Retrospective: Presentation Slides

A Decade's Progress

James B. Ramsey

New York University

October 2011

Introduction

A personal meander through wavelet land
OUTLINE

- Review of Bases in Functional Approximation
- Comparison of Fourier and Wavelet Projections
- Succinct Discussion of Empirical Use of Wavelets
- Conclusions

Note: All the approximating systems reviewed are characterized by reference to vector spaces

Regressands are projections onto appropriate bases

Consider Fourier Series:

$$\mathrm{Re}^{i2\pi ft} \text{ or alternatively expressed } : \quad 1, \sin(k\omega t), \; \cos(k\omega t), \; k = 1,2,3,...$$

Functional Representation and Bases Spaces

An aside on "orderings" and regressions

Usual practice: regressand in an N-K dimensional subspace.

Consider simple non-linear differential function

$$
\begin{aligned}
y &= f(x|\theta) = f(a_1|\theta) + f^1(a_1|\theta)(x - a_1) + \frac{f^2(a_1|\theta)(x - a_1)^2}{2!} \\
&+ \frac{f^3(a_1|\theta)(x - a_1)^3}{3!} + \frac{R(\xi)}{4!}
\end{aligned}
$$

Solution by least squares for data inputs:

$$\min_{\theta, a_1}\{\Sigma_1^N (y_i - f(x_i|\theta))^2\}$$

$$\begin{aligned} & y_i \ \{x_i, x_i^2 \ x_i^3\} \\ i &= 1, 2, 3....N \end{aligned}$$

Estimators obtained by projections onto space spanned by regressors more suitable spaces can be used instead. Different choices generate different parameterizations.

An alternative ancient procedure:

$$\{1, t^2, t^3, t^4, t^5, \ldots\}$$

or its orthogonal components
Projections of regressands on these monomials can only be EXPLORATORY
Exponential and power bases

$$\{e^{\lambda_1 t},\ e^{\lambda_2 t},\ e^{\lambda_3 t}.....e^{\lambda_k t}\}$$

$$\{t^{\lambda_1},\ t^{\lambda_2},\ t^{\lambda_3},\ ....t^{\lambda_k}\}$$

Fourier Bases

Usual practice:

$$
\begin{aligned}
& 1, \sin(k\omega t), \cos(k\omega t), \\
k &= 1, 2, 3, \ldots
\end{aligned}
$$

The approximating sequences are:

$$
y = f(t) \cong \sum_{k=1}^{K} c_k \phi_k
$$

$$
c_k = \int f(t)\phi_k(t)dt
$$

We gain in understanding if:

- Number of coefficients are few i.e. K is small;
- "f" is restricted to functions of a compatible class e.g. trig functions.
- Continuous functions not differentiable are **not well fitted.**
- Gibbs Phenomenon! Think box function.
- Economy of coefficients obtained by windows;

$$\hat{h}(\omega) = \frac{1}{2\pi} \sum_{-M}^{M} \lambda(s) \hat{R}(s) \cos(s\omega)$$

$$\lambda(s) = \begin{cases} 1, & |s| \leq M \\ 0, & |s| > M \end{cases}$$

Distant correlations are smoothed, short term oscillations are enhanced.

Spline Bases

A standard definition of B-spline is:

$$S(t) = \sum_{k=1}^{m+L-1} c_k B_{k,(t,\tau)}$$

- S(t) is the spline approximation
- $\{c_k\}$ are the coefficients of the projection
- $B_{k,(t,\tau)}$ is the B-spline function at position k, with knot structure, τ.
- τ designates the number of knots, L, and their distribution
- Subintervals are modeled in terms of polynomials of degree m
- B-splines easily fit locally complex functions.

Wavelets

Wavelets are based on functions of type:

- $\Phi(t)$ otherwise known as father
- $\Psi(t)$ otherwise known as mother
- Both are well localized in terms of time and scale
- Inner product Φ with respect to f is a low pass filter that produces a moving average.
- The inner product Ψ with respect to f is a high pass filter producing moving differences.
- Separately high and low pass filters are not invertible.
- Together they separate the signal into frequency bands.

Father & mother wavelets satisfy similar scaling and translation laws:

$$\phi_{J,k}(t) = 2^{-\frac{J}{2}}\phi\left(\frac{t-2^J k}{2^J}\right)$$

$$\psi_{J,k}(t) = 2^{-\frac{J}{2}}\psi\left(\frac{t-2^J k}{2^J}\right)$$

Functions underlying Ψ, Φ are related through the implied filter banks.
Recall Φ yields averages while Ψ yields moving differences;
but defined with respect to the same function.

Atoms

However the coefficients are obtained they approximate f(t) from an over sampled collection of atoms known as a "dictionary".

OBJECTIVE

Seek the best M atoms for a given f(t) out of a dictionary of P atoms
Three standard methods for choosing M:

- Matching pursuit: a greedy and suboptimal procedure
- Best basis algorithm
- Basis pursuit: synthesis of f(t) in terms of ϕ_i is under determined

Choosing a Good Basis

- Linear independence, unique representation
- Completeness, any f(t) represented
- Adding vectors destroys independence
- Removing vectors destroys completeness

The experienced Waveletor chooses his filter bank, i.e.:Ψ, Φ :

- To optimize continuity
- To optimize number of zero moments
- Investigates symmetry
- Shape of the wavelet function, e.g. Haar, Gaussian, Mexican Hat Daubechies series, Mallat series,etc..
- We seek a "scarce transformation matrix", i.e. most coefficients are zero.

Waveform Dictionaries

Consider $g_\gamma(t)$

$$
\begin{aligned}
g_\gamma(t) &= \frac{1}{\sqrt{s}} g\left(\frac{t-u}{s}\right) e^{i\xi t} \\
\gamma &= (s, u, \xi)
\end{aligned}
$$

for any s, u, ξ

- $1/\sqrt{s}$ normalizes the norm of $g(t)$ to 1
- $g(t)$ is centered at the abscissa u
- Energy is concentrated in the nbhd of u, size is proportional to s
- The Fourier transform is centered at the frequency ξ
- Its energy is concentrated in the nbhd of ξ and size is proportional to $\frac{1}{s}$
- Matching pursuit is used to determine the values of the coefficients.

Foreign Exchange Example

Using raw tick by tick data for a year obtained evidence of low frequencies that wax and wane in intensity.
Most of the energy of the system is localized in energy frequency bursts.
Only one hundred top structures provide a good fit, but the top one hundred varies from sample to sample unpredictably.

Instrumental Variables

Consider the errors in variables problem.
Traditionally solution is by assumption.
Wavelets can resolve the issue.

$$\begin{aligned} y_i &= y_i^* + \varepsilon_i \\ x_i &= x^* + \eta_i \\ z_i &= z_i^* + \omega_i \end{aligned}$$

Starred items structural, others random
Because we can separate structural and random components:
we obtain ideal instruments.

Structural Rates and Outlier Detection

Given the filter bank generated by Haar wavelet function provides a unique analysis of temporary structural change.

Comments on Forecasting

Wavelets provide a convenient refinement and flexibility not available using conventional methods.

But wavelets indicate that forecasting is more subtle than conventionally realized.

Wavelets indicate that forecasts need to be conditional on scale either separately or in groups.

Further developments along these lines would be fruitful.

Conclusions

Emphasizing the role of basis spaces insightful comparisons between Fourier series, B-Splines and wavelets were evaluated. This analysis was extended to Waveform dictionaries.
Brief reviews of a wide variety of Economic examples were examined:
Wavelets provide a useful generalization of Fourier transforms.
Wavelets in empirical work provide interesting insights into the structure of economic relationships.
In brief, wavelets:

- solve the instrumental variable problem
- are robust to specification errors.
- generalize the concept of forecasting
- are adaptable to complex basis spaces