Frequency-domain analysis of debt service in a macro-finance model of the euro-area

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¹Banque de France. The views expressed in the following are those of the authors and do not necessarily reflect those of the Banque de France.
To propose a novel approach based on spectral analysis and Macro-Finance Affine Term Structure Models (MF-ATSM) that may contribute to a better taking into account of fiscal-insurance principles.
Motivation

What is debt management?

Figure: Debt management
Motivation
Debt portfolio in G7 countries (1/2)

Figure: Average public-debt maturity in G7 countries

Source: OECD and national DMOs
Motivation
Debt portfolio in G7 countries (2/2)

Figure: Share of inflation-linked bonds in total sovereign bond outstanding

Source: OECD and national DMOs
Motivation

About the relevance of fiscal insurance

- Faraglia, Marcet and Scott (2009): previous attempts of normative analysis (tax-smoothing or fiscal insurance) regarding optimal debt management lead to liabilities structures that are dramatically different from observed ones.

- When you try to derive optimal debt structures from normative frameworks, the results...
  - ... are not robust,
  - ... are often based on unrealistic government’s abilities (e.g. to costlessly repurchase all outstanding debt at any period),
  - ... and often suggest that optimal asset positions are huge multiples of GDP (Buera and Nicoloni, 2004)

Tax smoothing:

The governments have an incentive to smooth taxes across time in order to minimize the excess burden stemming from distortionary taxes.
Motivation

About the relevance of fiscal insurance

- Should debt managers ignore tax-smoothing or fiscal insurance considerations?
  - About the extreme (and often practically unfeasible) results recommended by normative analysis: these may be interpreted as “direction” (Barro, 1999 and 2003, Bohn, 1990);
  - If not convinced about theoretical underpinnings of tax-smoothing, think of fiscal insurance as a form of Asset-Liability Management (ALM)

- While fiscal insurance principles do not constitute a primary concern for public debt managers, some DMOs have taken them into consideration at some point (see e.g. Coeuré, 2004 or Dudley, 2007)
From a practical point of view, what do we need?

The ability of public debt managers to include fiscal insurance principles into account depends on the availability of a framework that

- comprehends the joint modeling of macroeconomic variables and asset prices dynamics
- remains tractable (in order to make it possible to simulate a large number of strategies)
- models various funding-instrument prices

⇒ Associated with Macro-finance affine term-structure models, spectral analysis proves to be a relevant tool to represent and measure the funding-strategies performances
Methodology

The frequency-domain approach

On the long run, debt service is high when the primary deficit is high

→ no long-run budget smoothing.
Methodology

The frequency-domain approach

On the short run, debt service is high when the primary deficit is low → short-run budget smoothing.
ATSM are factor models of the yield curve, so only a small number of sources of variation underlie the pricing of the entire term structure of interest rates (Duffie and Kan, 1996)

ATSM impose no-arbitrage restrictions: the dynamic evolution of yields over time and across state of nature is consistent with the cross-sectional shape of the yield curve (after accounting for risk)

In macro-finance models (pioneered by Ang and Piazzesi, 2003), the factors are closely linked with macroeconomic variables
Methodology
Macro-Finance ATSM (2/2)

- Ingredients:
  - The dynamics of factors (observable or not) $F_t$
  - The specifications of the pricing kernel $m_{t+1}$ (or stochastic discount factor, SDF)

- Price $P_t$ of an asset providing the payoff $g(F_T)$ in period $T$:

$$P_t = E_t \left( m_{t+1}m_{t+2} \ldots m_{T-1}m_Tg(F_T) \right)$$

- In Affine Term Structure Models (ATSM), zero-coupon bond yields of maturity $\tau$ are given by:

$$\begin{bmatrix}
i_{1,t} \\
i_{2,t} \\
\vdots \\
i_{n,t}
\end{bmatrix} = A + BF_t$$
Application

- The model is exploited in order to analyse the implications of specific funding strategies on the debt-service variability
  - The model makes it possible to analyze the pro- or counter-cyclicality of debt service that are associated with a given financing strategy
  - Cyclicality of debt service is key if debt management is aimed at fiscal insurance

- Financing-strategy performances are analysed in the frequency domain
  - Spectral analysis decomposes covariances into components at different frequencies
  - The approach leads to a comprehensive view of the variable (co-)dynamics
Figure: Frequency-domain properties of macroeconomic variables
Application
Debt service analysis

- Once the frequency-domain representations of some variables are known, it is straightforward to carry out the frequency analysis of linear combinations of these variables and their lags.
- This is exploited here in order to analyze the frequency domain properties of debt servicing.

Examples

If (a) one chooses to fund an amount $D$ of debt on a monthly basis and (b) the debt stock is not fed back by interest payments, then the debt service of the rolling strategy is proportional to $i_{1,t-1}$.

Suppose that funding is based on 3-month bills and that the debt redemptions are evenly spread over the quarter, then interest payments are proportional to $1/3 \times (i_{3,t-1} + i_{3,t-2} + i_{3,t-3})$. 
Application
Debt service analysis

- Assumptions:
  - Potential output $GDP^*_t$ and bond issuances $I_t$, are assumed to grow at a constant positive pace of $g\%$

- Defining a financing strategy consists in determining what kinds of bonds are issued at each period in order to face the financing needs of government (see e.g. Bolder, 2003)

- Only simple funding strategies, that consist in issuing a constant fraction of different types of bonds at each period, are considered

- In this context, the debt-to-$GDP^*$ ratio is constant
Figure: Frequency-domain properties of debt service

Debt service

- Strat. 7-yr
- Strat. 5-yr
- Strat. 2-yr
- Strat. 6-mth

Debt service

- Strat 10-year indexed
- Strat 10-year nominal

Cycles (in months)
Application
Variance decomposition of debt service

- The unconditional variance of debt charges is more important when shorter-maturity bonds are issued.
- The lower the maturity of bonds issued, the larger the share of debt service variations explained by business-cycle components.
- At all frequencies, debt service is more variable when ILBs are issued.
- The share of the debt-servicing variance explained by high-frequencies components tends to be higher when funding is based on ILBs.
- Once those components whose period is lower than a year are removed, the differences in variances between nominal and inflation-linked bonds are less dramatic.
Figure: Debt service and real activity: cospectrum and phase

Cospectrum

- Strat. 6-mth
- Strat. 2-yr
- Strat. 10-yr indexed
- Strat. 10-yr

Phase

expressed as a share of cycle

in phase
Application
Covariance decomposition (debt service and real activity)

- The shorter the maturity of issued bonds, the larger the covariance between debt service and real activity.
- While the correlation between real activity and debt service is negative when long-term nominal bonds (maturities > 5 yrs) are issued, it is positive if shorter-term nominal bonds or ILBs are issued.
- When focussing on business-cycle frequencies, the correlations between real activity and interest payments are respectively equal to 0.17 and -0.18 for strategies that are based on 10-year ILBs and 10-year nominal bonds.
- When issuing bills or ILBs, the phase difference (between debt service and real activity) is lower than a quarter of cycle, which makes debt charges pro-cyclical.
Conclusion, improvements and extensions

- The approach suggests that issuing nominal short-term bonds or ILBs results in pro-cyclical debt service, which may contribute to smoothen the budget balance over the business cycle.
- Limits of the approach:
  - All series are assumed to be stationary, so are the simulated yields...
  - Funding requirements are exogenous.
- The analysis could be carried out on a larger number of –possibly more sophisticated– strategies.
- Optimization, for instance in a mean-variance framework: given relative preferences for “cost” and “risk”, what should be the optimal debt structure?...
Thank you!

Model

Specifications: The short-term rate

- The central bank sets the one-period nominal interest-rate \((i_{1,t})\), according to

\[
    i_{1,t} = \delta_0 + \underbrace{L_t}_{\text{medium-run inflation}} + \underbrace{S_t}_{\text{cyclically-responsive component}}
\]

- where \((\sim\text{Taylor rule})\)

\[
    S_t = \rho_S S_{t-1} + (1 - \rho_S) [g_y y_{t-1} + g_\pi (\pi_{t-1} - L_{t-1})] + \varepsilon_{S,t}
\]

- and

\[
    L_t = \rho_L L_{t-1} + (1 - \rho_L) \chi \pi_{t-1} + \varepsilon_{L,t}
\]
Model

Specifications: Phillips and IS curves

- Phillips curve:
  \[ \tilde{\pi}_t = L_t + \alpha_\pi (\tilde{\pi}_{t-1} - L_{t-1}) + \alpha_y y_{t-1} + \varepsilon_{\pi,t} \]

- Investment-saving (IS) curve:
  \[ y_t = \beta_y (L) y_{t-1} - \beta_r (i_{1,t-1} - E_{t-1}(\tilde{\pi}_t)) + \varepsilon_{y,t} \]

- These first equations form a VAR that reads:
  \[ F_t = \Psi F_{t-1} + \Sigma \varepsilon_t \]

- The stochastic shocks \( \varepsilon_t \) are assumed to be normally \( i.i.d. \).
Model
Specifications: SDF and price of risk

- The stochastic discount factor (SDF) is given by

\[ m_{t+1} = \exp \left[ -\frac{1}{2} \Lambda_t' \Lambda_t - \Lambda_t' \varepsilon_{t+1} - i_{1,t} \right] \]

where the price of risk is given by

\[ \Lambda_t = \lambda_0 + \lambda_1 F_t \]

- If \( b_{j,t} \) denote the price of a nominal \( j \)-period zero-coupon bond, it is given by:

\[ \ln b_{j,t} = \overline{A}_j + \overline{B}_j' F_t \]

where \( \overline{A}_j \) and \( \overline{B}_j \) are calculated numerically by solving a series of linear difference equations.

- All yields are assumed to be observed with a measurement error.
Model
Data: Macroeconomic data

- The data cover the period from January 1999 to June 2009 at the monthly frequency (Eurozone data)
- Real activity is represented by the first principal component of a set of 5 business and consumer confidence indicators (source: European Commission qualitative survey): industrial, construction, retail trade, service and consumer confidence
- The inflation series (HICP excl. tobacco, source: Eurostat) is seasonally adjusted using Census X12
- Inflation forecasts of the ECB *Survey of Professional Forecasters* are included amongst the estimation series (3 additional measurement equations: \( SPF \text{ forecasts} = \text{model-implied expectations} + \text{error term} \), for 1-, 2- and 5-year horizons)
Model
Data: Interest rates

- Zero-coupon nominal and real (end-of-month) interest rates are derived from
  - Government-bond yields (bootstrap on a spline-smoothened French TEC yield curve) and
  - inflation swap quotes (source: Bloomberg)

- Real yields are obtained as the difference between nominal yields and inflation swap rates (corrected from lags inherent in Eurozone inflation swaps)

- The maturities of the zero-coupon bonds are as follows:
  - Nominal: 1, 3 and 6 months, 1, 2, 3, 5, 7 and 10 years
  - Real: 1, 2, 5 and 10 years
Model

Estimation: A 2-step estimation procedure

- In the **first step**, the macro-model parameters (+ the SPF error-term standard deviation) are estimated by maximizing the log-likelihood (the log-L is obtained by applying the Kalman filter)

- Three parameters have been calibrated (the inflation parameter $g_{\pi}$ entering the Taylor rule, the two parameters defining the dynamics of medium-term inflation, $\rho_L$ and $\chi$)

- In a **second step**, the state-space model is enlarged by adding nominal and real yields amongst the observed variables (the state-space model is enlarged; all yields are assumed to be measured with errors)

- The coefficients of the market price of risk ($\lambda_1$ matrix) that load on lagged macro variables are set to zero
Model
Estimation: Missing-data treatment

- Missing-data problems stem from the fact that
  1. real yields are only available from 2004 onwards
  2. SPF inflation forecast are at the quarterly frequency
- For each period, the Kalman filter calculates a prediction of the state variables and computes the covariance matrix of the errors (prediction step)
- The filter then incorporates the new information given by the vector of observable variables (updating step)

⇒ The updating step can be carried out even if the number of observations varies with time
## Model

### Estimation: Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimation</th>
<th>Standard Error</th>
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</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>0.21</td>
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<td>$\alpha_y$</td>
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<td>0.013</td>
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<tr>
<td>$\sigma_\pi \times 10^3$</td>
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<td>$\beta_4$</td>
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<td>$g_\pi$</td>
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<td>$g_y$</td>
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<td>$\rho_L$</td>
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<td>$\chi$</td>
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<td>-</td>
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<tr>
<td>$\sigma_L \times 10^3$</td>
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<td>0.01</td>
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<tr>
<th>Parameter</th>
<th>Estimation</th>
<th>Standard Error</th>
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<tr>
<td>$\sigma_{3mth} \times 10^4$</td>
<td>0.78</td>
<td>0.05</td>
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<td>$\sigma_{6mth} \times 10^4$</td>
<td>1.21</td>
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<td>$\sigma_{1yr} \times 10^4$</td>
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<td>0.13</td>
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<td>$\sigma_{2yr} \times 10^4$</td>
<td>2.35</td>
<td>0.15</td>
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<td>$\sigma_{3yr} \times 10^4$</td>
<td>1.83</td>
<td>0.12</td>
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<td>$\sigma_{5yr} \times 10^4$</td>
<td>1.27</td>
<td>0.09</td>
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<td>$\sigma_{7yr} \times 10^4$</td>
<td>0.84</td>
<td>0.08</td>
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<td>$\sigma_{10yr} \times 10^4$</td>
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<td>0.15</td>
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<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimation</th>
<th>Standard Error</th>
</tr>
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<td>$\sigma_{1yr}^r \times 10^4$</td>
<td>4.82</td>
<td>0.44</td>
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<td>$\sigma_{2yr}^r \times 10^4$</td>
<td>4.35</td>
<td>0.37</td>
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<td>$\sigma_{5yr}^r \times 10^4$</td>
<td>2.91</td>
<td>0.26</td>
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<td>$\sigma_{10yr}^r \times 10^4$</td>
<td>2.20</td>
<td>0.2</td>
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<tr>
<td>$\sigma_{1yr}^{SPF} \times 10^4$</td>
<td>0.88</td>
<td>0.13</td>
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<tr>
<td>$\sigma_{2yr}^{SPF} \times 10^4$</td>
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<td>0.07</td>
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<td>$\sigma_{5yr}^{SPF} \times 10^4$</td>
<td>0.38</td>
<td>0.05</td>
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## Variance decomposition of debt service

<table>
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<tr>
<th>Bonds issued:</th>
<th>Nominal</th>
<th>Indexed</th>
<th>6-mth</th>
<th>1-yr</th>
<th>2-yr</th>
<th>5-yr</th>
<th>10-yr</th>
<th>5-yr</th>
<th>10-yr</th>
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<tbody>
<tr>
<td>Frequencies</td>
<td>Standard deviation of debt charges (in basis points)</td>
<td>All</td>
<td>185</td>
<td>167</td>
<td>132</td>
<td>93</td>
<td>97</td>
<td>202</td>
<td>201</td>
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<tr>
<td></td>
<td></td>
<td>Business-cycle</td>
<td>96</td>
<td>84</td>
<td>58</td>
<td>16</td>
<td>9</td>
<td>66</td>
<td>67</td>
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<tr>
<td></td>
<td></td>
<td>Excl. infra-year</td>
<td>181</td>
<td>163</td>
<td>127</td>
<td>85</td>
<td>84</td>
<td>115</td>
<td>111</td>
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<tr>
<td>Cycle's length</td>
<td>Variance decomposition</td>
<td>&gt; 8 yrs</td>
<td>0.68</td>
<td>0.69</td>
<td>0.73</td>
<td>0.79</td>
<td>0.71</td>
<td>0.11</td>
<td>0.09</td>
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<tr>
<td></td>
<td></td>
<td>1.5 yr &lt;&lt; 8 yrs</td>
<td>0.27</td>
<td>0.25</td>
<td>0.19</td>
<td>0.03</td>
<td>0.01</td>
<td>0.11</td>
<td>0.11</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1 yr &lt;&lt; 1.5 yr</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.02</td>
<td>0.03</td>
<td>0.10</td>
<td>0.11</td>
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<tr>
<td></td>
<td></td>
<td>&lt; 1 yr</td>
<td>0.05</td>
<td>0.05</td>
<td>0.07</td>
<td>0.17</td>
<td>0.25</td>
<td>0.68</td>
<td>0.69</td>
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</table>
Covariance decomposition (debt service and real activity)

<table>
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<tr>
<th>Bonds issued:</th>
<th>Nominal</th>
<th>Indexed</th>
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<tbody>
<tr>
<td></td>
<td>6-mth</td>
<td>1-yr</td>
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<th>Frequencies</th>
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<tbody>
<tr>
<td>All</td>
<td>21.0</td>
<td>16.0</td>
<td>6.0</td>
<td>-1.0</td>
<td>0.1</td>
<td>3.0</td>
<td>4.0</td>
</tr>
<tr>
<td>Business-cycle</td>
<td>5.0</td>
<td>3.0</td>
<td>-1.0</td>
<td>-1.0</td>
<td>-0.2</td>
<td>0.9</td>
<td>2.0</td>
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<tr>
<td>Excl. infra-year</td>
<td>20.0</td>
<td>15.0</td>
<td>6.0</td>
<td>-2.0</td>
<td>-0.3</td>
<td>3.0</td>
<td>4.0</td>
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</table>

<table>
<thead>
<tr>
<th>Frequencies</th>
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<td>All</td>
<td>0.51</td>
<td>0.43</td>
<td>0.22</td>
<td>-0.07</td>
<td>0.00</td>
<td>0.06</td>
<td>0.09</td>
</tr>
<tr>
<td>Business-cycle</td>
<td>0.34</td>
<td>0.21</td>
<td>-0.12</td>
<td>-0.57</td>
<td>-0.18</td>
<td>0.09</td>
<td>0.17</td>
</tr>
<tr>
<td>Excl. infra-year</td>
<td>0.51</td>
<td>0.43</td>
<td>0.21</td>
<td>-0.10</td>
<td>-0.01</td>
<td>0.10</td>
<td>0.16</td>
</tr>
</tbody>
</table>
**Figure:** Spectral density: Illustration

Area = share of \( \text{Var}(X) \) explained by cycles with periods between \( T_1 \) and \( T_2 \)

Area = \( \text{Var}(X) \)
Figure: Cospectrum of variables X and Y

Area = Covariance(X,Y)

Figure: Phase: illustration

$\varphi = 2\pi/2 \rightarrow \text{correl} = -1$

$\varphi = 2\pi/4 \rightarrow \text{correl} = 0$
Example 1: Pricing of a 2-period bond

- If
  - Factors’ dynamics: \( F_t = \Psi F_{t-1} + \sum \varepsilon_t \) (\( \varepsilon_t \sim N(0, I_d) \))
  - Pricing kernel: \( m_{t+1} = \exp \left[ -\frac{1}{2} \Lambda_t \Lambda_t - \Lambda_t \varepsilon_{t+1} - i_{1,t} \right] \) with \( \Lambda_t = \lambda_0 + \lambda_1 F_t \)
  - 1-period bond (short rate): \( i_{1,t} = \delta F_t \)

- Payoff: \( g(F_{t+2}) = 1 \), therefore \( P_t = E_t (m_{t+1} \times m_{t+2} \times 1) \)

\[
\Rightarrow i_{2,t} = \frac{\delta}{2} (I + \Psi) F_t - \frac{1}{2} \delta \sum \Sigma' \delta' - \frac{1}{2} \delta \sum \left( \lambda_0 + \frac{1}{2} \lambda_1 F_t \right)
\]

- average short rate
- convexity adjustment
- risk premium

\( \left( \frac{1}{2} [i_{1,t} + E_t (i_{1,t+1})] \right) \)
### Example 2: Pricing of a 1-period inflation-linked bond

- The framework makes it possible to price inflation-linked bonds (ILBs) as soon as inflation is one of the factors $F_t$.
  - Let’s inflation $\pi_t = \ln(CPI_t/CPI_{t-1})$ be the first component of $F_t$.
  - Then $\pi_t = \Gamma F_t$ where $\Gamma = [1 \ 0 \ldots \ 0]$.

- Payoff: $g(F_{t+1}) = \frac{CPI_{t+1}}{CPI_t}$, therefore $P_t = E_t\left(m_{t+1} \times \frac{CPI_{t+1}}{CPI_t}\right)$

<table>
<thead>
<tr>
<th>Expected inflation $\left(= E_t(\pi_{t+1})\right)$</th>
<th>Convexity adjustment $\left(-\frac{1}{2}\Gamma \Sigma \Sigma' \Gamma'\right)$</th>
<th>Risk premium $\left(-\Gamma \Sigma (\lambda_0 + \lambda_1 F_t)\right)$</th>
</tr>
</thead>
</table>

$$i_{1,t} - r_{1,t} = \Gamma \Psi F_t$$
Figure: Yield fit
Figure: Model properties: Risk premiums

5Y risk premiums

- Inflation risk premium
- Term premium

5Y model-implied expected inflation and SPF

- Model-implied expected infl.

Unconditional risk premiums

Unconditional yields

maturity (in months)
## Spectral analysis: basics

- **Time series** = Weighted sum of many cosine or sine functions of time with different periodicities

- The **spectral density** (function of frequency $\omega$): measures the importance of the $\omega$-frequency component in the variance of a given variable

- The cross-spectral density: complex function whose real and imaginary part are respectively called **cospectrum** and quadrature spectrum (in polar coordinate form: gain $R(\omega)$ and phase $\varphi(\omega)$)

- The cospectrum is proportional to the portion of the covariance between two variables that is attributable to cycles with frequency $\omega$
The model broadly follows the lines of Rudebusch and Wu’s (2008) model.
Figure: Estimation data

Inflation and SPF expected inflation

- Monthly inflation
- Y-o-y inflation

Real activity

Nominal yields

Real yields
A financing strategy that consists in issuing at each period a constant fraction, defined by weights $w_p$ (with $p \in \{1, \ldots, q\}$), of $\tau_p$-period bonds results in the following debt service (in percentage of $GDP^*$):

$$
\eta_t = \gamma \sum_{p=1}^{q} w_p \sum_{j=1}^{\tau_p} \frac{\vartheta_{\tau_p, t-j}}{(1+g)^j},
$$

Affine in the factors $F_t$

with

- $\vartheta_{\tau_p, t-j} = i_{\tau_p, t-j}$ if class-$p$ bonds are nominal $\tau_p$-period bonds and
- $\vartheta_{\tau_p, t-j} = r_{\tau_p, t-j} + \bar{\pi}_t$ if class-$p$ bonds are $\tau_p$-period ILBs and
- $\gamma = l_t / GDP^*_t$