t$ and $-\lambda\boldsymbol{\upsilon}'\hat{\boldsymbol{p}}_t = 0$, where $\kappa \equiv \Theta\,\frac{(1-\theta)(1-\beta\,\theta)}{\theta}\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)$. Consequently, the NKPC for aggregate inflation is expressed independent of the relative price.

B.4 Proof of Proposition 4

From Eq. (19) and equilibrium conditions, the relative in a sector j with flexible prices is a function of the output gap

$$\hat{p}_{j,t} = p_{j,t} + e_{j,t} - p_t = \Theta \left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right) \tilde{y}_t \tag{B.5}$$

The contemporaneous sectoral inflation is

$$\pi_{j,t} = \Theta \left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right) \Delta\tilde{y}_t + \pi_t - \Delta e_{j,t} \tag{B.6}$$

The contemporaneous aggregate inflation is

$$\pi_t = \upsilon_j \left[\Theta \left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right) \Delta\tilde{y}_t + \pi_t \right] + \sum_{k \neq j} \upsilon_k \left(\pi_{k,t} + \Delta e_{k,t} \right) \tag{B.7}$$

$$= \zeta_{\upsilon_j} \Delta\tilde{y}_t + \sum_{k \neq j} \frac{\upsilon_k}{1 - \upsilon_j} \left(\pi_{k,t} + \Delta e_{k,t} \right) \tag{B.8}$$

where $\zeta_{\upsilon_j} \equiv \frac{\upsilon_j \Theta}{1 - \upsilon_j} \left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right)$. Take expectations on both sides of Eq. (B.6). The expected sectoral inflation is

$$\mathrm{E}_t[\pi_{j,t+1}] = \Theta \left(\sigma + \frac{\alpha + \varphi}{1 - \alpha} \right) \left(\mathrm{E}_t[\tilde{y}_{t+1}] - \tilde{y}_t \right) + \mathrm{E}_t[\pi_{t+1}] \tag{B.9}$$

Take expectations on both sides of Eq. (B.8). The expected aggregate inflation is

$$\mathrm{E}_t[\pi_{t+1}] = \zeta_{\upsilon_j} \mathrm{E}_t[\Delta\tilde{y}_{t+1}] + \sum_{k \neq j} \frac{\upsilon_k}{1 - \upsilon_j} \mathrm{E}_t[\pi_{k,t+1}] \tag{B.10}$$

Substitute Eq. (B.10) into the IS curve:

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] - \sigma^{-1} \left(\hat{i}_t - \mathrm{E}_t[\pi_{t+1}] \right) \tag{B.11}$$

$$= \mathrm{E}_t[\tilde{y}_{t+1}] + \zeta_{\upsilon_j} \sigma^{-1} \left(\mathrm{E}_t[\tilde{y}_{t+1}] - \tilde{y}_t \right) - \sigma^{-1} \left(\hat{i}_t - \sum_{k \neq j} \frac{\upsilon_k}{1 - \upsilon_j} \mathrm{E}_t[\pi_{k,t+1}] \right) \tag{B.12}$$

$$= \mathrm{E}_t[\tilde{y}_{t+1}] - \left(\sigma + \zeta_{\upsilon_j} \right)^{-1} \left(\hat{i}_t - \sum_{k \neq j} \frac{\upsilon_k}{1 - \upsilon_j} \mathrm{E}_t[\pi_{k,t+1}] \right) \tag{B.13}$$

The NK framework with sector j being the flexible one has the following $2K$ equations, without the NKPC for sector j. The subscript $-j$ indicate vectors without the jth element.

$$\tilde{y}_t = \mathrm{E}_t[\tilde{y}_{t+1}] - \left(\sigma + \zeta_{\upsilon_j}\right)^{-1} \left(\hat{i}_t - \frac{\boldsymbol{\upsilon}'_{-j}\boldsymbol{\pi}_{-j,t+1}}{1-\upsilon_j}\right) \tag{B.14}$$

$$\boldsymbol{\pi}_{-j,t} = \beta\,\mathrm{E}_t\left[\boldsymbol{\pi}_{-j,t+1}\right] + \boldsymbol{\kappa}_{-j}\,\tilde{y}_t - \boldsymbol{\lambda}_{-j} \circ \hat{\boldsymbol{p}}_{-j,t} \tag{B.15}$$

$$\hat{\boldsymbol{p}}_{-j,t} = \hat{\boldsymbol{p}}_{-j,t-1} + \left(\mathbf{I} - \frac{\mathbf{1}\,\boldsymbol{\upsilon}'_{-j}}{1-\upsilon_j}\right)\left(\boldsymbol{\pi}_{-j,t} + \Delta\boldsymbol{e}_{-j,t}\right) - \zeta_{\upsilon_j}\Delta\tilde{y}_t\mathbf{1} \tag{B.16}$$

$$\hat{i}_t = \phi_\pi\,\pi_t + \phi_y\,\tilde{y}_t \tag{B.17}$$

where

$$\pi_t = \zeta_{\upsilon_j}\Delta\tilde{y}_t + \frac{\boldsymbol{\upsilon}'_{-j}\left(\boldsymbol{\pi}_{-j,t} + \Delta\boldsymbol{e}_{-j,t}\right)}{1-\upsilon_j} \tag{B.18}$$

The exchange-rate shock from sector j, $\Delta e_{j,t}$, does not enter the equation system, so it does not lead to changes in aggregate inflation and output gap. However, any shock from another sticky sector leads to changes in aggregate output gap. According to Eq. (B.5), price level in sector j changes only when there are changes in the aggregate output gap.

B.5 Proof of Proposition 5

For a two-sector economy, let $\gamma_1 = 1$, and $\gamma_2 = \gamma$. The probability of pricing in dollar is

$$\mathrm{Pr}_t = \frac{\mathcal{V}^*_{1,t|t}}{\mathcal{V}^*_{1,t|t} + \gamma\,\mathcal{V}^*_{2,t|t}} \tag{B.19}$$

When evaluated at desired price, the value functions are given by:

$$\tilde{\mathcal{V}}_{1,t|t} = \tilde{\Xi}_t + \beta\theta\mathrm{E}_t\tilde{\Xi}_{t+1} + \beta\left(1-\theta\right)\mathrm{E}_t\tilde{\mathcal{V}}_{t+1} \tag{B.20}$$

$$\tilde{\mathcal{V}}_{2,t|t} = \tilde{\Xi}_t + \beta\theta\mathrm{E}_t\tilde{\Xi}_{t+1} + \beta\left(1-\theta\right)\mathrm{E}_t\tilde{\mathcal{V}}_{t+1} \tag{B.21}$$

Note that the two value functions are identical. Hence,

$$\tilde{\mathcal{V}}_{1,t|t} = \tilde{\mathcal{V}}_{2,t|t} = \tilde{\mathcal{V}}_t \tag{B.22}$$

The probability of pricing in dollar evaluated at flexible price is a constant:

$$\tilde{\mathrm{Pr}}_t = \frac{1}{1+\gamma} \tag{B.23}$$

Taylor expansion of Eq. (B.19) up to the second order around the desired price gives

$$\mathrm{Pr}_t \approx \tilde{\mathrm{Pr}}_t + \frac{1}{2}\frac{\partial \mathrm{Pr}_t}{\partial \mathcal{V}^*_{1,t|t}} \sum_{\ell=0}^{\infty} \tilde{\Xi}_{pp}\left(x_{t+\ell}\right)\left(p^*_{1,t} - \tilde{p}_{t+\ell}\right)^2 + \frac{1}{2}\frac{\partial \mathrm{Pr}_t}{\partial \mathcal{V}^*_{2,t|t}} \sum_{\ell=0}^{\infty} \tilde{\Xi}_{pp}\left(x_{t+\ell}\right)\left(p^*_{2,t} + e_{t+\ell} - \tilde{p}_{t+\ell}\right)^2 \tag{B.24}$$

Evaluate the derivatives at the desired prices

$$\frac{\partial \mathrm{Pr}_t}{\partial \mathcal{V}^*_{1,t|t}} = \frac{\left(\mathcal{V}^*_{1,t|t} + \gamma \mathcal{V}^*_{2,t|t}\right) - \mathcal{V}^*_{1,t|t}}{\left(\mathcal{V}^*_{1,t|t} + \gamma \mathcal{V}^*_{2,t|t}\right)^2} = \frac{\gamma \mathcal{V}^*_{2,t|t}}{\left(\mathcal{V}^*_{1,t|t} + \gamma \mathcal{V}^*_{2,t|t}\right)^2} = \frac{\gamma}{(1+\gamma)^2 \tilde{\mathcal{V}}_t} \tag{B.25}$$

$$\frac{\partial \mathrm{Pr}_t}{\partial \mathcal{V}^*_{2,t|t}} = \frac{-\gamma \mathcal{V}^*_{1,t|t}}{\left(\mathcal{V}^*_{1,t|t} + \gamma \mathcal{V}^*_{2,t|t}\right)^2} = -\frac{\gamma}{(1+\gamma)^2 \tilde{\mathcal{V}}_t} \tag{B.26}$$

Substitute into Eq. (B.24):

$$\mathrm{Pr}_t \approx \frac{1}{1+\gamma} + \frac{1}{2}\frac{\gamma}{(1+\gamma)^2 \tilde{\mathcal{V}}_t} \sum_{\ell=0}^{\infty} (\beta\theta)^{\ell}\, \tilde{\Xi}_{pp}\left(x_{t+\ell}\right)\left[\left(p^*_{1,t} - \tilde{p}_{t+\ell}\right)^2 - \left(p^*_{2,t} + e_{t+\ell} - \tilde{p}_{t+\ell}\right)^2\right] \tag{B.27}$$

The terms in the summation resembles Gopinath et al. (2010). Using the proof for their Proposition 2, we have

$$\mathrm{Pr}_t \approx \frac{1}{1+\gamma} + \frac{1}{2}\frac{\gamma \tilde{\Xi}_{pp}\left(x_t\right)}{(1+\gamma)^2 \tilde{\mathcal{V}}_t} \sum_{\ell=0}^{\infty} (\beta\theta)^{\ell} \left[\left(p^*_{1,t} - \tilde{p}_{t+\ell}\right)^2 - \left(p^*_{2,t} + e_{t+\ell} - \tilde{p}_{t+\ell}\right)^2\right] \tag{B.28}$$

$$= \frac{1}{1+\gamma} + \frac{\gamma K\left(x_t\right)}{(1+\gamma)^2 \tilde{\mathcal{V}}_t} \sum_{\ell=0}^{\infty} (\beta\theta)^{\ell}\, \mathrm{Var}_t\left(e_{t+\ell}\right)\left[\frac{1}{2} - \frac{\mathrm{Cov}_t\left(\tilde{p}_{t+\ell}, e_{t+\ell}\right)}{\mathrm{Var}_t\left(e_{t+\ell}\right)}\right] \tag{B.29}$$

where $K(x_t) \equiv -\tilde{\Xi}_{pp}(x_t) > 0$.

C Solutions to a two-sector economy

C.1 AIAO

Since the exchange-rate shock is the only exogenous variable in the equation system, all endogenous variables can be expressed in terms of it:

$$\tilde{y}_t = \psi^{aiao}_{ye} \Delta e_t; \quad \pi_t = \psi^{aiao}_{\pi e} \Delta e_t; \quad \hat{i}_t = \psi^{aiao}_{ie} \Delta e_t. \tag{C.1}$$

where ψ_{ye}^{aiao}, $\psi_{\pi e}^{aiao}$, and ψ_{ie}^{aiao} are unknown coefficients to be determined. Because $\mathrm{E}_t\Delta e_{t+1} = 0$, the forecasts of the endogenous variables one period ahead are:

$$\mathrm{E}_t[\tilde{y}_{t+1}] = 0; \quad \mathrm{E}_t[\pi_{t+1}] = 0. \tag{C.2}$$

Substitute $\mathrm{E}_t[\tilde{y}_{t+1}]$ and $\mathrm{E}_t[\pi_{t+1}]$ into the three-equation system, and use the Taylor rule to substitute out the nominal interest rate in the dynamic IS curve, the equation system reduces to:

$$\tilde{y}_t = -\sigma^{-1}\left(\phi_\pi \pi_t + \phi_y \tilde{y}_t\right) \tag{C.3}$$

$$\pi_t = \kappa \tilde{y}_t + \upsilon \Delta e_t \tag{C.4}$$

from which π_t and $\tilde{y}_t$ can be solved in terms of Δe_t:

$$\tilde{y}_t = -\frac{\upsilon \phi_\pi}{\sigma + \phi_y + \kappa \phi_\pi} \Delta e_t \tag{C.5}$$

$$\pi_t = \frac{\upsilon\left(\sigma + \phi_y\right)}{\sigma + \phi_y + \kappa \phi_\pi} \Delta e_t \tag{C.6}$$

From the Taylor rule, the nominal interest rate is

$$\hat{i}_t = \frac{\upsilon \sigma \phi_\pi}{\sigma + \phi_y + \kappa \phi_\pi} \Delta e_t. \tag{C.7}$$

Define $\Omega \equiv \sigma + \phi_y + \kappa \phi_\pi$. The coefficients are

$$\psi_{ye}^{aiao} = -\upsilon \phi_\pi \Omega, \tag{C.8}$$

$$\psi_{\pi e}^{aiao} = \upsilon\left(\sigma + \phi_y\right)\Omega, \tag{C.9}$$

$$\psi_{ie}^{aiao} = \upsilon \sigma \phi_\pi \Omega. \tag{C.10}$$

The sectoral dynamics involve the bilateral relative price as a state variable.

$$\pi_{1,t} = \psi_{\pi_1 s}^{aiao} s_{t-1} + \psi_{\pi_1 e}^{aiao} \Delta e_t; \quad \pi_{2,t} = \psi_{\pi_2 s}^{aiao} s_{t-1} + \psi_{\pi_2 e}^{aiao} \Delta e_t. \tag{C.11}$$

The forecast for the sectoral inflation are

$$\mathrm{E}_t \pi_{1,t+1} = \psi_{\pi_1 s}^{aiao} s_t; \quad \mathrm{E}_t \pi_{2,t+1} = \psi_{\pi_2 s}^{aiao} s_t \tag{C.12}$$

From the dollar sector's NKPC:

$$\pi_{1,t} = \beta\psi_{\pi_1 s}^{aiao} s_t + \kappa\psi_{ye}^{aiao}\Delta e_t + \lambda\upsilon s_t \tag{C.13}$$

$$= \left(\beta\psi_{\pi_1 s}^{aiao} + \lambda\upsilon\right) s_t + \kappa\psi_{ye}^{aiao}\Delta e_t \tag{C.14}$$

$$= \left(\beta\psi_{\pi_1 s}^{aiao} + \lambda\upsilon\right) \theta s_{t-1} + \left[\left(\beta\psi_{\pi_1 s}^{aiao} + \lambda\upsilon\right)\theta + \kappa\psi_{ye}^{aiao}\right]\Delta e_t \tag{C.15}$$

The last equation comes from the autoregressive representation of the bilateral relative price. From comparison with the unknown coefficients:

$$\psi_{\pi_1 s}^{aiao} = \left(\beta\psi_{\pi_1 s}^{aiao} + \lambda\upsilon\right)\theta \tag{C.16}$$

$$\psi_{\pi_1 e}^{aiao} = \left(\beta\psi_{\pi_1 s}^{aiao} + \lambda\upsilon\right)\theta + \kappa\psi_{ye}^{aiao} \tag{C.17}$$

The coefficients $\psi_{\pi_1 s}^{aiao}$ and $\psi_{\pi_1 e}^{aiao}$ can be solved as:

$$\psi_{\pi_1 s}^{aiao} = \upsilon\left(1-\theta\right) \tag{C.18}$$

$$\psi_{\pi_1 e}^{aiao} = \upsilon\left(1-\theta\right) - \upsilon\kappa\phi_\pi\Omega \tag{C.19}$$

By combining the terms with common coefficients, the dollar-sector inflation can be rewritten as a function of contemporary bilateral price and the exchange-rate shock:

$$\pi_{1,t} = \frac{\upsilon\left(1-\theta\right)}{\theta} s_t - \upsilon\kappa\phi_\pi\Omega\Delta e_t \tag{C.20}$$

$$= \upsilon\left(1-\theta\right) s_{t-1} + \upsilon\left(1-\theta-\kappa\phi_\pi\Omega\right)\Delta e_t \tag{C.21}$$

The dollar-sector inflation is above its steady state if

$$1-\theta > \kappa\phi_\pi\Omega \tag{C.22}$$

$$\sigma + \phi_y + \kappa\phi_\pi > \frac{\kappa}{1-\theta}\phi_\pi \tag{C.23}$$

$$\sigma + \phi_y > \phi_\pi\left(1-\beta\theta\right)\Theta\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right) \tag{C.24}$$

$$\left(1-\beta\theta\right)\Theta\left(\sigma + \frac{\alpha+\varphi}{1-\alpha}\right) < \frac{\sigma+\phi_y}{\phi_\pi} \tag{C.25}$$

From Proposition 2, inflation in the non-dollar sector is

$$\pi_{2,t} = \pi_{1,t} - \frac{1-\theta}{\theta} s_t \tag{C.26}$$

$$= -\frac{(1-\upsilon)(1-\theta)}{\theta} s_t - \upsilon\kappa\phi_\pi \Omega \Delta e_t \tag{C.27}$$

The sectoral output gaps are derived from the demand functions:

$$\tilde{y}_{1,t} = \tilde{y}_t + \upsilon\epsilon s_t \tag{C.28}$$

$$= \upsilon\epsilon s_t - \upsilon\phi_\pi \Omega \Delta e_t \tag{C.29}$$

$$= \upsilon\epsilon\theta s_{t-1} + \upsilon\left(\epsilon\theta - \phi_\pi \Omega\right) \Delta e_t \tag{C.30}$$

$$\tilde{y}_{2,t} = \tilde{y}_t - (1-\upsilon)\,\epsilon s_t \tag{C.31}$$

$$= -(1-\upsilon)\,\epsilon s_t - \upsilon\phi_\pi \Omega \Delta e_t \tag{C.32}$$

The contemporaneous response of the dollar-sector output gap depends on the parameters. The sector produces above the natural level if

$$\epsilon\theta > \phi_\pi \Omega \tag{C.33}$$

$$\sigma + \phi_y + \kappa\phi_\pi > \frac{\phi_\pi}{\epsilon\theta} \tag{C.34}$$

$$\left(\frac{1}{\epsilon\theta} - \kappa\right)\phi_\pi < \sigma + \phi_y \tag{C.35}$$

$$\frac{1}{\epsilon\theta} - \kappa < \frac{\sigma + \phi_y}{\phi_\pi} \tag{C.36}$$

C.2 DIAO

The expected aggregate inflation in the dynamic IS curve is rewritted as:

$$\mathrm{E}_t[\pi_{t+1}] = \mathrm{E}_t\left[(1-\upsilon)\,\pi_{1,t+1} + \upsilon\pi_{2,t+1}\right] \tag{C.37}$$

$$= \mathrm{E}_t\left[\pi_{1,t+1} + \upsilon\left(s_{t+1} - s_t\right)\right] \tag{C.38}$$

$$= \mathrm{E}_t\left[\pi_{1,t+1}\right] - \upsilon\,(1-\theta)\, s_t \tag{C.39}$$

where the last equations follows from Eq. (40). The bilateral relative price s_t can be viewed as an exogenous autoregressive variable in a three-equation system, so all endogenous variables can be written as functions of s_t. Let $\tilde{y}_t = \psi_{ys}^{diao} s_t$, and $\pi_{1,t} = \psi_{\pi_1 s}^{diao} s_t$. Again, it follows from Eq. (40) that $\mathrm{E}_t[\tilde{y}_{t+1}] = \psi_{ys}^{diao}\theta s_t$, and $\mathrm{E}_t[\pi_{1,t+1}] = \psi_{\pi_1 s}^{diao}\theta s_t$. Substitute these, together with the Taylor rule, into the dynamic IS curve and the dollar-sector NKPC yields the following equation system for ψ_{ys}^{diao} and $\psi_{\pi_1 s}^{diao}$:

$$(1-\beta\theta)\,\psi_{\pi_1 s}^{diao} = \kappa\psi_{ys}^{diao} + \lambda\upsilon \tag{C.40}$$

$$(\phi_\pi - \theta)\,\psi_{\pi_1 s}^{diao} = -\left[\sigma(1-\theta)+\phi_y\right]\psi_{ys}^{diao} - \upsilon(1-\theta) \tag{C.41}$$

Define $\Lambda \equiv \frac{1}{(1-\beta\theta)[\sigma(1-\theta)+\phi_y]+\kappa(\phi_\pi-\theta)}$. The solutions to the equation system are:

$$\psi_{ys}^{diao} = -\lambda\upsilon\phi_\pi\Lambda \tag{C.42}$$

$$\psi_{\pi_1 s}^{diao} = \frac{\upsilon(1-\theta)}{\theta}\left(1-\kappa\phi_\pi\Lambda\right) \tag{C.43}$$

The response of dollar-sector inflation depends on the parameters. It is negative if $\psi_{\pi_1 s}^{diao} < 0$, and positive otherwise. Taken into account the expression of Λ, this condition is elaborated as:

$$(1-\beta\theta)\left[\sigma(1-\theta)+\phi_y\right] + \kappa(\phi_\pi-\theta) < \kappa\phi_\pi \tag{C.44}$$

$$(1-\beta\theta)\left[\sigma(1-\theta)+\phi_y\right] < \kappa\theta \tag{C.45}$$

$$\lambda\sigma + \frac{(1-\beta\theta)\,\phi_y}{\theta} < \kappa \tag{C.46}$$

$$\phi_y < \frac{(\kappa-\lambda\sigma)\,\theta}{1-\beta\theta} \tag{C.47}$$

The dynamics of the nominal interest rate is obtained by substituting the solutions to output gap and dollar-sector inflation into the Taylor rule:

$$\hat{i}_t = \phi_\pi\psi_{\pi_1 s}^{diao}s_t + \phi_y\psi_{ys}^{diao}s_t \tag{C.48}$$

$$= -\upsilon(\kappa-\lambda\sigma)(1-\theta)\,\phi_\pi\Lambda s_t \tag{C.49}$$

Its response if negative if:

$$\kappa\sigma^{-1} > \lambda \tag{C.50}$$

$$\lambda\Theta\left(1+\frac{\sigma^{-1}(\alpha+\varphi)}{1-\alpha}\right) > \lambda \tag{C.51}$$

$$\sigma^{-1} > \frac{1-\Theta}{\Theta}\frac{1-\alpha}{\alpha+\varphi} \tag{C.52}$$

The inflation in the non-dollar sector is obtained using Proposition 2:

$$\pi_{2,t} = \pi_{1,t} - \frac{1-\theta}{\theta} s_t \tag{C.53}$$

$$= -\frac{1-\theta}{\theta} \left(1 - \upsilon + \upsilon\kappa\phi_\pi\Lambda\right) s_t \tag{C.54}$$

The aggregate inflation is the weighted sum of the sectoral inflation:

$$\pi_t = (1-\upsilon)\,\pi_{1,t} + \upsilon\pi_{2,t} \tag{C.55}$$

$$= -\frac{1-\theta}{\theta}\upsilon\kappa\phi_\pi\Lambda s_t + \upsilon\Delta e_t \tag{C.56}$$

$$= -(1-\theta)\,\upsilon\kappa\phi_\pi\Lambda s_{t-1} + \upsilon\left[1 - (1-\theta)\,\kappa\phi_\pi\Lambda\right]\Delta e_t \tag{C.57}$$

$$\tag{C.58}$$

The coefficient of the exchange-rate shock is

$$\upsilon\left\{\kappa\theta\left(\phi_\pi - 1\right) + (1-\beta\theta)\left[\sigma\,(1-\theta) + \phi_y\right]\right\}\Lambda > 0 \tag{C.59}$$

The real interest rate is:

$$\hat{r}_t = \hat{i}_t - \mathrm{E}_t[\pi_{t+1}] \tag{C.60}$$

$$= -\upsilon\,(\kappa - \lambda\sigma)\,(1-\theta)\,\phi_\pi\Lambda s_t + \frac{1-\theta}{\theta}\upsilon\kappa\phi_\pi\Lambda\theta s_t \tag{C.61}$$

$$= \upsilon\lambda\sigma\,(1-\theta)\,\phi_\pi\Lambda s_t \tag{C.62}$$

The sectoral output gap dynamics can be derived from the demand functions:

$$\tilde{y}_{1,t} = \tilde{y}_t + \upsilon\epsilon s_t = -\upsilon\left(\lambda\phi_\pi\Lambda - \epsilon\right) s_t \tag{C.63}$$

$$\tilde{y}_{2,t} = \tilde{y}_t - (1-\upsilon)\,\epsilon s_t = -\left[\lambda\upsilon\phi_\pi\Lambda + \epsilon\,(1-\upsilon)\right] s_t \tag{C.64}$$

For the dollar sector to produce above the natural level, $\epsilon > \lambda\phi_\pi\Lambda$. Otherwise, it produces below the natural level.

C.3 DIDO

Follow the same method as in the case of DIAO. Solutions to the three-equation system are:

$$\tilde{y}_t = -\lambda\,\upsilon\,\left(\phi_\pi + \frac{\theta\,\epsilon}{1-\theta}\,\phi_y\right)\Lambda\,s_t \tag{C.65}$$

$$\pi_{1,t} = \frac{\upsilon\,(1-\theta)}{\theta}\left[1-\kappa\,\left(\phi_\pi + \frac{\theta\,\epsilon}{1-\theta}\phi_y\right)\Lambda\right]\,s_t \tag{C.66}$$

$$\hat{i}_t = -\upsilon\,(\kappa-\lambda\,\sigma)\,(1-\theta)\left(\phi_\pi + \frac{\theta\,\epsilon}{1-\theta}\,\phi_y\right)\Lambda\,s_t \tag{C.67}$$

From the IS curve, the real interest rate is positive:

$$\hat{r}_t = \upsilon\,\lambda\,\sigma\,(1-\theta)\left(\phi_\pi + \frac{\theta\,\epsilon}{1-\theta}\,\phi_y\right)\Lambda\,s_t \tag{C.68}$$

Inflation in the non-dollar sector is negative:

$$\pi_{2,t} = -\frac{1-\theta}{\theta}\left[1-\upsilon+\upsilon\,\kappa\,\left(\phi_\pi + \frac{\theta\,\epsilon}{1-\theta}\phi_y\right)\Lambda\right]\,s_t < 0 \tag{C.69}$$

Sectoral output gaps are given by:

$$\tilde{y}_{1,t} = -\upsilon\,\left[\lambda\left(\phi_\pi + \frac{\theta\,\epsilon}{1-\theta}\,\phi_y\right)\Lambda-\epsilon\right]\,s_t \tag{C.70}$$

$$\tilde{y}_{2,t} = -\left[\lambda\,\upsilon\,\left(\phi_\pi + \frac{\theta\,\epsilon}{1-\theta}\,\phi_y\right)\,\Lambda+\epsilon\,(1-\upsilon)\right]\,s_t \tag{C.71}$$

Aggregate inflation is:

$$\pi_t = -\frac{1-\theta}{\theta}\,\upsilon\,\kappa\,\left(\phi_\pi + \frac{\theta\,\epsilon}{1-\theta}\,\phi_y\right)\,\Lambda\,s_t + \upsilon\,\Delta e_t \tag{C.72}$$

The coefficient of the exchange-rate shock is

$$\upsilon\,\{\kappa\theta\,(\phi_\pi-1-\epsilon\phi_y)+(1-\beta\theta)\,[\sigma\,(1-\theta)+\phi_y]\}\,\Lambda \tag{C.73}$$

whose sign depends on the parameters. In the case of large elasticity of substitution, the aggregate inflation response is negative.