The Continuous Wavelet Transform: A Primer

Maria Joana Soares[1]   Luís Aguiar-Conraria[2]

[1]Department of Mathematics and Applications/NIPE
[2]Department of Economics/NIPE
University of Minho, Portugal

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- Ramsey and Lampart (1998a and 1998b), Ramsey (1999 and 2002)
- Gençay, Selçuk and B. Withcher (2001a,b and 2005)
- Wong, Ip, Xie and Lui (2003)
- Lee (2004)
- Connor and Rossiter (2005)
- Crowley and Lee (2005)
- Fernandez (2005)
- Gallegati and Gallegati (2007)
- ...

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★ Review paper by Crowley (2007)

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Main contribution of the paper

The development of the concepts of wavelet multiple coherency and wavelet partial coherency.

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4. to describe how the transforms can be implemented in practice
5. to provide a user-friendly Matlab toolbox implementing the referred wavelet tools

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     - *Constructed example*
     - *Stock markets and oil prices*

Continuous Wavelet Transform

Definition

A function $\psi \in L^2(\mathbb{R})$ is a **wavelet** if it satisfies the following *admissibility condition*

$$C_\psi := \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|^2} d\omega < \infty,$$

where $\hat{\psi}$ is the Fourier transform of ψ.

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Definition

Given a function $x \in L^2(\mathbb{R})$, its **continuous wavelet transform (CWT)** with respect to the wavelet ψ is the function of two variables given by

$$\mathsf{W}_{\psi;x}(\tau, s) = |s|^{-1/2} \int_{-\infty}^{\infty} x(t) \overline{\psi\left(\frac{t-\tau}{s}\right)} dt$$

Wavelet Power and Wavelet Phase

- ψ complex-valued

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To be able to separate the phase and amplitude information of a time-series, we must use a complex wavelet (e.g. a Morlet wavelet).

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To study the relationship between two time series x and y, we can use the following generalizations of the basic wavelet tools: cross-wavelet transform (and cross-wavelet power), complex wavelet coherency and phase-difference.

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- The **cross-wavelet transform (XWT)** of two time-series x and y is defined as

$$\mathrm{W}_{xy} = \mathrm{W}_x \overline{\mathrm{W}_y},$$

  where W_x and W_y are the wavelet transforms of x and y, respectively.
- The **cross-wavelet power** is the absolute value of the XWT.

Complex Wavelet Coherency

Definition

Given two time-series x and y we define their **complex wavelet coherency** ϱ_{xy} by:

$$\varrho_{xy} = \frac{S\left(\mathsf{W}_{xy}\right)}{\left[S\left(|\mathsf{W}_x|^2\right) S\left(|\mathsf{W}_y|^2\right)\right]^{1/2}},$$

where S denotes a smoothing operator in both time and scale (e.g. convolution with appropriate windows).

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$\star$ Phase-difference only meaningful when coherency is high.

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Idea

Formulas to compute wavelet multiple coherency and wavelet partial coherency can be obtained by simply adapting formulas for multiple correlation and partial correlation, respectively.

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(Reference: Classical book by Kendall and Stuart, *The Advanced Theory of Statistics* (1966)).

Notations

Let p time series $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p$, be given.

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- $\boldsymbol{q}_j := \{2, \ldots, p\} \setminus \{j\}, \quad (2 \leq j \leq p)$

Multiple and Complex Partial Wavelet Coherencies

Definition

The **multiple wavelet coherency** between the series $\boldsymbol{x_1}$ and all the other series $\boldsymbol{x_2}, \ldots, \boldsymbol{x_p}$ will be denoted by $R_{1\,(\mathbf{q})}$ and is given by

$$R_{1\,(\mathbf{q})} = \sqrt{1 - \frac{\mathscr{S}^d}{\mathsf{S}_{11}\,\mathscr{S}^d_{11}}}$$

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Definition

The **complex partial wavelet coherency** of $\boldsymbol{x}_1$ and $\boldsymbol{x}_j$, $2 \leq j \leq p$, (controlling for all the other series) will be denoted by $\varrho_{1\,j.\boldsymbol{q}_j}$ and is given by

$$\varrho_{1\,j.\boldsymbol{q}_j} = -\frac{\mathscr{S}^d_{j1}}{\sqrt{\mathscr{S}^d_{11}\,\mathscr{S}^d_{jj}}}.$$

Partial Wavelet Coherency and Partial Phase-Difference

We can write the complex partial wavelet coherency $\varrho_{1\,j.\mathbf{q}_j}$ in polar form:

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The angle $\phi_{1\,j.\mathbf{q}_j}$ of the complex partial wavelet coherency is called the **partial phase-difference** of $\boldsymbol{x}_1$ and $\boldsymbol{x}_j$, given all the other series.

Formulas in Terms of Complex Coherencies: Example

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For example, in the case where we just have three series $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$, the multiple wavelet coherency and the complex partial wavelet coherency are given by the following "more familiar" formulas:

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$$R^2_{1(2\,3)} = \frac{R^2_{12} + R^2_{13} - 2\Re\left(\varrho_{12}\,\varrho_{23}\,\overline{\varrho_{13}}\right)}{1 - R^2_{23}}$$

$$\varrho_{1\,2.3} = \frac{\varrho_{12} - \varrho_{13}\overline{\varrho_{23}}}{\sqrt{\left(1 - R^2_{13}\right)\,\left(1 - R^2_{23}\right)}}$$

Comovement of Stock Returns

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- King, Sentana and Wadhwani (1992)
- Forbes and Rigobon (2002)
- Brooks and Del Negro (2004)
- Rua and Nunes (2009), Rua (2010) $\hookrightarrow$ analyse the comovement in the time-frequency space, by resorting to wavelet analysis.

Stock Markets: Who is the Leader?

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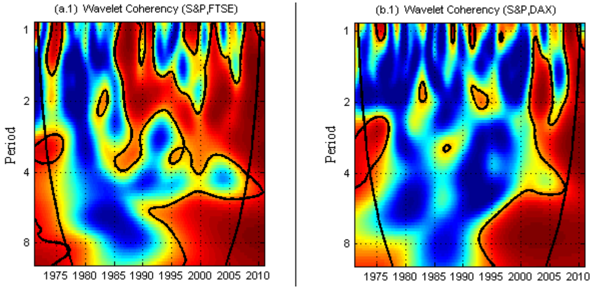
Here, we only show the results for S&P/FTSE and for S&P/ DAX.

Wavelet Coherencies

Left: S&P (US) and FTSE (UK); Right: S&P (US) and DAX (Germany)



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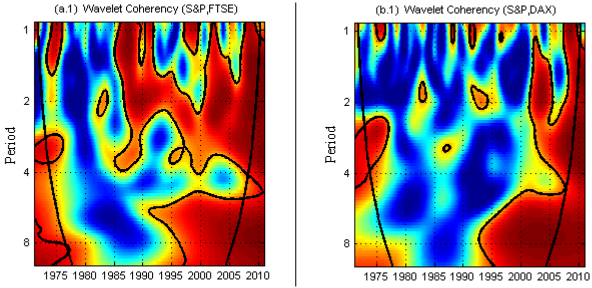

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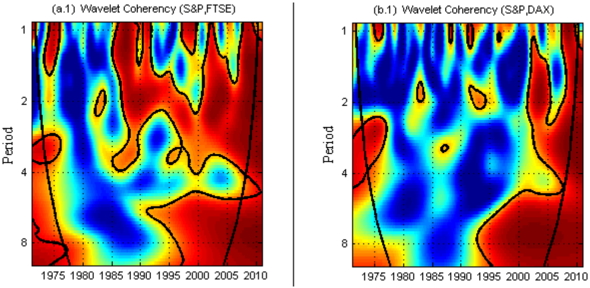

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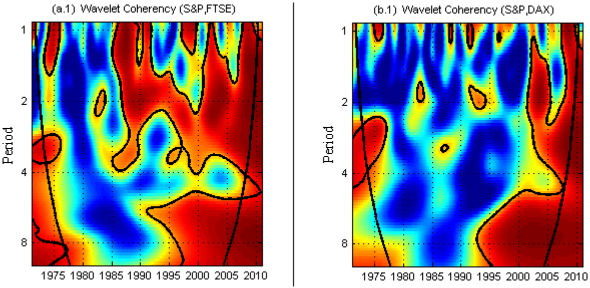

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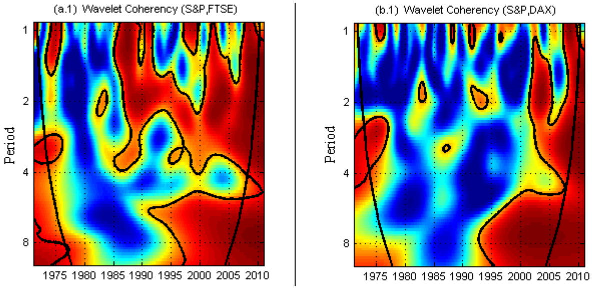

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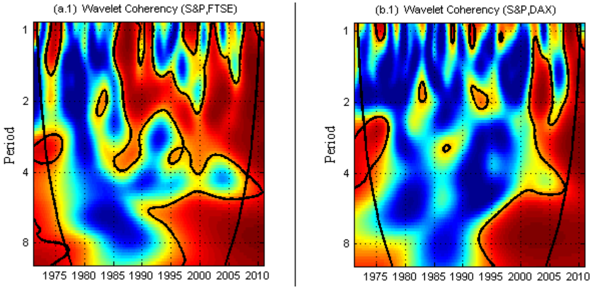

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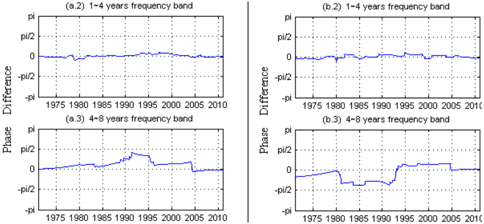
Wavelet Phase-Differences

Left: S&P (US) and FTSE (UK); Right: S&P (US) and DAX (Germany)

An interesting conclusion (not present in Rua and Nunes (2009)) arises when one looks at the phase-differences.



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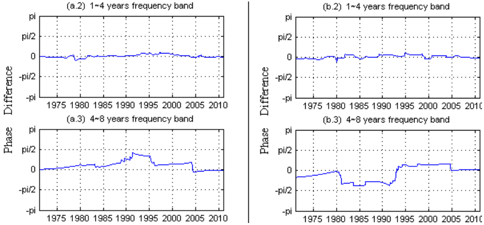

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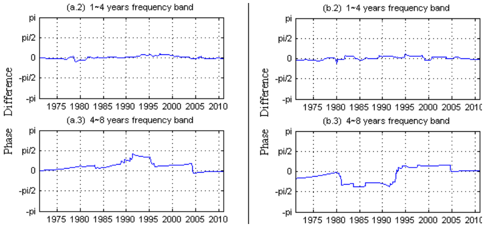

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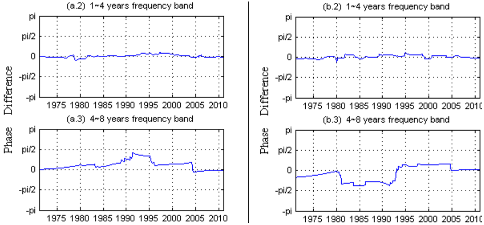

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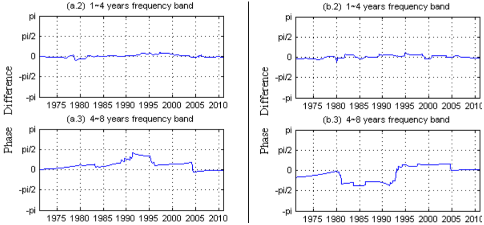

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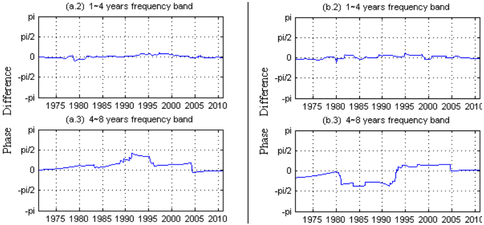

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  ($*$) In fact, until early 1990s, phase-difference between S&P and DAX is negative; but this is a period of low coherency $\Rightarrow$ phase-difference not meaningful.

Constructed Example

Consider three time series that share two common cycles, with some leads and delays:

$$\begin{cases} x_t = \sin\left(\frac{2\pi}{3}t\right) + 3\sin\left(\frac{2\pi}{6}t\right) + \varepsilon_{x,t} \\ y_t = 4\sin\left(\frac{2\pi}{3}(t+\frac{5}{12})\right) + 3\sin\left(\frac{2\pi}{6}(t-\frac{10}{12})\right) + \varepsilon_{y,t}, \quad t=0,\frac{1}{12},\frac{2}{12},\ldots,50. \\ z_t = 3\cos\left(\frac{2\pi}{6}t\right) \end{cases}$$

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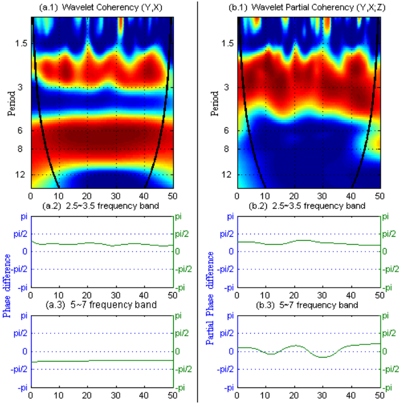
- Series x_t and y_t share 3-year and 6-year cycles; while y_t leads x_t in the shorter period cycle, the opposite happens in the longer period cycle
- The third variable, z_t, shares the 6-year cycle both with x_t and y_t.

Wavelet Coherency *vs* Partial Wavelet Coherency



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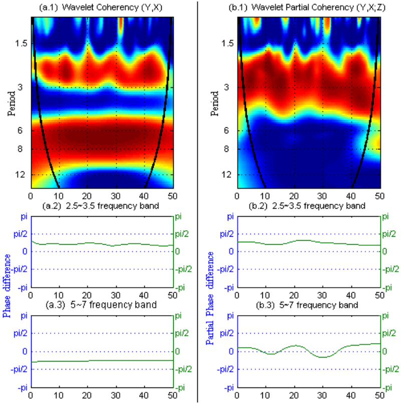

- Left: Wavelet coherency and phase-difference

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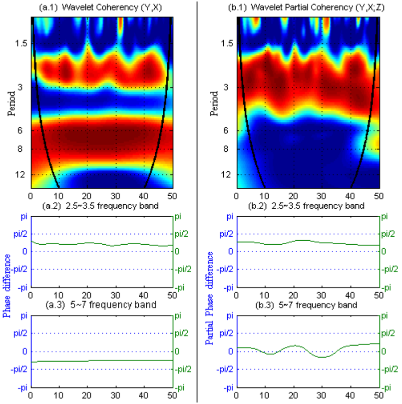

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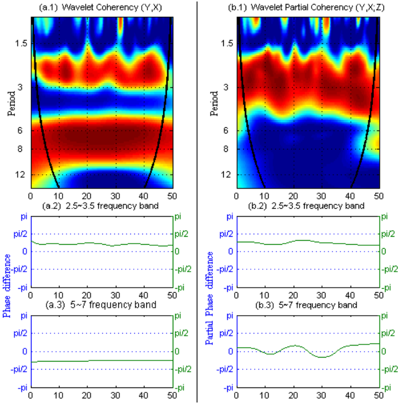

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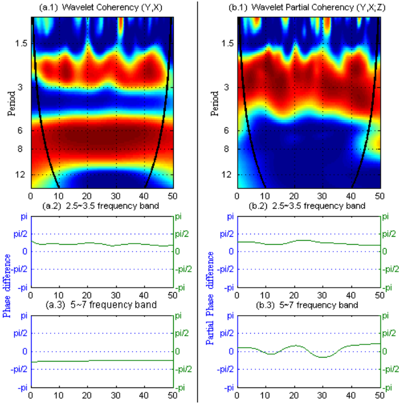

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- Right: Partial wavelet coherency and partial phase-difference, after controlling for z_t $\hookrightarrow$ capture only the 3-year cycle relation.

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  - however, an increase in global aggregate demand will result in both higher real oil prices and higher stock prices.

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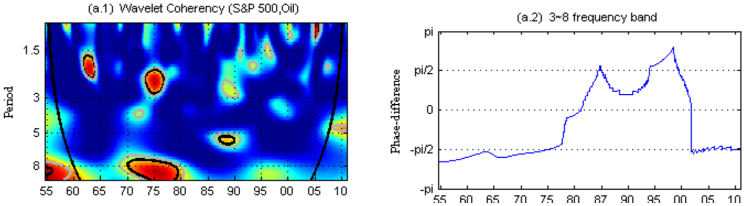
Data

We gathered monthly data, running from July 1954 to December 2010, on several variables:

- S&P-500 Stock Returns (S&P)
- Oil Prices (Oil),
- Industrial Production (IP)
- CPI inflation (π)
- Effective Federal Funds Real Rate (r)

Wavelet Coherency and Phase-Difference
(S&P and Oil)



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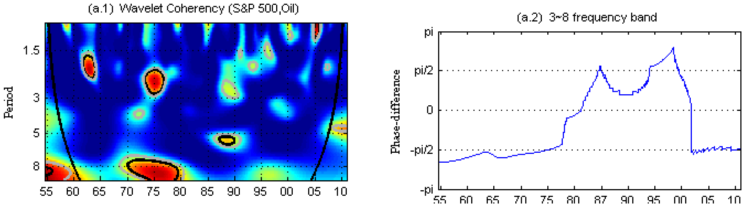

- Regions of high coherency are very scarce (phase-difference is not meaningful in these situations)

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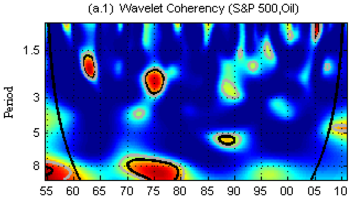

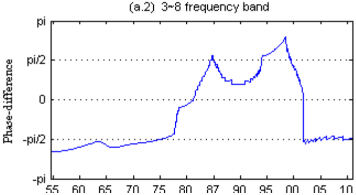

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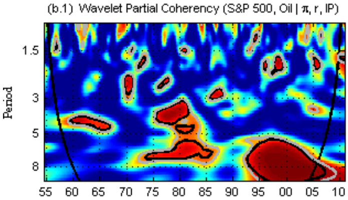
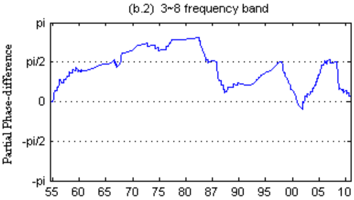
  $\overset{?}{\Longrightarrow}$ no relevant linkages between oil prices and the stock market.

Partial Wavelet Coherency and Partial Phase-Difference
(S&P and Oil, controlling for the other variables)





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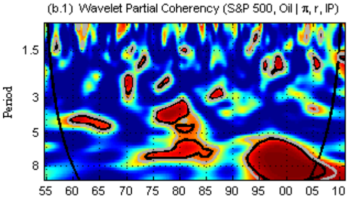

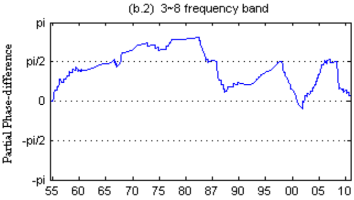

- Regions of high coherency between the mid-1970s and mid-1980s along the $3 \sim 8$ years period frequency band and again, at lower frequencies, after the early 1990s.

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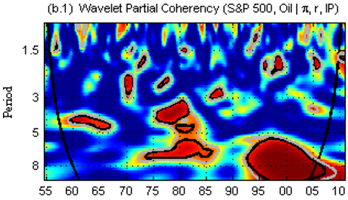

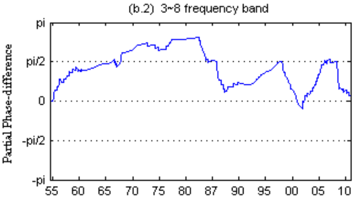

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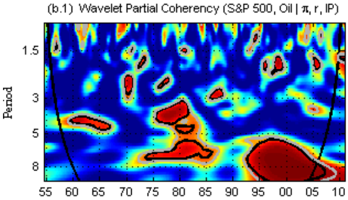

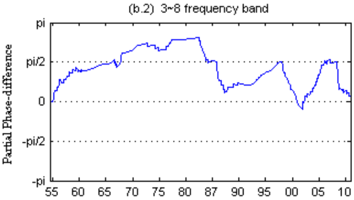

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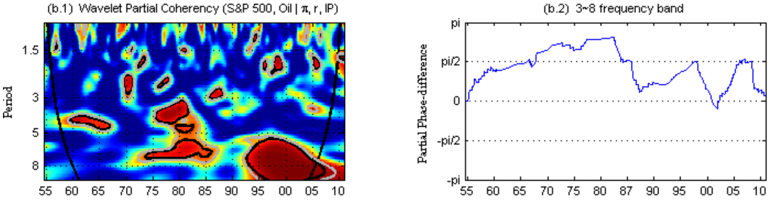

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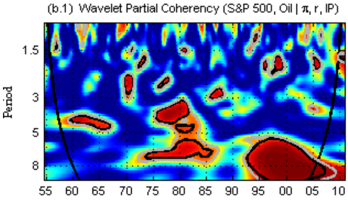

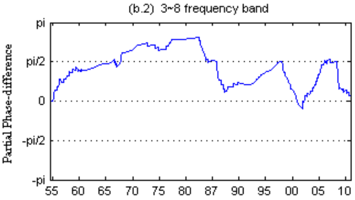

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Partial Wavelet Coherency and Partial Phase-Difference

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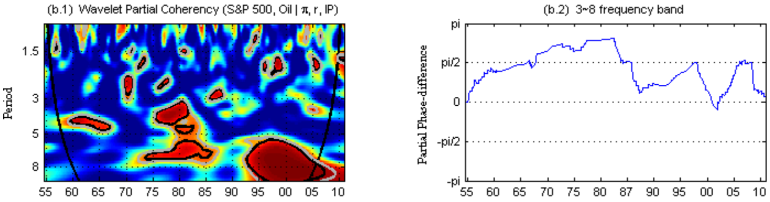

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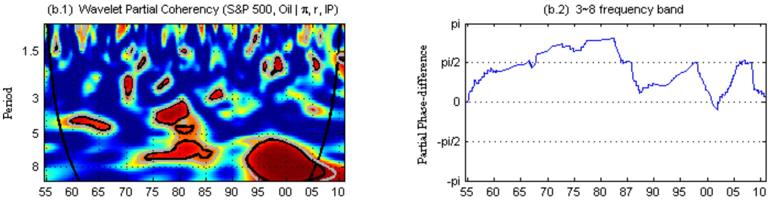

- In the 1990s and 2000s regions of high coherency are at lower frequencies and the phase-difference is consistently between 0 and $\pi/2$

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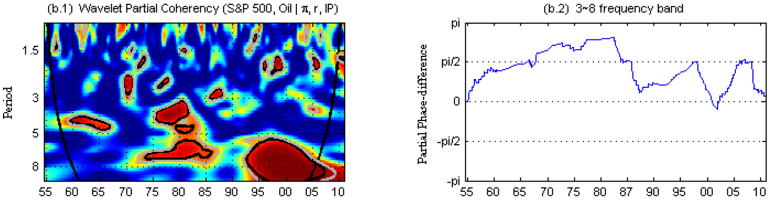

- In the 1990s and 2000s regions of high coherency are at lower frequencies and the phase-difference is consistently between 0 and $\pi/2$ $\Rightarrow$ the series are now in phase, with the stock market leading

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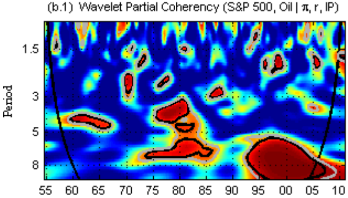

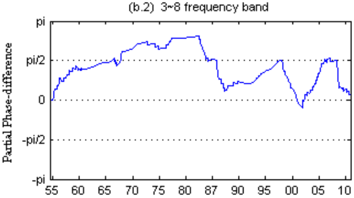

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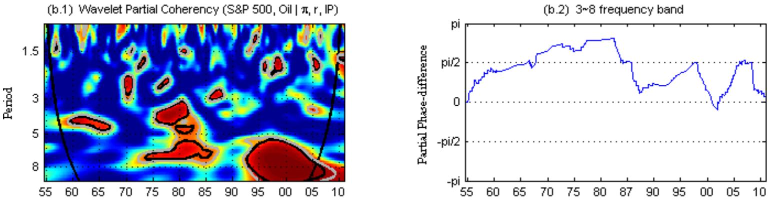

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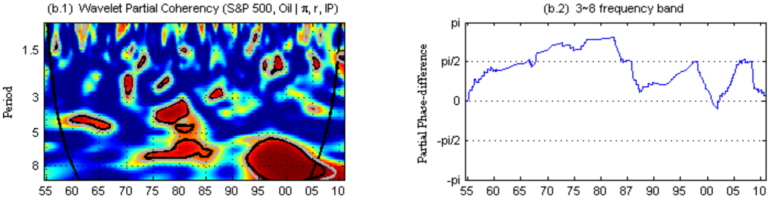

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