Financial Instability via Adaptive Learning

(Very Incomplete)

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This paper develops a simple model in which adaptive learning by investors leads to recurrent booms and busts in asset prices. The model captures aspects of Minsky's "financial instability hypothesis" in which periods of tranquility lead investors to increase their estimates of expected returns and reduce their estimates of return volatility. The changes of beliefs drive up asset prices and hence realized returns. However once agents invest a significant fraction of their wealth in stocks, the economy becomes fragile and so small negative shocks can lead to large declines in prices. I show how this process recurs over time, and discuss the features of the model which drive the boom-bust cycles in asset prices.

The first theorem of the financial instability hypothesis is that the economy has financing regimes under which it is stable, and financing regimes in which it is unstable.

The second theorem of the financial instability hypothesis is that over periods of prolonged prosperity, the economy transits from financial relations that make for a stable system to financial relations that make for an unstable system.

Hyman Minsky, 1992

Many of Minsky's colleagues regarded his "financial-instability hypothesis," which he first developed in the nineteen-sixties as radical, if not crackpot. Today with the subprime crisis on the verge of metamorphosing into a recession, references to it have become commonplace.

John Cassidy, The New Yorker, Feb. 8, 2008

1. INTRODUCTION

1.1. Overview

The financial crisis of 2008 has led to a wealth of recent research on the causes and consequences of financial and credit market fluctuations. While the literature has branched off in a number of different directions, one result of the crisis has been that ideas which had fallen out of fashion or have not been fully developed have begun to be re-examined. One particular theory which has received renewed attention in both the popular press and the academic literature following the crisis is Hyman Minksy's financial instability hypothesis (see Minsky, 1975, 1986, 1992). This theory, which describes a cycle of expanding and contracting credit driven by changes in expectations and risk assessments, has garnered renewed interest following the crisis. In this

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project I develop a simple model with adaptive learning which can generate recurrent, endogenous fluctuations in credit market conditions and asset prices characteristic of episodes of financial instability.

In particular, this paper formalizes and rationalizes certain parts of the content and intuition of Minsky's financial instability hypothesis. Minsky developed his theory initially in his 1975 interpretation of Keynes, and it was used to explain asset price bubbles and crashes in the famous book of Kindleberger (1978). Briefly, the narrative of the financial instability hypothesis begins with a period of growth, perhaps in the aftermath of a relatively recent crisis. Investors are rather conservative based on this past experience, but a string of good investment outcomes leads them to revise down their risk estimates and increase their expectations of future returns. In turn, they then take on more leverage, which lenders are willing to supply because they too share the more optimistic assessments. As the growth of credit expands, prices of risky assets increase, and investors find that even speculative investments are profitable. Eventually, the growth of demand for credit leads to highly levered positions which can be difficult to sustain if there is an increase in the cost of debt service.[1] As investors sell assets to finance their debt, asset prices begin to fall, making more of their debt-financed investments become unprofitable. This leads to more selling, and thus further exacerbates the downward asset price spiral.

While this story has intuitive appeal, relatively little work has been done within mainstream (neoclassical) economics to formalize and incorporate the insights of this hypothesis, apart from the relatively small related literature which is discussed below. Perhaps a central reason why the financial instability hypothesis has not been more widely studied is that it relies on temporary changes in investor sentiment, a concept which is difficult to address in a rational expectations model. On the other hand, if agents only have bounded rationality and update their beliefs over time as they observe more data, then the current state of beliefs and expectations changes over time and can have an independent influence on economic outcomes. In particular, I show how sequences of temporary exogenous shocks can interact with the adaptive learning process, and lead to large swings in asset positions, with corresponding booms and busts in asset prices.

I study a simple model where agents can invest in either borrow or lend at fixed rate of return, or they can purchase a risky stock which provides a claim on a dividend process. While the stock price is determined in equilibrium, I assume that investors believe that the risky asset returns are stochastic and i.i.d., but do not know the mean or variance of the returns. They form estimates of the expected returns and return volatility based on their observations of past asset returns, and then update these estimates adaptively over time. If agents had rational expectations and knew the asset return distribution, returns would indeed be i.i.d. with a constant mean and variance. The key to the model is the positive feedback that results from the way agents' beliefs about returns affect their demand for risky assets, and hence the

[1]Minsky emphasized how increased demand for credit can drive up interest rates and so increase the costs of high leverage. In this paper interest rates are constant, so this channel is shut down. Instead, we show how negative return realizations provide agents with less income to service their debt.

equilibrium asset prices, which in turn affect future beliefs. This feedback is locally stable, so that there is tendency for agents' beliefs to settle down around constant expected return rational expectations equilibrium. However I show that there are occasional but repeated episodes of relative instability, characterized by a runup in the price of the risk asset which is followed by a rapid stock market crash.

In particular, I show how a sequence of positive shocks to dividends can lead agents to believe that mean returns have increased and the variability of returns has fallen. In response, investors increase their demand for stocks, selling off some of their bond holdings in order to increase their position in risky assets. As the asset demand increases, the new inflow of funds further drives up the realized prices of risky assets. Agents then observe these higher realized returns, which further increase their estimates of expected returns, and hence lead them to demand even more risky assets. This process resembles an asset price bubble, where expectations of higher returns lead, at least temporarily, to higher realized returns and prices rapidly increase. Thus agents' beliefs "escape" from the constant rate of return equilibrium, with increasing mean and decreasing variance estimates being self-reinforcing and generating a large increase in asset price valuation.

However once agents' positions in stocks increase sufficiently, the economy becomes rather fragile. In particular, once agents invest nearly all of their wealth in stocks, or even borrow to invest even more, then small changes in portfolio choices have very large impact on prices. Thus any accumulation of negative shocks, which leads to decreases in estimates of the mean of returns, will lead to a reduction in the portfolio share of stocks, and a sizeable fall in prices. After observing asset prices trending upward, this reduction in prices will also be accompanied by an increase in the volatility estimate, which leads to a further reduction in the demand for stocks. Thus once the inflow of funds to risky assets slows, so does the growth of prices. As agents revise their estimates and seek to sell off stocks, this outflow of funds leads to puts more downward pressure on asset prices, until they stabilize at a lower level. I show how this process, resembling in broad outline aspects of the recent financial crisis, repeats over time. I use analytic methods I have developed to characterize the "escape dynamics" which drive this process. These methods allow me to show how the expected interval between crises and the severity of the asset boom and subsequent crash depend on the features of the model.

1.2. Related Literature

As discussed above, there has been relatively little work formalizing or utilizing the ideas which Minsky developed in a series of papers Minsky (1975, 1986, 1989, 1992). While Kindleberger (1978) used the financial instability hypothesis to great effect as an organizing principle in his classic study of asset pricing bubbles, he did not formally model the mechanism. Other references include Taylor and O'Connell (1985), who develop a version of a traditional Keynesian model which incorporates a Minsky style crisis, and Sethi (1992) who studies a learning model where the financial instability can result from switches between multiple equilibria. In a similar spirit to this paper, Friedman and Laibson (1989) lay out a portfolio choice model with learning which they use to interpret their empirical evidence of extraordinary market volatility. However our approaches are different, as they focus solely on the implications of learning for agents' beliefs and choices given observed asset price data. By contrast, we study a dynamic equilibrium model where agents' choices determine the equilibrium prices. More recently, Eggertsson and Krugman (2012) develop a modified New Keynesian model where changes in borrowing constraints can lead to a Minsky-style crisis of deleveraging. In addition to the differences in approach, there is a difference in focus as they study output dynamics in response to an exogenous deleveraging shock, while I study asset price dynamics where the deleveraging effects occur endogenously due to negative dividend realizations.

Perhaps the most extensive analysis of the financial instability hypothesis has been in the Post-Keynesian literature, including Keen (1995). But this approach is rather different from the neoclassical approach built on tastes, technologies, and optimization, which has perhaps limited its impact on policy and the broader profession. Unlike such previous formalizations of Minsky's ideas, my work uses a relatively standard economic model, the classic Lucas (1978) asset pricing model. My only significant departure is that I do not assume that agents have rational expectations, but rather assume that they form forecasts of future events using simple statistical rules which they update over time.

My research draws on the literature of adaptive learning in macroeconomics. This literature has typically focused on the question of whether adaptive agents, who base their actions on simple learning rules, could eventually learn a rational expectations equilibrium. This provides a foundation for rational expectations models, and limits focus to equilibria which are "learnable." Evans and Honkapohja (2001) provide an excellent overview of this literature. But rather than focus on eventual limit points, in this paper I show how the learning process itself can have important effects on outcomes. Previous papers considering the impact of learning on asset prices include Timmermann (1993), Adam, Marcet, and Nicolini (AMN) (2009), and Branch and Evans (2011). In terms of approach, my paper is related to these latter two papers, as AMN use a very similar Lucas asset pricing model and focus on how learning can lead to increased volatility in prices and price-dividend ratios. Branch and Evans (2011) also consider agents who learn about risk and returns, but their setting is fairly different. Neither consider portfolio choice dynamics which are crucial to my model and to Minsky's story.

In my model, as discussed above there is a general tendency for beliefs to fluctuate around the limiting rational expectations equilibrium. However the interesting feature of the model is that it gives rise to occasional episodes in which beliefs "escape" from the limit, as this drives the boom-bust cycles. Sargent (1999) introduced the study of such escape dynamics in macroeconomics, and jointly and solely I have analyzed these dynamics theoretically in previous work such as Cho, Williams, and Sargent (2002), and Williams (2001, 2009). In general terms, the basic mechanism is that agents occasionally misperceive an accumulation of random stochastic shocks as reflecting a change in the economic environment. This causes them to change their behavior, and so leads to outcomes which reinforce their perceptions. These endogenous dynamics lead agents' beliefs to escape from the limit. However the shifts are only temporary and agents are again drawn toward the limit point. This pattern of convergence and escape leads to recurrent episodes in which there are large changes in outcomes. Sargent, Williams, and Zha (2006, 2008) have estimated empirical models to show how these dynamics can explain the history of inflation in the United States and the experience of recurrent hyperinflations in Latin America. In this project, I show how the escape dynamics can lead to recurrent episodes of financial instability.

2. THE MODEL AND RATIONAL EXPECTATIONS EQUILIBRIUM

2.1. The Model

The model is a simple variation on a standard Lucas (1978) type consumption-based model. One difference with a standard setup is that I study an "open economy" version of the model, where agents can borrow and lend at a constant risk-free world interest rate. This is a simple way of introducing (net) borrowing and lending while retaining the simplicity of the representative agent approach. Clearly, modeling the supply of loanable funds and endogenizing interest rates would be an important topic for future work. In my model, there are two assets, stocks denoted S_t which trade at the price P_t, and bonds B_t that have a constant one-period return R. There is unit supply of stock (at most): $S_t \leq 1$. Unlike most "closed-economy" versions of the model, I assume that banks can borrow and lend freely at a fixed return R, and they have "deep pockets" so that there is no constraint on the supply of funds. Thus I take R as exogenous, making the domestic holdings of bonds B_t endogenous. The driving stochastic process is an exogenous dividend stream D_t which accrues to stockholders. For simplicity I assume that dividend growth is log-normal:

$$\log D_{t+1} = \log D_t + d + \sigma W_{t+1}$$

where $d > 0$ is the constant mean rate of dividend growth, $\sigma > 0$ is its standard deviation, and W_{t+1} is an i.i.d. Gaussian shock.

The agents have identical standard time separable power utility over consumption C_t with parameter γ and discount factor β. Agents choose how much to consume each period, and given their current portfolio they decide what their holdings of stocks and bonds should be for the next period. Thus the agent's problem is to solve:

$$\max E_0 \sum_{t=0}^{\infty} \frac{C_t^{1-\gamma}}{1-\gamma}$$

subject to the budget constraint:

$$P_t S_t + B_t + C_t = (P_t + D_t) S_{t-1} + R B_{t-1}. \tag{1}$$

Defining the gross return on stocks as $Z_t = (P_t + D_t)/P_{t-1}$, the optimality conditions lead to the standard Euler equations:

$$C_t^{-\gamma} = \beta R E_t \left[C_{t+1}^{-\gamma}\right], \tag{2}$$
$$C_t^{-\gamma} = \beta E_t \left[C_{t+1}^{-\gamma} Z_{t+1}\right]. \tag{3}$$

2.2. Rational Expectations Equilibrium

We now solve for a rational expectations equilibrium with zero debt. In equilibrium $S_t \equiv 1$, so the aggregate resource constraint is:

$$C_t = D_t + RB_{t-1} - B_t.$$

That is, total consumption is equal to dividends plus the net borrowing from abroad (after accounting for interest costs). For arbitrary interest rates R, the economy may run a current account surplus or deficit, but if we fix the interest rate R on bonds at its closed economy equilibrium level then the current account will balance. In this case the equilibrium in our environment matches the closed economy equilibrium where bonds are in zero net supply.

The simple structure of our economy, with power utility and log-normal growth, allows us to completely solve for the zero debt equilibrium.[2] In particular, in equilibrium consumption is equal to dividends: $C_t = D_t$, and the price-dividend ratio on stocks is constant:

$$P_t = \delta D_t$$
$$\delta = \frac{\beta\theta}{1-\beta\theta}, \text{ where: } \theta = \exp((1-\gamma)d + 0.5(1-\gamma^2)\sigma^2)$$

The rate of return on bonds and stocks are respectively given by:

$$\log R = -\log(\beta) + \gamma d - 0.5\sigma^2\gamma^2, \tag{4}$$
$$\log Z_t = \log\left(\frac{1+\delta}{\delta}\right) + d + \sigma W_t$$
$$= \log R + 0.5\sigma^2(2\gamma - 1) + \sigma W_t. \tag{5}$$

Even though directly solving for the equilibrium is rather straightforward in this environment, it will be useful for later reference to describe the agent's decision rules. For this, we define the agent's wealth X_t as the right side of the budget constraint (1):

$$X_t = RB_{t-1} + (P_t + D_t)S_{t-1}.$$

[2] Adam, Marcet, and Nicolini (2009) this as well, and focus on learning in a zero debt environment.

Also, define v_t as the share of the agent's post-consumption wealth that is invested at date t in stocks:

$$v_t = \frac{P_t S_t}{X_t - C_t}$$

Then the agent's decision rules in the rational expectations equilibrium can be written:

$$\begin{aligned} C_t &= \frac{1}{1+\delta} X_t \\ v_t &= \frac{E\log(Z_t) - \log(R)}{0.5\sigma^2(2\gamma - 1)} = 1 \end{aligned}$$

That is, agents consume a constant fraction of their wealth and invest a constant fraction of their post-consumption wealth in stocks. These decision rules are just the same as in the classic paper of Hakansson (1970) (which itself is a discrete time version of the model of Merton (1973)). This equivalence is natural, as Hakansson (1970) studied consumption and portfolio choice problems in an environment with i.i.d. returns, and in our environment the rational expectations equilibrium stock returns Z_t are i.i.d. as can be seen in equation (5) above. Of course, the zero-debt equilibrium returns are also determined so that the agent always invests all his wealth in stocks.

3. EQUILIBRIUM FOR ARBITRARY BELIEFS

3.1. Self-Confirming Equilibrium

As a bridge to study our adaptive learning model, we first analyze the consequences of agents having arbitrary (but fixed) beliefs. In order to solve their consumption and portfolio decisions, agents do not necessarily need to know the process for dividends or to understand the determinants of equilibrium prices. All they need is to have a forecast of the distribution of expected returns on stocks. Therefore, we now suppose that agents choose decision rules for consumption and their portfolio allocations assuming that risky asset returns are i.i.d lognormally distributed, with mean m and standard deviation s. We have already seen that in the rational expectations equilibrium returns are indeed i.i.d. with mean μ and standard deviation σ, where:

$$\mu \equiv E\log(Z_t) = \log\left(\frac{1+\delta}{\delta}\right) + d.$$

Thus if $m = \mu$ and $s = \sigma$ the returns are indeed i.i.d., so agents' beliefs are correct and the outcomes of the economy are the same as under rational expectations.

Following Sargent (1999), I adopt the notion of a self-confirming equilibrium to describe such a situation.[3] In a self-confirming equilibrium (SCE), agents optimize given their beliefs and their beliefs are correct *along the equilibrium path* (but may

[3]The self-confirming equilibrium concept originally arose in game theory. See Fudenberg and Levine (1998) for more discussion. Sargent (1999) formulated the concept in a parametric form for adaptive learning models like mine.

be incorrect for events not observed in equilibrium). Therefore $(m, s) = (\mu, \sigma)$ is a self-confirming equilibrium in this model, and in fact is the unique SCE. When $m \neq \mu$ and/or $s \neq \sigma$ then equilibrium returns are not i.i.d., as they depend on the aggregate asset positions as we show below. In such cases, agents beliefs are incorrect, as they fail to account for the serial correlations in returns. Nonetheless, we can still solve for an equilibrium in such an environment, as we now do.

Note that the uniqueness result holds for self-confirming equilibrium defined in a strong sense. That is, we take as part of the specification of agents' beliefs that returns are i.i.d. In previous work, such as Cho, Williams, and Sargent (2002) and Williams (2009), we have used a weaker version of self-confirming equilibrium, which simply requires that agents' expectations be correct in equilibrium. In our setting, it is possible for there to be multiple (m, s) values that are consistent with the implied equilibrium mean and standard deviation of returns, but only one of these will have i.i.d. returns.

3.2. Equilibrium for Arbitrary Beliefs

As we noted above, the solutions of agents' decision rules for the consumption and portfolio choice problem with i.i.d. returns follows from Hakansson (1970). In particular, the decision rules with arbitrary beliefs (m, s) have the same form as before: $C_t = c(m, s)X_t$, and $v_t = v(m, s)$. That is, agents consume a constant fraction of their wealth X_t and invest a constant fraction of their post-consumption wealth in stocks, where the fractions depend on their perceptions of stock returns. However now with arbitrary (m, s) and nonzero bond holdings, there are not explicit solutions for the constants c and v. Instead, the decision rules are determined implicitly by the following expressions. The optimal portfolio fraction $v(m, s)$ solves the static maximization problem:

$$U(v(m, s)) = \max_v \int \frac{((x - R)v + R)^{1-\gamma}}{1 - \gamma} dF(x; m, s), \tag{6}$$

where $F(\cdot; m, s)$ is the lognormal cdf with parameters m and s. Then the optimal fraction of wealth to consume $c(m, s)$ is:

$$c(m, s) = 1 - [\beta U(v(m, s))(1 - \gamma)]^{\frac{1}{\gamma}}. \tag{7}$$

Figure 1 plots the agent's optimal decision rule for the portfolio share $v(m, s)$ versus the expected return m and the perceived volatility s. In the figure we also impose a portfolio constraint $v(m, s) \leq \overline{v}$, which is discussed in more detail below. The figure clearly shows that, as expected, the optimal fraction invested in stocks is increasing in expected returns and decreasing in volatility. Moreover, even though there are not explicit expressions for the optimal decision rules, the figure shows that $v(m, s)$ increases in nearly linear manner in m (up to the constraint) and increases nearly linearly in $1/s^2$, just as in the rational expectations decision rules. For some of our

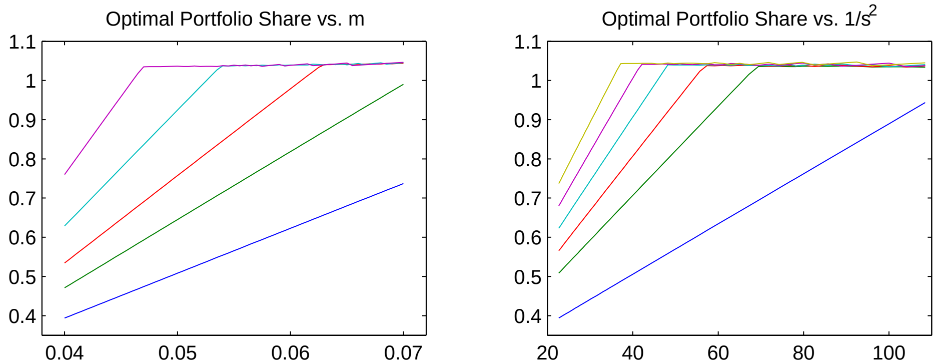

FIGURE 1. Agents optimal portfolio decision rules $v(m,s)$. The left panel plots $v(m,s)$ versus m for different levels of s, while the right panel plots $v(m,s)$ versus s for different levels of m

simulation results below, we find it useful to thus approximate $v(m,s)$ by the following:

$$v(m,s) \approx v_0 + v_1 \frac{1}{s^2} + v_2 m + v_3 \frac{m}{s^2}.$$

Although not exact, this provides a very good approximation to the optimal decision rules.

With the agents' decision rules, it is rather easy to solve for an equilibrium. First, note that the value of the stock holdings at date t is:

$$P_t S_t = (1-c)vX_t = (1-c)v(RB_{t-1} + S_{t-1}(P_t + D_t)),$$

where we suppress the dependence of c and v on the beliefs (m,s). Therefore the stock price must satisfy:

$$P_t = \frac{(1-c)v(RB_{t-1} + S_{t-1}D_t)}{S_t - (1-c)vS_{t-1}}. \tag{8}$$

Imposing the equilibrium conditions that the agent holds the entire supply of stocks, $S_t = S_{t-1} = 1$, we then have the equilibrium price:

$$P_t = \frac{(1-c)v}{1-(1-c)v}(RB_{t-1} + D_t) \equiv \delta^*(RB_{t-1} + D_t), \tag{9}$$

as long as this expression is well defined. We return to this issue and impose restrictions which insure that the equilibrium price is well-defined in the next section.

In the zero-debt rational expectations equilibrium, stock prices are driven entirely by dividends and the price-dividend ratio was constant. Now, as agents can take either long or short positions in bonds, the total demand for stocks depends on the asset positions and this is reflected in prices and returns. In particular, stock returns are now given by:

$$Z_t = \frac{P_t + D_t}{P_{t-1}} = \frac{\delta^* RB_{t-1} + (1+\delta^*)D_t}{\delta^*(RB_{t-2} + D_{t-1})}.$$

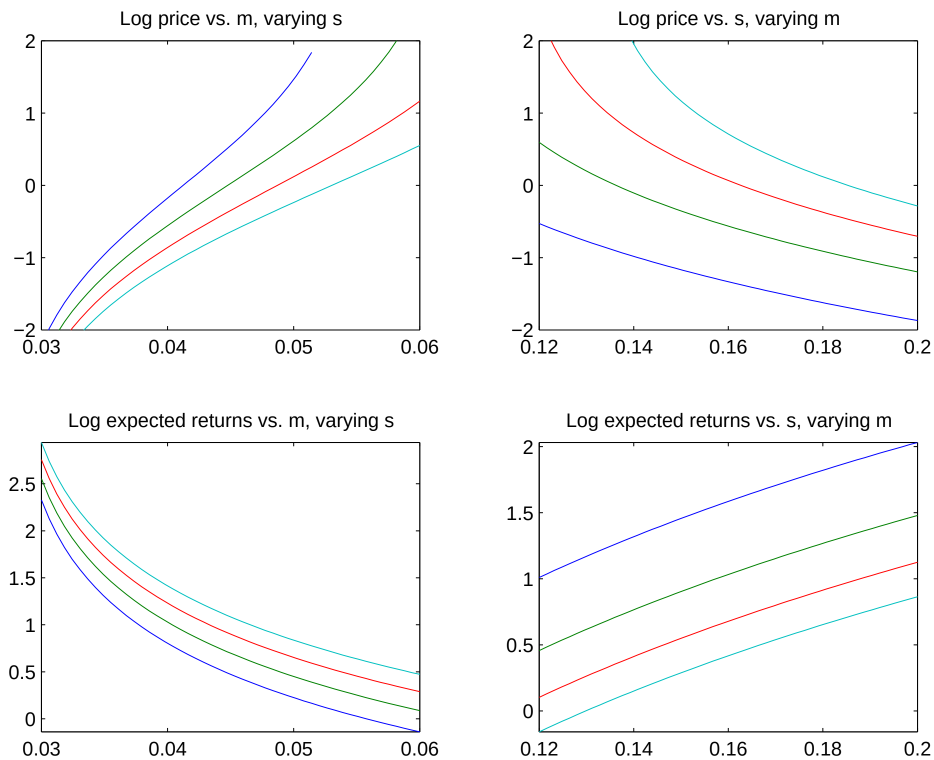

FIGURE 2. The effects of beliefs (m, s) on expected returns $E_t Z_{t+1}$. The left panel plots $E_t Z_{t+1}$ versus m for different levels of s, while the right panel plots $E_t Z_{t+1}$ versus s for different levels of m

When $B_{t-1} = B_{t-2} = 0$ this reduces to the rational expectations expression, with returns driven by dividend growth. But in general, agents' bond positions from the previous period B_{t-1} affect their wealth X_t in period t through interest payments received (or made if $B_{t-1} < 0$), and thus their demand for current stock purchases. Therefore prices and returns are in general correlated over time, and depend on the composite factor $RB_{t-1} + D_t$.

The dependence of stock prices and expected stock returns on beliefs is illustrated in Figure 2. The top row of panels shows the log of the stock price P_t while the bottom row shows the log of the expected stock return $E_t Z_{t+1}$. The left column of panels shows the values versus the perceived mean return m for different levels of the perceived volatility s, while the right shows the values versus the perceived standard deviation s for different levels of the perceived mean m. The plot assumes $B_t = B_{t-1} = 0$ and $D_t = 1$, so that $E_t D_{t+1} = \exp(d + .5\sigma^2)$. As would expected, returns are decreasing in m and increasing in s. Increases in the perceived mean or decreases in the perceived volatility of stocks lead to an increase in the portfolio share v, which bids up their prices, and lowers expected future returns. However, current *realized returns* are higher due to the increase in prices. In our learning model below, agents compare their perceptions to realized returns, and thus increases in the perceived mean return can be self-sustaining.

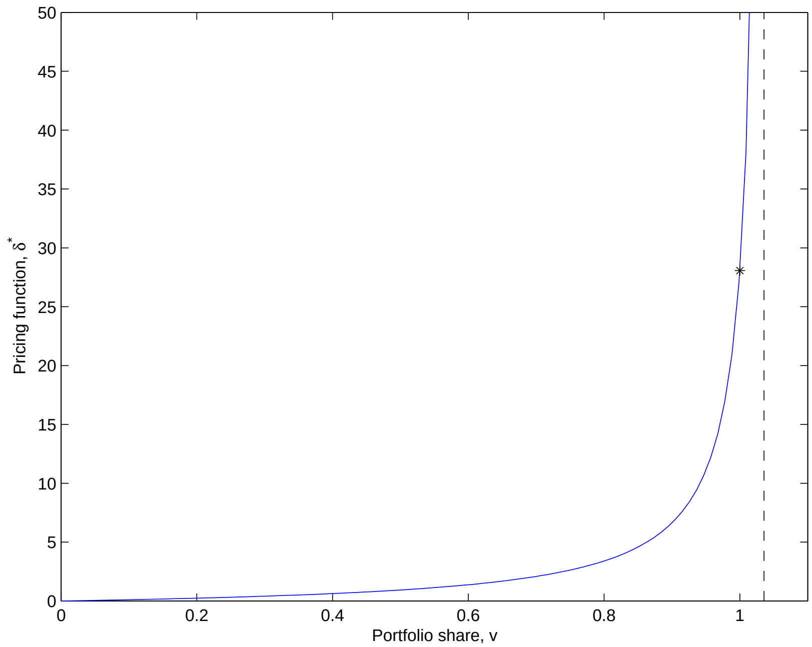

FIGURE 3. The pricing function δ^* versus the portfolio share v. The SCE level of $v = 1$ is shown with $*$, while the dotted line is maximal level $v = 1/(1-c)$.

3.3. Price Determination and Borrowing Constraints

In this model stock prices are very sensitive to the portfolio share in risky assets. This is particularly true for v near or greater than one, because as agents almost all of their funds in stocks, or even borrow invest more, the price increases dramatically. This effect is illustrated in Figure 3, which plots the pricing function term δ^* as a function of the portfolio share v. The figure illustrates what (9) makes clear, namely that as $v \to \frac{1}{1-c}$ the stock price P_t blows up as $\delta^* \to \infty$. Thus there are equilibria with debt and $v > 1$, but there is an overall limit on how much leverage the economy can sustain. Moreover, as will become important later, stock prices are very sensitive to changes in the portfolio share near the SCE (and rational expectations) value of $v = 1$. Very small changes in portfolio positions lead to large changes in stock prices. Note that this discussion has assumed c is constant, but as equation (7) makes clear, the optimal consumption varies with v. Thus the plot actually shows the pricing function $\delta^*(v)$ with $c = c(v)$.

However there is an additional complication which also is exacerbated as agents borrow in order to invest in the risky asset. When agents enter the period in debt, there is always a chance that the realization of this period's dividend may be less than the total value of their debt. That is, when $B_{t-1} < 0$ there is positive probability that $RB_{t-1} + D_t < 0$. This means that the agents would not be able to repay the principal and interest out of flows, a situation which Minsky (1986) refers to as "speculative finance". In such situations, agents would typically need to roll over their liabilities, and thus the stock of debt would increase. Minsky emphasizes how this can lead to fragility of the financial system, as balance sheets become very sensitive to fluctuations in principal. Minsky (1986) also discusses a further stage of "Ponzi finance," where agents cannot even repay interest on their (longer term) debt and so would need to sell off assets. In our setting with one-period debt, principal must be repaid each period, so these the speculative and Ponzi stages merge. That is, if $RB_{t-1} + D_t < 0$ agents would want to sell off their stock holdings in order to repay their debt. However in our representative agent environment, they cannot actually sell off stocks in equilibrium as they must hold the entire unit supply. Thus this need for liquidity puts extreme downward pressure on prices, and in fact would drive the price of stocks to zero. As (9) shows, the stock price is $P_t = 0$ at $RB_{t-1} + D_t = 0$ and P_t would nonsensically go negative if $RB_{t-1} + D_t < 0$.

Rather than ruling out leverage and debt altogether, we impose a borrowing constraint of the form:

$$B_t \geq -\frac{\phi}{R} D_t \tag{10}$$

where ϕ governs the tightness of the constraint. This constraint does not entirely rule out the possibility of $RB_{t-1} + D_t < 0$, which is difficult to achieve since it entails a restriction between choices made in the previous period and the current period's shock realizations. Nonetheless, the constraint (10) can make it very unlikely that the key restriction is violated. For example, since $D_t = D_{t-1}e^{d+\sigma W_t}$, by setting $\phi = e^{d-3\sigma}$ we can ensure that the restriction is only violated with probability 0.0013. For the very small probability events when the restriction is violated, we assume that the supply of assets S_t is cut enough to ensure a positive price. The stock price in such periods is then determined by (8) with the restricted $S_t < 1$. One interpretation of such interventions, which were not encountered in our simulations below, would be that outside forces such as the government intervene and buy up the excess shares in order to support the market. However once the market recovers, the government unloads its share holdings and we again have $S_t = 1$. We emphasize that such events were not observed in our simulations below.

Finally, note that (10) can be rewritten as a joint restriction on agents' portfolio choices v, which in turn implies a restriction on the consumption share c by equation (7). In particular in equilibrium (with $S_t = 1$) we have:

$$\begin{aligned} B_t &= (1-c)(1-v)X_t = (1-c)(1-v)[RB_{t-1} + P_t + D_t] \\ &= \frac{(1-c)(1-v)}{1-(1-c)v}(RB_{t-1} + D_t), \end{aligned}$$

where the last equality follows from equation (9). Thus we can rewrite the borrowing constraint (10) as:

$$\frac{(1-c)(1-v)}{1-(1-c)v} \geq -\frac{\phi}{R}\left(\frac{D_t}{RB_{t-1} + D_t}\right).$$

By equation (7) this can be written as a constraint on v alone.

For simplicity, we impose this restriction on a period-by-period basis. That is, we solve for the optimal v as in (6) and if it violates the borrowing constraint we impose the constrained upper bound. This is admittedly ad-hoc, as in their optimization problems agents do not take the borrowing constraint into account and do not foresee the possibility of future constraints binding. Solving an agent's full optimization problem with occasionally binding constraints would ruin significantly complicate matters. Such computations would be prohibitively costly in the learning formulation we turn to next, as it would require re-solving a complex numerical problem at each date as beliefs are updated. The lack of foresight about borrowing constraints is thus another aspect of the bounded rationality that we assume.

4. ADAPTIVE LEARNING

4.1. Belief Dynamics

Thus far we have analyzed the determination of equilibrium prices and returns for arbitrary fixed beliefs. We now describe how agents update their beliefs over time. We assume that agents learn adaptively, meaning that when they form their decision rules agents treat their beliefs as constant, but then they update their beliefs over time. Thus agents do not take into account that they are learning and that their future beliefs will differ from their current beliefs. The approach, which Kreps (1998) calls "anticipated utility," dominates the literature on adaptive learning in macroeconomics (see Evans and Honkapohja (2001) for an overview).

The anticipated utility approach is also consistent with the some of Minsky's discussion of the financial instability hypothesis. For example, Minsky (1975) states that, "Financing is often based on an assumption 'that the existing state of affairs will continue indefinitely', but of course this assumption proves false." In this way, Minsky was following Keynes (1953): "The essence of this convention – though it does not, of course, work out quite so simply – lies in assuming that the existing state of affairs will continue indefinitely, except in so far as we have specific reasons to expect a change. [...] We know from extensive experience that this is most unlikely. The actual results of an investment over a long term of years very seldom agree with the initial expectation."

In practice, this means that at each date we solve for the equilibrium prices as above with $(m, s) = (m_t, s_t)$, the agents' current estimates of the mean and standard deviations of returns as of date t. Then after observing the realized equilibrium returns, agents update their beliefs. Since agents simply estimate means and variances, they update their beliefs using the simple recursive procedure:

$$m_{t+1} = m_t + \varepsilon_t(\log(Z_t(m_t, s_t)) - m_t), \tag{11}$$

$$s^2_{t+1} = s^2_t + \varepsilon_t([\log(Z_t(m_t, s_t)) - m_t]^2 - s^2_t). \tag{12}$$

Here ε_t is known as the gain, and it reflects the weight that agents place on the most recent observations. We consider both the case where $\varepsilon_t = 1/(t + 1)$ so that the gain decreases over time with more observations, and the constant gain case where $\varepsilon_t = \varepsilon$.

Implicit in the assumption of $\varepsilon_t = 1/(t+1)$ is the subjective belief that mean asset returns and volatility are constant, so that more observations simply give more data on the unchanging underlying parameters. Thus in this case all observations are weighted equally, and the decay of the gain gives at least the possibility that agents' beliefs will eventually stabilize and converge to some constants.

On the other hand, if agents believe that the mean and variance of returns change over time, then more recent observations would be more informative about the current situation than observations further in the past. In such situations, constant gain learning rules are appropriate, as they discount past observations at rate ε, allowing beliefs to better track the underlying moving parameters. In particular, Sargent and Williams (2005) showed that (some) constant gain learning rules are Bayesian optimal filters when the underlying parameters follow a random walk. The size of the gain ε reflects the variance of the innovations in the random walk, which in turn governs how quickly the parameters drift. An alternative motivation for constant gain learning is that such learning rules are more robust to model misspecification, in the sense of Hansen and Sargent (2008). That is, if agents are uncertain about the true process generating the return data, they may opt to use learning rules which work well even when their model of returns is misspecified. Evans, Honkapojha, and Williams (2010) show that (some) constant gain learning rules are robust optimal filters. In this case the size of the gain ε reflects the size of the class of potential misspecifications.

Both of these motivations for constant gain learning, tracking potential changes and handling potential uncertainties, are reflected in further discussion of Minsky. For example, Minsky (1989) states, "[M]odels of system performance that help form the expectations of businessmen and bankers are affected by the recent performance of the economy... The critical agents are unsure how the economy will perform, because they are unsure of the effect of recent institutional and environmental changes." In this paper we mostly focus on constant gain learning rules, which capture aspects of the financial instability hypothesis.

4.2. Analysis of Beliefs

To analyze the dynamics of agents' beliefs, we apply some fundamental well-known results in the learning literature. Marcet and Sargent (1989), Evans and Honkapohja (2001), and others have applied results from the theory of stochastic approximation to show that in the limit as the gain gets small, the evolution of beliefs can be approximated by a deterministic ordinary differential equation.

The basic idea of the results is that we can rewrite the updating equation (11) as:

$$\frac{m_{t+1} - m_t}{\varepsilon_t} = \log(Z_t(m_t, s_t)) - m_t.$$

Note that the left side of the equation can be interpreted as a finite-difference approximation of a time derivative, where we let ε_t be the notional "time" between observations. Then as $\varepsilon_t \to 0$ the left side of the equation converges to that time derivative. Also in this limit, we pack in more and more observations in any finite interval of time, so we effectively average over the shock realizations. Thus as $\varepsilon_t \to 0$, the beliefs converge to the trajectories of the ODEs which we call (following Sargent (1999)), the mean dynamics ODEs:

$$\begin{aligned} \dot{m} &= E[\log(Z_t(m,s))] - m, \\ \dot{s}^2 &= E[\log(Z_t(m,s)) - m]^2 - s^2. \end{aligned}$$

The same ODEs govern the limiting behavior of beliefs in both the constant and decreasing gain cases, but the notion of convergence differs. When $\varepsilon_t = 1/(t+1)$, the beliefs converge with probability one over time to the solution of the ordinary differential equations. But when $\varepsilon_t = \varepsilon$, the beliefs converge weakly (or in distribution) to the paths of the ODEs. Moreover, the converge is across sequences of beliefs, each of which are indexed by the constant level of the gain.

Although the mean dynamics ODEs characterize the limiting behavior of the beliefs, the return dynamics are sufficiently complex that we cannot express them analytically. In particular, the moments of the log return process Z_t cannot be calculated explicitly, due to the nonlinearities in the return dynamics, along with the occasional explosive behavior which triggers the borrowing constraints. We show how to compute the mean dynamics numerically below. However it is clear that the self-confirming equilibrium is an equilibrium point of the mean dynamics. That is, by setting $\dot{m} = \dot{s}^2 = 0$ we can easily see that:

$$\begin{aligned} m = \mu &= E[\log(Z_t(\mu,\sigma))] \\ s^2 = \sigma^2 &= E[\log(Z_t(\mu,\sigma)) - \mu]^2 \end{aligned}$$

Again, given the complexity of the mean dynamics, analyzing global stability properties is a daunting task analytically. We show some numerical results below. However the local stability of the mean dynamics can be analyzed analytically. As in Evans and Honkapohja (2001), the SCE is locally expectationally stable if the eigenvalues of the Jacobian matrix of the mean dynamics have negative real parts when evaluated at the SCE. Since the variance estimate s^2 is of 2nd order, it does matter for the local stability. Thus it is enough to simply verify the local stability of the mean estimate, and for this we simply require:

$$\frac{\partial}{\partial m} E[\log(Z_t(m,\sigma^2))]\bigg|_{m=\mu} - 1 < 0.$$

Tedious calculations show that we actually have:

$$\frac{\partial}{\partial m} E[\log(Z_t(m,\sigma^2))]\bigg|_{m=\mu} < 0$$

and so therefore the SCE is clearly locally stable. Thus we expect that (at least) for beliefs (m_t, s_t) which start near (μ,σ) we will have convergence to the SCE.

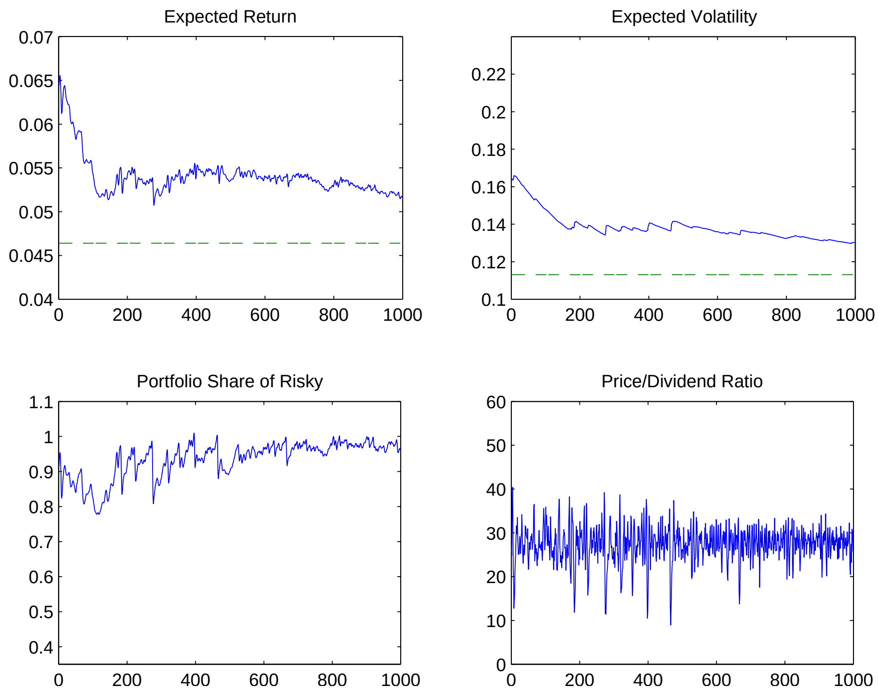

FIGURE 4. Representative simulation with decreasing gain $\varepsilon_t \propto 1/(t+1)$. The top left panel plots m_t, the top right plots s_t, the bottom left plots v_t, and the bottom right plots P_t/D_t

This is, in fact, what we observe in simulations of the learning model with decreasing gain, as Figure 4 shows. For all simulations with $\varepsilon_t \propto 1/(t+1)$, of which the figure is representative, we observe some volatility in beliefs early in the sample, but over time they stabilize at the SCE values. Because learning is relatively slow in this model and the effect of initial conditions take a long time to wear off, we actually set the gain to be constant for the first 250 periods, then switch to a decresing gain. The top two panels of the figure plot the beliefs (m_t, s_t), with the SCE values shown as dotted lines. Here we clearly see that after some initial volatility, beliefs converge to the SCE values. The bottom left panel shows v_t, the proportion of the agent's portfolio invested in stocks, while the bottom right panel shows the equilibrium price-dividend ratio. Here we see that there are some large fluctuations in portfolios early in the sample, which correspond to rather significant fluctuations in P/D ratios. However over time the portfolio share converges up toward unity, and the P/D ratio begins to look like i.i.d. fluctuations around a constant mean. Thus learning may lead to some additional variability early on, but the effects of learning vanish over time.

However with constant gain settings, no matter how small, we observe substantial fluctuations in beliefs, which results in substantial fluctuations asset prices. A representative sample is shown in Figure 5, which we discuss in the next section.

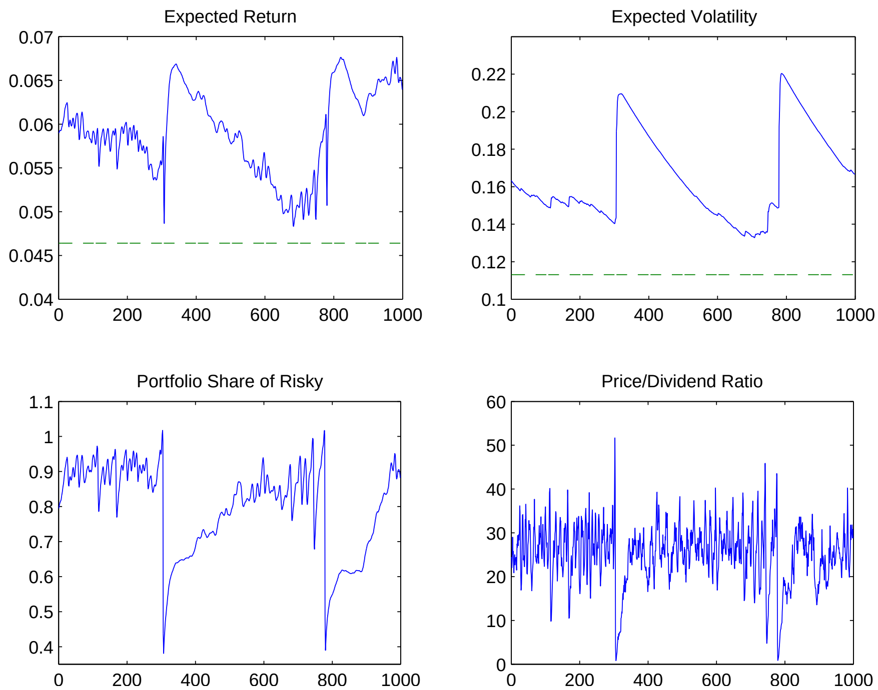

FIGURE 5. Representative simulation with constant gain $\varepsilon = 0.003$. The top left panel plots m_t, the top right plots s_t, the bottom left plots v_t, and the bottom right plots P_t/D_t

5. FINANCIAL INSTABILITY AND ESCAPE DYNAMICS

5.1. Episodes of Financial Instability

When agents discount past observations, we observe substantial fluctuations in asset prices and portfolio positions. In fact, the model with constant gain learning exhibits the fluctuations in beliefs and prices similar to those described by Minsky in his financial instability hypothesis. In Figure 5, we see periods of rapid increases in the price-dividend ratio driven by an increase in the portfolio share of assets going into stocks. But this boom period ends in a market crash characterized by a reduction in the portfolio share of assets and a rapid decline in the price-dividend ratio.

Clearly these asset price movements are driven by substantial fluctuations in beliefs. For example, we see expected returns increasing in the aftermath of a collapse and then declining thereafter as the economy experiences more stable times. Once asset prices begin their rapid ascent, expected returns increase once again, and plummet during the asset price collapse. Similarly agents' expected volatility spikes up rapidly in the periods around a crisis, then declines during more stable times, before increasing once again as prices rise. These changes are roughly consistent with much of the narrative of the financial instability hypothesis that we discussed above. We now provide a bit more detail on the mechanics of what drives the booms and busts in our model.

To focus in more detail on the mechanics of the model, we consider one single episode of a runup and collapse in asset prices, which is shown in Figure 6. The figure

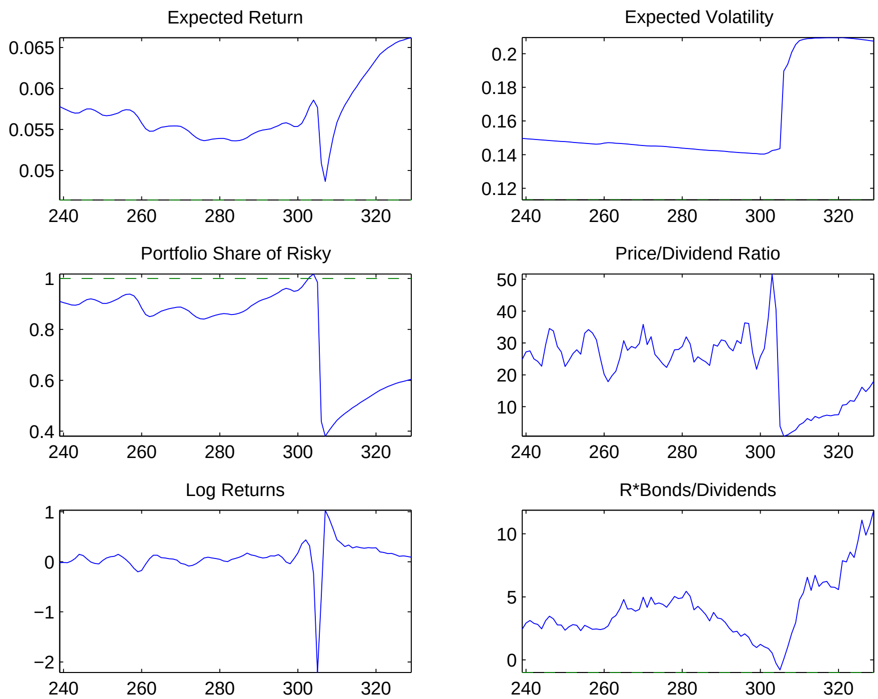

FIGURE 6. Detail of a simulation with constant gain $\varepsilon = 0.003$. The top left panel plots m_t, the top right plots s_t, the middle left plots v_t, the middle right plots P_t/D_t, the bottom left panel shows $\log R_t$, and the bottom right panel plots RB_{t-1}/D_t.

shows a closeup on the data in Figure 5, but also shows the log gross return and a key composite indicator, cum-interest bond holdings relative to dividends, RB_{t-1}/D_t. By our discussion above, we expect agents' beliefs (m_t, s_t) to converge to the SCE, which would also imply that the portfolio share v_t should converge to one. We observe the force of these mean dynamics in Figure 6 in the first several periods, up until around period 300. As in Minsky's story, this figure starts in the aftermath of a previous collapse. Volatility estimates were high, as agents had been through a period of large fluctuations, and thus they decline over this span. Mean estimates drift downward early on, as returns recover quickly following a price collapse and spike upward rapidly, then gradually drift downard. Throughout the aftermath of the crash and the convergence back to the SCE, price-dividend ratios remain relatively low. As volatility drifts downward and expected returns stabilizie or turn upward slightly, agents start gradually increase their share of wealth going into stocks. However around period 300 this starts to accelerate, and mean returns trend upward as agents increase the share of their portfolios going to risky assets. As this share v_t increases to near 1, bond holdings B_t decline and even go slightly negative, as the bottom right panel shows. That is, agents start to forget the past experience of rapid fluctuations, and so they become more willing to invest in stocks. As they do, they bid up asset prices and returns. Since agents are backward looking, these higher realized returns lead to increases in expected returns going forward. This drives the rapid runup in asset prices.

A variety of factors combine to end the boom and lead to the crash in asset prices. Although in general price fluctuations depend on agent's bond holdings, as these holdings shrink, prices are driven mainly by the fluctuations in dividends D_t. Therefore once the portfolio share v_t is high enough, negative shocks to dividends can have a significant impact on the stock price. So with enough wealth concentration in stocks, a string of negative shocks can lead to a substantial fall in asset prices.

In addition, as we have seen in Figure 3, as v_t gets close to or above unity, small changes in portfolio shares have very large impacts on prices. Thus near $v_t = 1$ small changes in beliefs, which lead to small reductions in v_t can have a big impact on prices. In Figure 6, we see that the volatility estimate s_t drifts downward around period 300 even as m_t starts to increase. However as prices begin their rapid ascent, these (positive) return fluctuations lead to an increase in the estimated volatility of returns, which leads to a fall in v_t. Thus even when prices are still booming, agents become more cautious about future returns and so may cut back on their holdings of stocks.

Finally, in this simulation the borrowing constraint binds in period 304, which means that prices are not able to increase as rapidly as they would otherwise. If this constraint were not imposed, we would have $RB_{t-1}/D_t < -1$, which as we saw above characterized Minsky's "speculative finance" regime. However such a situation is incompatible with equilibrium in our environment. Thus we must restrict v_t in this period. It is still above unity, so agents are borrowing in order to invest in stocks, but they cannot borrow as much as they would like. Thus agents had been anticipating a more rapid growth in prices, but the constraint limits this growth, which leads to a negative return suprise. The combination of these factors: negative dividend shocks, increases in volatility estimates, and restrictions on borrowing, are what lead to the end of the runup in asset prices. Although all of the factors are present here, the borrowing constraint need not bind in order for a price collapse to occur, as we see in some simulations below.

5.2. Escape Dynamics

[TO BE COMPLETED]

Even though beliefs tend to be drawn toward SCE, there are occasional "escapes" away from it.

Sequences of unlikely shocks lead agents to change beliefs, in turn they change their behavior, which then can lead to rapid further changes in beliefs and outcomes.

Can use large deviation theory to characterize the most probable escape paths. "If an unlikely event happens, it is very likely to happen in the most likely way."

In Williams (2001, 2009) I have shown previously that we can analyze escapes as a perturbation of the mean dynamics.

$$\begin{aligned} \dot{m} &= E[\log(Z_t(m,s))] - m + v_m \\ \dot{s}^2 &= E[\log(Z_t(m,s)) - m]^2 - s^2 + v_s \end{aligned}$$

where $m(0) = \mu, s(0) = \sigma$, and (v_m, v_s) force (m, s) to exit a neighborhood of (μ, σ). The most probable escape path can be found by solving a cost minimization problem to determine optimal (v_m, v_s). The minimized value $\bar{S}$ of the cost minimization problem determines the rate at which the escapes take place. In particular, as $\varepsilon \to 0$ time to escape τ^ε is exponential in $1/\varepsilon$ at rate $\bar{S}$:

$$\tau^\varepsilon \sim \exp(-\bar{S}/\varepsilon)$$

6. QUANTITATIVE RESULTS

[TO BE COMPLETED]

Thus far we have illustrated and discussed the mechanism by which learning can lead to financial instability. However the parameters we have used have been rather arbitrary, just serving to illustrate the properties of the model. We now present a calibrated version.

To calibrate the model, we use Robert Shiller's data set on the S & P 500 prices and dividends from 1871-2011. The empirical price-dividend ratio is shown in Figure 10. We calibrate the dividend process to match the mean and standard deviation of monthly dividend growth rates over this sample. This gives us $\mu = 0.001$ and $\sigma = 0.0146$, which correspond to an annual mean growth rate of 1.46% with standard deviation 5.06%. We choose preference parameters β and γ to match an annual risk free interest rate of 1%, thus a quarterly rate of 0.25%. More precisely, we choose the

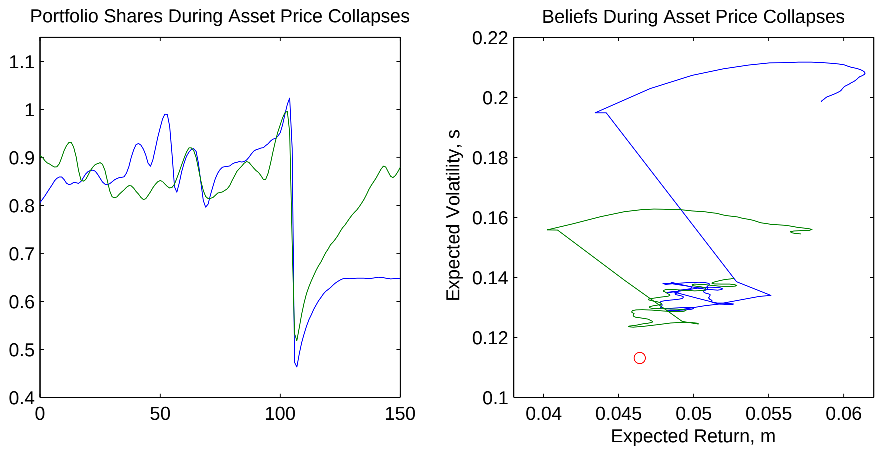

FIGURE 7. Portfolio shares v_t and beliefs s_t vs. m_t for periods surrounding asset price collapse, $\varepsilon = 0.004$.

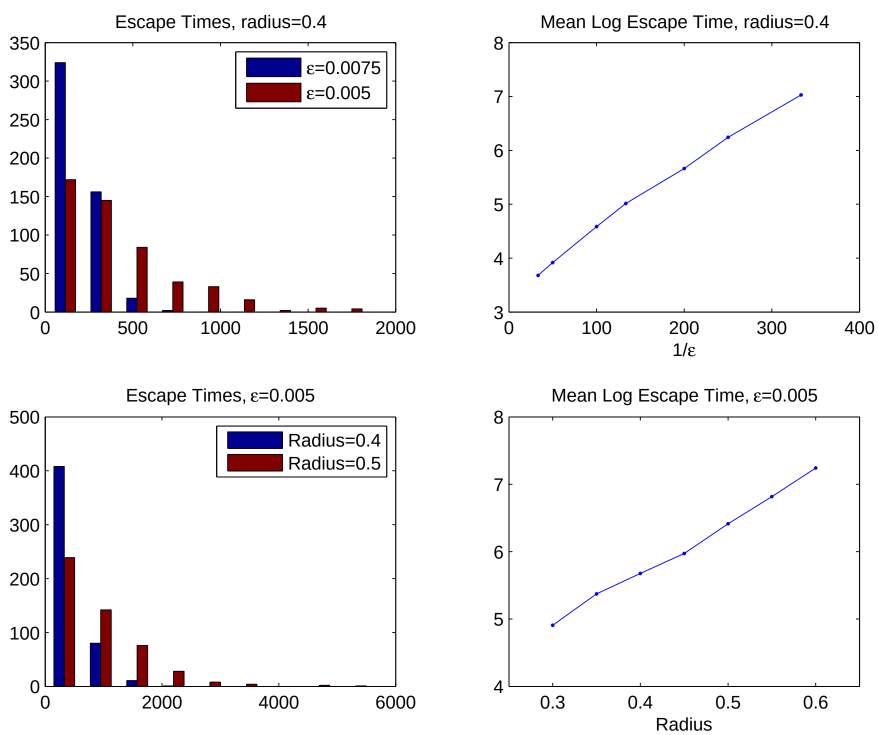

FIGURE 8. The distribution of escape times for differing values of the gain ε and the radius r.

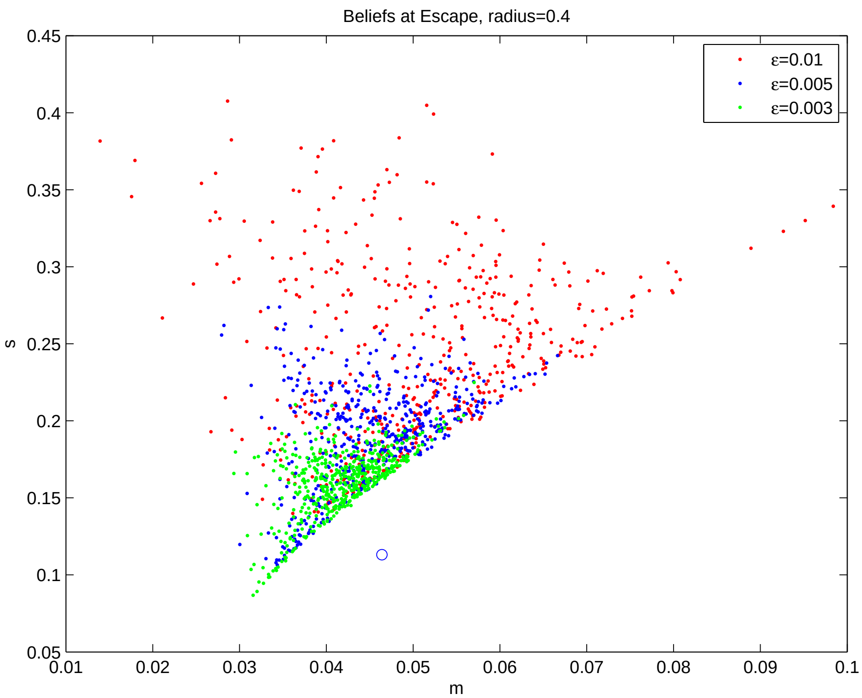

FIGURE 9. The distribution of beliefs m and s during escape episodes for different values of the gain ε.

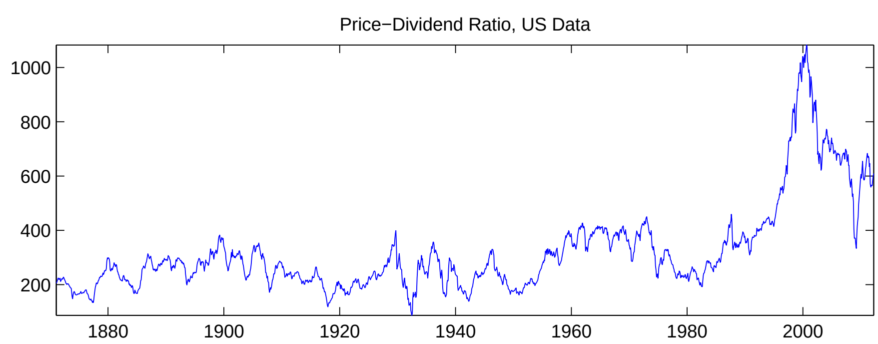

FIGURE 10. Price-dividend ratio (monthly) in US data, Shiller dataset.

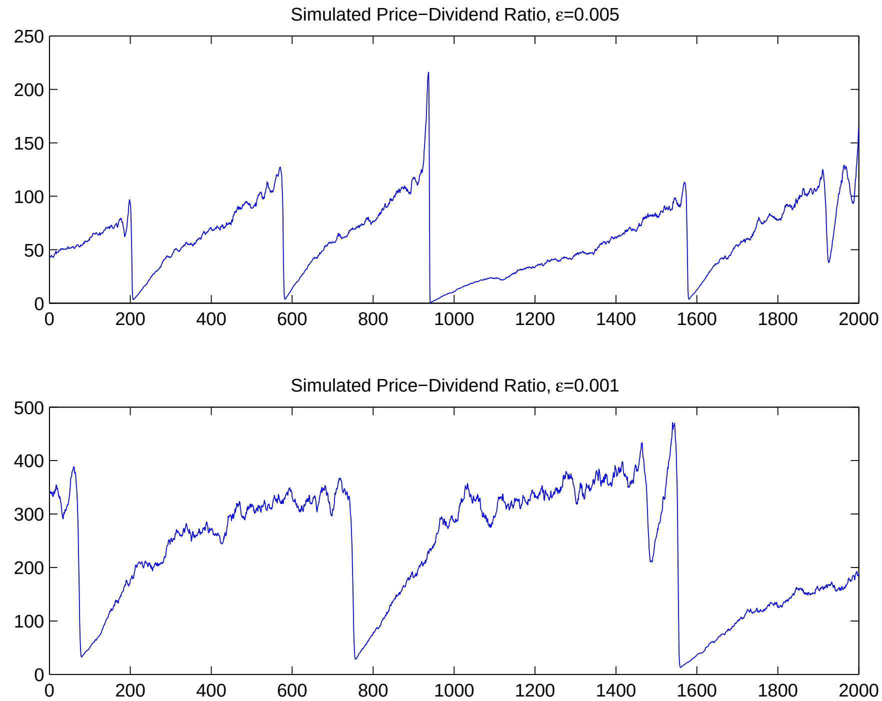

FIGURE 11. Simulated time series of the price-dividend ratio for two different values of the gain ε.

smallest whole number for γ which allows us to match the interest rate and still have $\beta < 1$. This gives us $\gamma = 9$, and $\beta = 0.998$.

Simulations of calibrated model: effect of gain ε.

Some suggestive evidence: variation of volatility estimates in the VIX

7. CONCLUSION

In this paper I have developed a simple model in which adaptive learning by investors leads to recurrent booms and busts in asset prices. The model captures aspects of Minsky's "financial instability hypothesis" in which periods of tranquility lead investors to increase their estimates of expected returns, driving up asset prices and hence realized returns. However once agents invest a significant fraction of their wealth in stocks, the economy becomes fragile and so small negative shocks can lead to large declines in prices. I have shown how this process recurs over time, and which features of the model which drive the boom-bust cycles. In particular, the speed of learning plays a key role.

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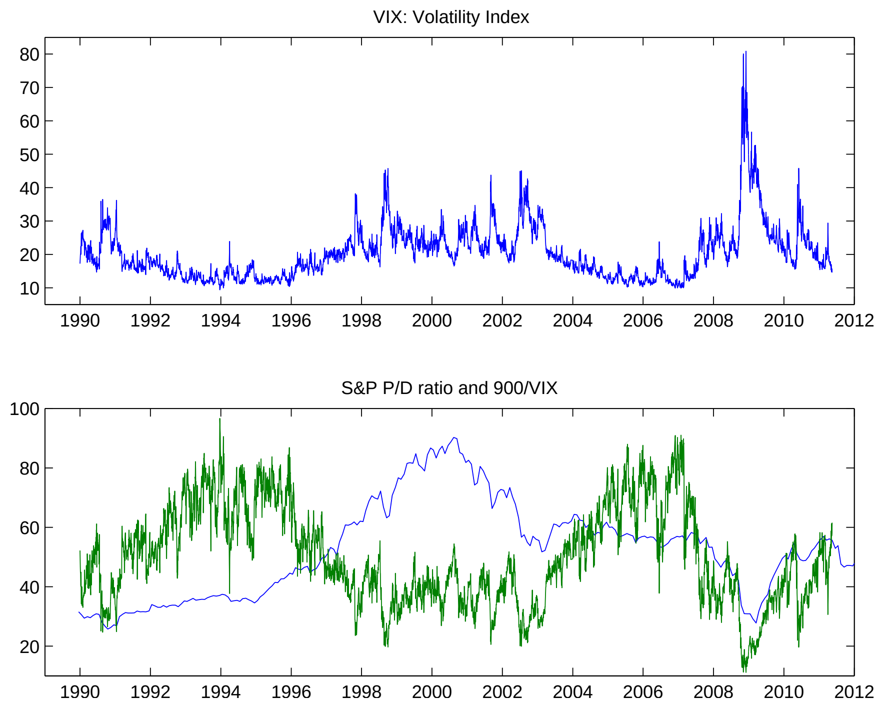

FIGURE 12. Top panel: S & P 500 Volatility Index (VIX) from CBOE. Bottom panel: S & P 500 price dividend ratio along with 900 times the inverse of the VIX.

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