Modeling Sovereign Risk with Correlated Stochastic Processes (DRAFT VERSION)

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Abstract

In this work we use stochastic processes and correlation networks to model systemic risk between the economies of the European monetary union, in the post crisis period.

For each country we consider, as a financial leverage measure, the Debt/GDP ratio. We then model the time dynamic of both the Debt and the GDP by means of a linear combination of two stochastic equations: an eurozone systematic process and a country specific idiosyncratic process.

Doing so, we model debt sustainability from both the financial and the real side, and in terms of both common and country-specific factors.

We provide an estimation model for the parameters of the processes, and we derive the implied default probabilities for each country. Systemic risk is estimated by means of the estimated partial correlation matrix that is a function of the estimated parameters.

1 Introduction

1.1 Motivation

The last few years have witnessed an increasing research literature in the field of systemic risk. Most of these studies are concerned with the financial side, and try to explain how the default probability of a country, or of a company, expressed in terms of a financial variable, depends on that of the others, or on systematic risk factors, that may include macroeconomic ones, such as the growth rate.

The dependence of systemic risk on the real side of the economy has typically been studied using causal models, in which the financial and the real components of an economy are modelled separately. Noticeable reference papers are Billio et al. (2015) and Schwaab et al. (2015), who model systemic risk in terms of econometric regression models based on the correlations with systemic and idiosyncratic risk factors. Another stream of research is described by Ang and Longstaff (2012) and Brownlees et al. (2014), who model systemic risk in terms of stochastic processes, that may depend on a common systematic factor, that incorporates variations in the real side of the economy.

The recent financial crisis has however shown that shocks in the financial and in the real side of an economy are strongly interrelated, and should be jointly modelled, not only within a country, but also across them.

Recently, Ramsay and Sarlin (2015) have introduced to the field of systemic risk measurement a number of financial leverage measures used in corporate finance, such as the ratio Debt/GDP and Debt/Cash Flow as early warning indicators of financial crisis. Doing so, they introduce joint modelling of the financial and the real sides.

Our aim here is to extend, in a stochastic framework, the approach of Ramsay and Sarlin, considering the Debt/GDP ratio as a financial leverage measure, on which to base the calculations of country's default probabilities.

We model each term of the ratio by means of stochastic processes, as in And and Longstaff (2012) and Brownless et al. (2014), employing a linear combination of two stochastic processes: a systematic and an idiosyncratic one.

In addition, like Billio et al. (2015) and Schwaab et al. (2015), we explicitly model the partial correlation matrix between different systematic and idiosyncratic factors and embed this modeling into a

stochastic process framework.

In terms of relevance, our model should be useful, in particular, to study the impact of economic policies on the default probability of a country, within a multi-correlated framework in which such policies affect the systematic components of the model, at the real or at the financial level.

In the European Union, characterized by one monetary authority (the European Central Bank), that regulates still fragmented national markets, the importance of this study is particularly evident: for example, southern european countries, differently from northern ones, have benefited very little from the drop of monetary rates that has followed the financial crisis. By explicitly modeling the correlation process between countries and between companies in a given country, we aim at capturing the main factors that may constrain the transmission of the monetary impulse.

1.2   Background

From a methodological viewpoint, to estimate our proposed model, we extend the work in Kalogeropoulos et al. (2011), who has introduced a multivariate Cox-Ingersoll-Ross (CIR) process to model the dynamics of exchange rates.

Their model can be specified starting from a general family of non-parametric, time-homogeneous and continuous models for the dynamic of the interest rate Y_t:

$$\mathrm{d}\, Y_t = (\theta_1 - \theta_2 Y_t)\, \mathrm{d}\, t + \theta_3 (Y_t)^\beta\, \mathrm{d}\, W_t, \tag{1.1}$$

where $\beta = 0.5$ corresponds to the CIR process, while $\beta = 0$ represents the Vasicek model.

The previous process can be applied to model the joint dynamic of the interest rates of a group of countries. For example, we can represent the variations of the bond rates in a group of countries as functions of the variation of monetary rates, described as a Wiener process, represented by a geometric Brownian motion $\mathrm{d}\, W_t$, as in the CIR formulation.

Mathematically, for a group $Y_t = (y_t^1, ..., y_t^N)$ of countries, each of the N stochastic processes can be expressed as follows:

$$\mathrm{d}\, y_t^i = (\theta_1^i - \theta_2^i y_t^i)\, \mathrm{d}\, t + \theta_3^i \sqrt{y_t^i}\, \mathrm{d}\, W_t, \qquad \mathrm{i} = 1,\, ...,\, \mathrm{N} \tag{1.2}$$

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where each parameter $\theta^i_{\{1,2,3\}}$ is process-specific.

The structure of (1.2) can be enriched by introducing correlation coefficients between the N stochastic processes, leading to a multi-variate CIR:

$$
\begin{cases}
\text{Corr}(\mathrm{d}\,y^i, \mathrm{d}\,y^j) = \rho^{ij} = \rho^{ji}, & \\
\rho^{ij} \neq 1 & \text{for } i \neq j, \\
\rho^{ij} = 1 & \text{for } i = j.
\end{cases}
\tag{1.3}
$$

The variance of each CIR process can be calculated, and the result is the following:

$$
Var\left[y^i_t | y^i_0\right] = y^i_0 \left(\frac{\theta^i_3}{\theta^i_2}\right)^2 \left(e^{-\theta^i_2 t} - e^{-2\theta^i_2 t}\right) + \frac{\theta^i_1}{2}\left(\frac{\theta^i_3}{\theta^i_2}\right)^2.
\tag{1.4}
$$

The limit of the above variance can be calculated for an adjustment speed that tends to zero:

$$
\lim_{\theta^i_2 \to 0} Var\left[y^i_t | y^i_0\right] = y^i_0 (\theta^i_3)^2 t.
\tag{1.5}
$$

Then, using the correlation coefficients (1.3), the instantaneous covariance matrix can be derived as:

$$
A = \begin{bmatrix}
y^1_0(\theta^1_3)^2 & \rho^{12}\sqrt{y^1_0 y^2_0}\theta^1_3\theta^2_3 & \dots & \rho^{1N}\sqrt{y^1_0 y^N_0}\theta^1_3\theta^N_3 \\
\rho^{21}\sqrt{y^2_0 y^1_0}\theta^2_3\theta^1_3 & y^2_0(\theta^2_3)^2 & \dots & \rho^{2N}\sqrt{y^2_0 y^N_0}\theta^2_3\theta^N_3 \\
\vdots & \vdots & \ddots & \vdots \\
\rho^{N1}\sqrt{y^N_0 y^1_0}\theta^N_3\theta^1_3 & \rho^{N2}\sqrt{y^N_0 y^2_0}\theta^N_3\theta^2_3 & \dots & y^N_0(\theta^N_3)^2
\end{bmatrix}
\tag{1.6}
$$

Note that A is a symmetric matrix, so (a) $A = A^T$ and (b) A can be decomposed according to the spectral theorem.

Note also that the stochastic processes proposed by Kalogeropoulos et al. (2011) can be written in a compact multidimensional form:

$$
\mathrm{d}\,Y_t = M(Y_t, \Theta_{1,2})\,\mathrm{d}\,t + \Sigma(Y_t, \Theta_3)\,\mathrm{d}\,W_t,
\tag{1.7}
$$

where

$$[M]^i = \theta_1^i - \theta_2^i y_t^i, \quad [\Sigma]^i = \theta_3^i \sqrt{y_t^i \rho^i}, \quad \Theta_{1,2} = \begin{bmatrix} \theta_1^1 & \theta_2^1 \\ \vdots & \vdots \\ \theta_1^N & \theta_2^N \end{bmatrix}, \quad \Theta_3 = \begin{bmatrix} \theta_3^1 \\ \vdots \\ \theta_3^N \end{bmatrix}.$$

Our aim is to broaden Kalogeropoulos et al. (2011), extending their multivariate CIR stochastic process in a more general process able (a) to capture both the systematic and idiosyncratic components that may affect interest rate dynamics and (b) to describe the correlation structure by means of graphical Gaussian network modeling as in Giudici and Spelta (2015).

Our proposed models will be applied and compared to data that concern the recent post-crisis period (2010-2014) and the countries belonging to the Eurozone. As the validity of a model ought to be tested in terms of its predictive performance, we will also develop an appropriate model assessment methodology based on out-of-sample predictions of interest and growth rates, for a given Monte Carlo path of monetary and real reference rates.

The paper is structured as follows: Section 2 describes the proposed models and Section 3 presents the empirical evidence obtained from their application.

2 Proposal

We assume that the dynamic of the debt of each country expressed, for simplicity, by the evolution of the associated interest rate, can be described by a linear combination of stochastic processes. We assume, in fact, that all of them follow the same diffusion mechanism, that can be considered as the systematic process; and, in addition, we assume that they are also characterised by another stochastic equation, that can be considered as an idiosyncratic evolution. The complete process is the following:

$$Z_t^i = -\alpha^i S_t + \beta^i y_t^i, \qquad m < i, \tag{2.1}$$

where S_t stands for the systematic process, while y_t^i represents the idiosyncratic process referred to country i. Finally, α^i measures the weight of the systematic process on country i, while β^i is a weight variable which measures the influence of the idiosyncratic equation

on the general, complete process Z_t^i, that describes the evolution of interest rates. From an economic viewpoint, note that the above formulation expresses Z_t^i as a spread between the cost of debt and the cost of money.

Both the systematic and the idiosyncratic processes can be formulated as stochastic differential equations, through the CIR specification:

$$\begin{cases} \mathrm{d}\,S_t = (a - vS_t)\,\mathrm{d}\,t + b\sqrt{S_t}\,\mathrm{d}\,B_t, \\ \mathrm{d}\,y_t^i = (\theta_1^i - \theta_2^i y_t^i)\,\mathrm{d}\,t + \theta_3^i \sqrt{y_t^i}\,\mathrm{d}\,W_t, \end{cases} \qquad (2.2)$$

where $\mathrm{d}\,B_t$ and $\mathrm{d}\,W_t$ are two independent Brownian motions.

The previous equation derives from an important assumption: the systematic process is the same for all the countries considered in the sample, but it differently influences each generic country-specific process Z_t^i, through the weight α^i.

The next step consists in deriving the covariance matrix of the process. To achieve this objective we introduce the following assumptions on the correlation structure:

$$\begin{cases} \mathrm{Corr}[\mathrm{d}\,y_t^i, \mathrm{d}\,y_t^j] = \rho^{ij}, \\ \mathrm{Corr}[\mathrm{d}\,S_t, \mathrm{d}\,y_t^j] = \gamma^j. \end{cases} \qquad (2.3)$$

The first equation is consistent with the assumptions used in the formulation of multidimensional CIR processes; the second one describes the correlation between each idiosyncratic process and the systematic process S_t.

We can now obtain the covariance $\mathrm{Cov}(Z_t^i, Z_t^j)$, where

$$\begin{cases} \mathrm{d}\,Z_t^i = -\alpha^i\,\mathrm{d}\,S_t + \beta^i\,\mathrm{d}\,y_t^i, \\ \mathrm{d}\,Z_t^j = -\alpha^j\,\mathrm{d}\,S_t + \beta^j\,\mathrm{d}\,y_t^j, \\ i, j = \text{countries}. \end{cases} \qquad (2.4)$$

After some calculations the following expression for the instantaneous covariance can be demonstrated:

$$\mathrm{Cov}(Z^i, Z^j) = \alpha^i \alpha^j b^2 S_0 + \sqrt{S_0}\, b \cdot \left[\alpha^i \beta^j \gamma^j \sqrt{y_0^j} \theta_3^j + \alpha^j \beta^i \gamma^i \sqrt{y_0^i} \theta_3^i \right] + \beta^i \beta^j \sqrt{y_0^i y_0^j} \theta_3^i \theta_3^j \rho^{ij}.$$
$$(2.5)$$

Note that the previous equation can be simplified if the two countries coincide $(i = j)$:

$$\mathrm{Cov}(Z^i, Z^i) = (\alpha^i)^2 b^2 S_0 b + 2\sqrt{S_0}\sqrt{y_0^i}\alpha^i\beta^i b\theta_3^i\gamma^i + (\beta^i)^2 y_0^i(\theta_3^i)^2. \quad (2.6)$$

A further development can be achieved by deriving a compact formulation for the instantaneous covariance matrix. Consider the correlation matrix of the idiosyncratic processes:

$$P = \begin{bmatrix} 1 & \rho^{12} & \dots & \rho^{1N} \\ \rho^{21} & 1 & \dots & \rho^{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^{N1} & \rho^{N2} & \dots & 1 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} \gamma^1 \\ \vdots \\ \gamma^i \\ \vdots \\ \gamma^N \end{bmatrix}, \qquad (2.7)$$

where each element in P consists in the correlation coefficient between the idiosyncratic processes of two countries, while Γ is a column vector which includes the correlation coefficients between each institution i and the systematic process S_t.

Through the previous specification we can rewrite the instantaneous covariance matrix A in the following, simple decomposition:

$$A = \Phi \cdot \Theta^T, \qquad (2.8)$$

where

$$[\Phi]^i = \left[\alpha^i b\sqrt{S_0}, \quad \alpha^i, \quad \beta^i\sqrt{S_0 y_0^i}b\theta_3^i[\Gamma]^i, \quad \beta^i\sqrt{y_0^i}\theta_3^i\sqrt{[P]^i}\right],$$

$$[\Theta^T]^j = \begin{bmatrix} \alpha^j b\sqrt{S_0} \\ \beta^j\sqrt{S_0 y_0^j}b\theta_3^j[\Gamma]^j \\ \alpha^j \\ \beta^j\sqrt{y_0^j}\theta_3^j\sqrt{[P]^j} \end{bmatrix}.$$

We remark that the result expressed by (2.5) and (2.8) is thus very useful for the determination of the partial correlation matrix, based on the inverse of the correlation matrix. Using it we can derive the graphical network models described in Giudici and Spelta (2015) and

Brownlees et al. (2014) for a precise description of the systemic risk links between countries.

The model described in (2.1) can be generalised to consider the difference between two independent stochastic processes that capture, respectively, the evolution of the financial and of the real side of the economy.

Indeed, by considering that $f(t) = Debt(t)/GDP(t)$ is a two-variables function that depends on time just through the time dependence of its two components Debt and GDP, it can be easily shown that its total derivative is the following:

$$\mathrm{d}\, f = \frac{\partial f}{\partial Debt} \cdot \mathrm{d}\, Debt + \frac{\partial f}{\partial GDP} \cdot \mathrm{d}\, GDP = \frac{\frac{\partial Debt}{\partial t} \cdot GDP - Debt \cdot \frac{\partial GDP}{\partial t}}{GDP^2}.$$

(2.9)

Therefore, we model the time evolution of the Debt/GDP ratio and, therefore, the sustainability of a debt, by looking at the evolution of both the financial liability side and the real asset side.

More precisely, we assume that the overall stochastic process is given by $Z_{t,1}^i - Z_{t,2}^i$, where:

$Z_{t,1}^i$ is independent of $Z_{t,2}^i$;

the evolution of the Debt, simplified by the cost of the debt service, is described by the following process:

$$Z_{t,1}^i = -\alpha_1^i S_{t,1} + \beta_1^i y_{t,1}^i,$$

(2.10)

where $S_{t,1}$ represents the Euribor interest rate evolution, while $y_{t,1}^i$ describes the interest rate of 10-years maturity government bonds: thus, $Z_{t,1}^i$ measures the weighted spread between bond interest rates and monetary rates;

finally, the evolution of the GDP of a country can be modelled by the following process:

$$Z_{t,2}^i = -\alpha_2^i S_{t,2} + \beta_2^i y_{t,2}^i,$$

(2.11)

which represents the spread between the country-specific GDP growth rate ($y_{t,2}^i$) and the GDP growth rate of the Eurozone ($S_{t,2}$).

2.1 Model estimation and validation

All the proposed CIR time-homogeneous processes need a specific parameter estimation procedure. For this aim we can define the following

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variables:

$$c = \frac{2\theta_2}{\theta_3^2(1 - e^{-\theta_2 t})}, \quad u = cBR_t e^{-\theta_2 t}, \quad q = \frac{2\theta_1}{\theta_3^2} - 1, \quad v = cBR_{t+1}.$$

The log-likelihood function of the process can then be derived as:

$$\ln L(\Theta) = (N-1)\ln c + \sum_{j=1}^{N-1} \left[-u_{t_j} - v_{t_j} + \frac{q}{2}\ln\left(\frac{v_{t_j}}{u_{t_j}}\right) + \ln[I_q(2\sqrt{u_{t_j}v_{t_j}})] \right],$$

(2.12)

where $I_q(2\sqrt{uv})$ is the modified Bessel function of order q. The parameter vector $\hat{\Theta}$ is thus found by maximizing the log-likelihood function:

$$\hat{\Theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) = \arg\max_{\Theta} \ln L(\Theta). \tag{2.13}$$

To compare the proposed models with others, such as linear regression models, on the same playing field, we ought to develop a predictive assessment procedure. This is particularly meaningful especially in the light of the necessity to forecast ahead of time the levels of the systemic rates, that are the main explanatory components of the model.

In order to predict financial or economic spreads $(Z_{t,1}^i, Z_{t,2}^i)$ for the countries considered in the sample, we need to estimate also the weights of the systematic (α^i) and the idiosyncratic (β^i) processes. Let us call $d_{\{1,2\}}^i(real)$ as the observed spreads for the period under analysis. Then

$$\begin{cases} \beta_{\{1,2\}}^i = \text{Corr}\left(d_{\{1,2\}}^i(real), y_{t,\{1,2\}}^i\right), \\[2mm] \alpha_{\{1,2\}}^i = \mathbb{E}\left(\frac{\beta_{\{1,2\}}^i y_{t,\{1,2\}}^i - d_{\{1,2\}}^i(real)}{S_{t,\{1,2\}}}\right) \end{cases} \tag{2.14}$$

According to the standard cross-validation (backtesting) procedure, to evaluate the predictive performance of a model we can compare, for a given time period, the predictions of interest rate spreads $(Z_{t,1}^i)$ and of GDP growth rate spreads $(Z_{t,2}^i)$ obtained with the previous combinations of stochastic processes with the actual values. To obtain a robust measurement we can indeed generate N scenarios of

the general processes, using the estimated parameters and weights, and obtain the corresponding values using either (2.10) and (2.11). On the basis of them we can calculate and approximate Monte Carlo expected values and variances of the predictions, as follows.

Let $Z^i_{t,\{1,2\}}$ be a spread to be predicted at time t, with unknown density function $f_Y(y)$. The expected value of Y can then be approximated with

$$\widehat{\mathbb{E}(Y)} = \frac{1}{N} \sum_{k=1}^{N} y^{(k)}, \tag{2.15}$$

and its variance with

$$\widehat{var(Y)} = \frac{1}{N^2} \sum_{k=1}^{N} [y_i - E(\hat{Y})]^2. \tag{2.16}$$

Similarly, for each generated scenario we can calculate the corresponding default probability, according to equations (2.19) and (2.21).

2.2 Default probability estimation

The analysis of systemic risk has the general objective of estimating how the change in the probability of default PD (actual or perceived) of a country may affect other countries, and which of them.

The methodology described so far can be extended to derive default probabilities and, therefore, to build network models for default probabilities. The link between the processes introduced in this Section and the default probabilities of the corresponding country can be obtained as follows.

Let us assume that we are in an arbitrage-free context. According to the two specifications of the general process $Z^i_{t,\{1,2\}}$, two PDs can then be obtained. The first (PD_1) exclusively depends on the interest rate spread, and it can be derived considering:

$$D_{t+1} = (1 - PD_1)e^{S_{t,1}+d_1}D_t, \tag{2.17}$$

where D_{t+1} (D_t) is the total debt at time $t+1$ (t), and d_1 is the spread between the idiosyncratic and the systematic interest rate. The analogous risk-free expression is the following:

$$D_{t+1} = D_t e^{S_{t,1}}. \tag{2.18}$$

Equating (2.17) with (2.18) we can obtain PD_1:

$$PD_{t,1}^i = 1 - e^{-d_{t,1}^i} = 1 - e^{-Z_{t,1}^i}. \qquad (2.19)$$

The second expression of the PD (PD_2), can be obtained by considering both the processes $Z_{t,1}^i$ and $Z_{t,2}^i$ together and by deriving the probability of default from the ratio between the liability (debt) and the asset components. Through this procedure equations (2.17) and (2.18) become:

$$\begin{cases} \frac{D_{t+1}}{A_{t+1}} = (1 - PD_2)\frac{D_t}{A_t}\frac{e^{S_{t,1}+d_1}}{e^{S_{t,2}+d_2}}, \\ \\ \frac{D_{t+1}}{A_{t+1}} = \frac{D_t}{A_t}e^{S_{t,1}-S_{t,2}}. \end{cases} \qquad (2.20)$$

And, equating the two expressions, we obtain:

$$PD_{t,2}^i = 1 - e^{-(d_{t,1}^i - d_{t,2}^i)} = 1 - e^{-(Z_{t,1}^i - Z_{t,2}^i)}. \qquad (2.21)$$

From the above equation some comments can be made: (a) if d_1^i decreases, the probability of default decreases, which is consistent with the definition of d_1^i as the spread between the country government bond interest rates and the monetary rates (the higher $y_{t,1}^i$ and $Z_{t,1}^i$, the riskier the country); (b) similarly, if d_2^i decreases the probability of default increases, which is consistent with the definition of d_2^i as the spread between the idiosyncratic GDP growth rate and the European GDP growth rate.

3 Application

3.1 Data and descriptive statistics

The recent financial crisis, together with the sovereign crisis, has had a great impact. In particular, the volatility of the default probability of each country has significantly increased, as well as the relationships between countries considered as part of an interconnected network have substantially changed. In the Eurozone, characterized by one monetary authority (the European Central Bank), that regulates still fragmented national markets, this effect is particularly evident, as southern european countries are very close to each other, with northern economies also strongly interconnected and characterized by limited relations with southern countries.

To investigate the above issues we focus on five european countries, France, Germany, Greece, Italy and Spain, for the post-crisis period, ranging from January 2010 to December 2014.

For the purposes of our analysis, the systematic process is the 1-month Euribor, while the idiosyncratic process is defined by the interest rates of 10-years government bonds. All the data collected and used in this analysis have monthly frequencies. More precisely, as the GDP growth rates are quarterly released, in order to obtain monthly data we have interpolated the available values.

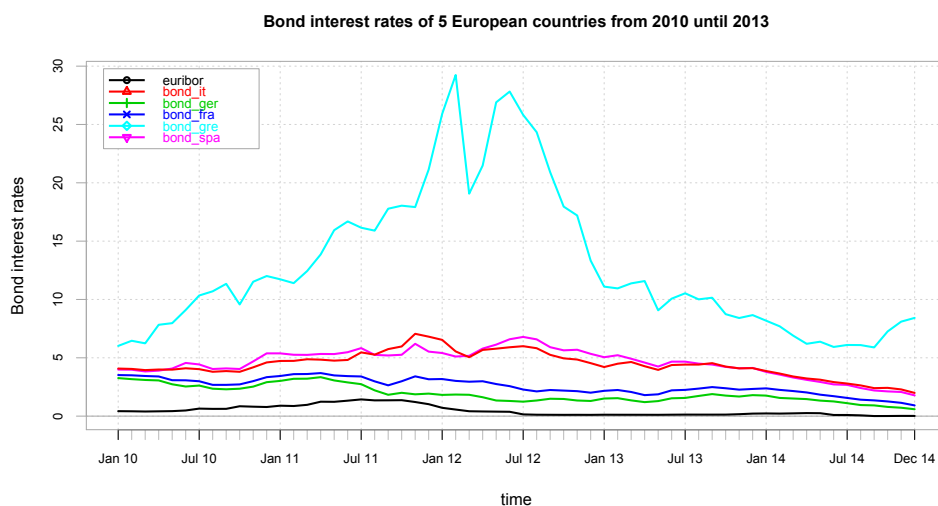The time-evolution of both processes can be observed in Figure 1.

Figure 1: Monthly time evolution of 10-years maturity bond interest rates, from January 2010 until December 2014.

From Figure 1 it is clear that the Euribor is the lowest interest rate (at the moment very close to zero); Greek bond rates, on the contrary, are characterized by the highest values for the whole period and by a strong volatility, with a strong peak during 2012. This feature is obviously consistent with the Greece sovereign crisis. Spain and Italy seem to have very similar behaviours and, finally, Germany and France curves are quite homogeneous.

The evolution of the GDP growth rates are represented in Figure 2.

Figure 2 shows that during the first post-crisis years almost all the

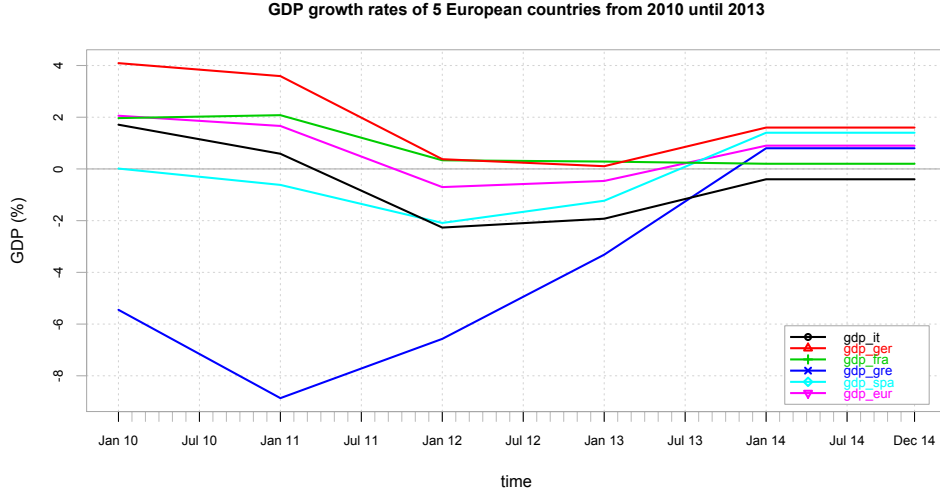GDP growth rates of 5 European countries from 2010 until 2013

Figure 2: Monthly time evolution of GDP growth rates, from January 2010 until December 2014.

GDP growth rates were negative, with a strong decrease for Greece; since 2013 the trend has changed and the GDPs have started increasing again for all the countries, with the exception of France.

The correlation matrices between the processes can be calculated, and are reported in Table 1 ($S_{t,1}$ and $y_{t,1}^i$) and in Table 2 ($S_{t,2}$ and $y_{t,2}^i$).

	France	Germany	Greece	Italy	Spain	Euribor
France	1.000					
Germany	0.908	1.000				
Greece	0.283	-0.066	1.000			
Italy	0.589	0.243	0.828	1.000		
Spain	0.562	0.284	0.793	0.918	1.000	
Euribor	0.747	0.658	0.308	0.520	0.426	1.000

Table 1: Correlation matrix between the interest rates on 10-years government bonds and the Euribor.

From Table 1 one can notice that almost all the correlation coefficients are positive, meaning a strong relationship between the bond interest rates of the five european countries considered in the sample. The most positive links are between France and Germany, and be-

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tween Greece, Italy and Spain. This result would suggest us to divide the sample into two, independent clusters, one composed by northern economies (France and Germany), while the other one including southern economies (Spain, Italy and Greece).

	France	Germany	Greece	Italy	Spain	Eurozone
France	1.000					
Germany	0.903	1.000				
Greece	-0.749	-0.414	1.000			
Italy	0.782	0.966	-0.180	1.000		
Spain	-0.171	0.256	0.763	0.467	1.000	
Eurozone	0.746	0.955	-0.130	0.993	0.528	1.000

Table 2: Correlation matrix between the GDP growth rates and the Eurozone GDP growth rate.

Table 2, on the contrary, shows a different scenario. By analyzing the correlations between the GDP growth rates one can notice that Germany and France are still positively related, but now Italy has radically changed its position: in fact, it is positively and significantly linked with both France and Germany, meaning that its GDP growth rate presents a behaviour much more similar to that of northern economies with respect to southern ones. Spain and Greece are still related. Note that Germany and Italy have the strongest relationship with the whole Eurozone behaviour.

3.2   Model estimation and validation

The first step consists in deriving the CIR coefficients for all the countries and for the two general processes $Z_{t,\{1,2\}}^i$ through the maximization of the log-likelihood function. In order to be able to do out-of-sample tests, we have used data from 2010 until 2013: in this way we can generate all the processes for 2014, and we can predict the values of the spreads for all the countries. The estimated parameter values obtained for the two systematic processes $S_{t,1}$ (Euribor interest rate) and $S_{t,2}$ (Eurozone GDP growth rate) are reported in Table 3.

	a_1	v_1	b_1	a_2	v_2	b_2
All countries	0.011	0.028	0.124	0.053	0.066	0.140

Table 3: Estimated parameters of the systematic processes $S_{t,1}$ and $S_{t,2}$.

In Table 4 are reported the estimated parameters of the idiosyncratic processes $y_{t,1}^i$ (10-years bond interest rate) and $y_{t,2}^i$ (GDP growth rate), where i refers to each of the five countries considered in this analysis.

	France	Germany	Greece	Italy	Spain
$\theta_{1,1}$	0.194	0.123	1.309	0.441	0.549
$\theta_{2,1}$	0.078	0.073	0.086	0.091	0.108
$\theta_{3,1}$	0.116	0.124	0.548	0.150	0.152
$\theta_{1,2}$	0.010	0.021	0.004	0.020	0.016
$\theta_{2,2}$	0.038	0.044	0.0001	0.113	0.0001
$\theta_{3,2}$	0.127	0.137	0.126	0.081	0.125

Table 4: Estimated parameters of the idiosyncratic processes $y_{t,1}^i$ and $y_{t,2}^i$.

Table 4 shows that Greece has the highest volatility parameter for the process that describes bond interest rates ($\theta_{3,1}$): this is consistent with the descriptive statistics and, in particular, with the graph shown in Figure 1.

Secondly, we have to derive the weight coefficients of the systematic and the idiosyncratic processes, consistently with equation (2.14): they are reported in Table 5.

	France	Germany	Greece	Italy	Spain
α_1	-0.978	0.126	0.935	-0.276	-0.366
β_1	0.612	0.772	0.998	0.855	0.851
α_2	-1.416	0.779	1.448	1.058	0.581
β_2	-0.267	0.936	0.958	0.959	0.278

Table 5: Weight coefficients of the two general processes $Z_{t,\{1,2\}}^i$.

Through the specification of the parameters obtained so far we are now able to generate the total processes $Z_{t,1}^i$ and $Z_{t,2}^i$ for the period 2010-2013 or for 2014, and for all the countries. An interesting point consists in the analysis of the correlation coefficients between them. Table 6 represents the correlation matrix between $Z_{t,1}^i$ for $t = 2010, ..., 2014$.

Table 6 (which shows the correlations between the processes that describe the spread between bond interest rates and monetary rates) is absolutely consistent with Table 1, showing again two distinct clusters characterized by a strong inner correlation: France and Germany on one side, and Spain, Italy and Greece on the other one.

	France	Germany	Greece	Italy	Spain	Euribor
France	1.000					
Germany	0.702	1.000				
Greece	0.008	-0.504	1.000			
Italy	0.410	-0.257	0.776	1.000		
Spain	0.300	-0.250	0.787	0.841	1.000	
Euribor	0.944	0.496	0.083	0.491	0.392	1.000

Table 6: Correlation coefficients between the processes $Z_{t,1}^i$.

Table 7, instead, reports the correlation coefficients between $Z_{t,2}^i$, which is the process that describes the difference between the idiosyncratic GDP growth rate and the global Eurozone GDP growth rate.

	France	Germany	Greece	Italy	Spain	Eurozone
France	1.000					
Germany	-0.927	1.000				
Greece	0.749	-0.931	1.000			
Italy	-0.859	0.814	-0.598	1.000		
Spain	0.940	-0.995	0.924	-0.825	1.000	
Eurozone	-0.998	0.906	-0.712	0.857	-0.920	1.000

Table 7: Correlation coefficients between the processes $Z_{t,2}^i$.

Comparing Table 6 with Table 7, it is interesting to note that some coefficients change sign, meaning that the relationship between some couples of countries changes depending on the variables under analyses. For example, if we look at the matrix referred to $Z_{t,1}^i$, Italy is positively correlated to Spain and negatively correlated to Germany; but if we change perspective and we look at the GDP growth rate, we can notice that those two relationships change in sign, still remaining significant. Another interesting case regards France: its bonds interest rates, in fact, are positively correlated to the German ones, but its GDP decreases when the German GDP increases. As in the previous case, the correlation matrix of Table 7 is consistent with Table 2.

3.3   Network analysis

From the correlation matrices reported in Tables 6 and 7 we can calculate their inverse and, therefore, obtain the partial correlations between countries. This, following Giudici and Spelta (2015) allows to build a graphical Gaussian network between the default probabilities

of different countries, which gives an important representation of systemic risk channels.

Such partial correlations, referred to $Z_{t,1}^i$, for $t = 2010, ..., 2013$, are reported in Table 8, along with the p-values that correspond to the hypotheses of them being equal to zero (no connection).

	France	Germany	Greece	Italy	Spain
France	1.000				
Germany	0.952	1.000			
	(429.71)				
Greece	0.338	-0.371	1.000		
	(5.68)	(7.04)			
Italy	0.723	-0.752	0.053	1.000	
	(48.19)	(57.22)	(0.123)		
Spain	-0.412	0.402	0.515	0.529	1.000
	(9.02)	(8.50)	(15.91)	(17.07)	

Table 8: Inverse correlation matrix for $Z_{t,1}^i$.

Table 9, instead, reports the partial correlations $Z_{t,2}$, again for $t = 2010, ..., 2013$, obtained from the inversion of the matrix in 7.

	France	Germany	Greece	Italy	Spain
France	1.000				
Germany	-0.854	1.000			
	(118.97)				
Greece	-0.689	-0.934	1.000		
	(39.81)	(298.71)			
Italy	-0.004	0.411	0.398	1.000	
	(0.001)	(8.94)	(8.31)		
Spain	-0.833	-0.996	-0.960	0.409	1.000
	(100.18)	(5177.3)	(515.77)	(8.86)	

Table 9: Inverse correlation matrix for $Z_{t,2}^i$.

Finally, Table 10 reports the partial correlations of the spread between the two processes $Z_{t,1}^i - Z_{t,2}^i$, again for $t = 2010, ..., 2013$, consistently with the formulation of PD_2^i.

By considering a significance level $\alpha = 0.01$, we can select the most significant correlations, and thus derive the graphical Gaussian networks for $PD_{t,1}$ and $PD_{t,2}$, with $t = 2010, ...2013$, as in Figure 3.

The comparison between the three networks, calculated on past data, reflects what has been underlined in the previous Section: the

	France	Germany	Greece	Italy	Spain
France	1.000				
Germany	-0.405	1.000			
	(8,65)				
Greece	-0.349	-0.434	1.000		
	(6.09)	(10.21)			
Italy	-0.457	0.294	0.349	1.000	
	(11.61)	(4.15)	(6.09)		
Spain	0.665	0.111	0.598	0.382	1.000
	(34.91)	(0.55)	(24.43)	(7.51)	

Table 10: Inverse correlation matrix for $Z_{t,2}^i$-$Z_{t,2}^i$
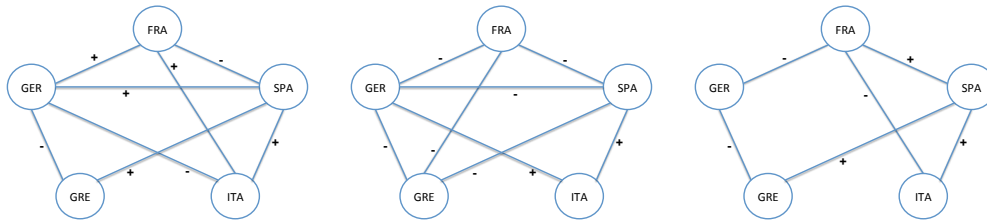
Figure 3: Networks for the five european countries considered in the sample: $Z_{t,1}$ (left), $Z_{t,2}$ (center), and $Z_{t,1} - Z_{t,2}$ (right).

inclusion of the GDP growth rate, and so of a macroeconomic variable, in the study of the default probability is important and necessary in order to capture all the correlations and the direct links between the countries.

In the network on the left one can conclude that two clusters are present: one composed by southern countries, Italy, Spain and Greece, and the other one composed by northern economies, such as France and Germany. Italy and Spain seem to act as intermediary countries, with the first positively correlated with France, and the latter positively linked with Germany.

The central network, referred to the GDP growth rate, shows a completely different situation: in particular, France becomes negatively correlated to Germany, and this because of its decreasing GDP during the last years, as well as Germany seems to be a different country, being negatively related to almost all the others.

Finally, the total network on the right shows the result obtained by combining the previous networks: as underlined before, France has completely changed its position, being now more related to south-

ern countries and acting as an intermediary between north and south economies. Spain and Italy are no more related to Germany, and this is probably due to a sort of compensation effect between financial and real processes.

3.4 Default probability estimation

After having analyzed the correlation coefficients, we can now calculate the two default probabilities of the 5 countries for 2014: the first probability $(PD_{t,1}^i)$ considers only the spread between bond interest rates and monetary rates and it is calculated with equation (2.19); the second default probability $(PD_{t,2}^i)$ incorporates both the spread between interest rates and the spread between the country-specific GDP growth rate and the Eurozone GDP growth rate, and it is based on equation (2.21). Both probabilities can be derived on past data (2010-2013) or can be predicted for the next year (2014).

More precisely, Figure 4 shows the probabilities of default calculated on past data (from 2010 until 2013): the first graph refers to $PD_{t,1}$, while the second one describes $PD_{t,2}$.
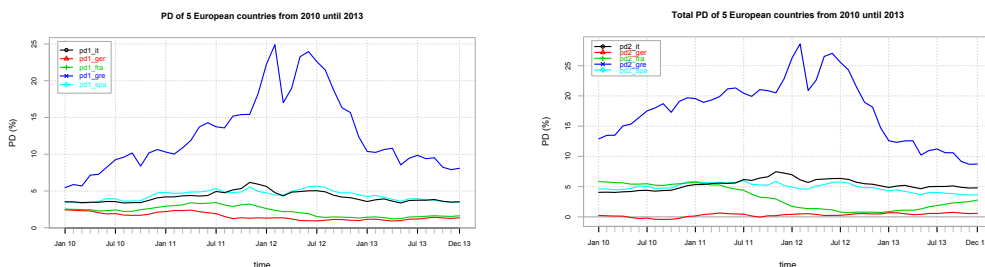

Figure 4: Default probabilities from 2010 until 2013: $PD_{t,1}$ (left) and $PD_{t,2}$ (right).

From Figure 4 it is clear that the inclusion of the GDP growth rate, together with the spread between interest rates, changes the default probabilities during the period 2010-2013. This is especially evident for Greece, which experienced an increase in the PD after the addition of the GDP. This is consistent with the fact that its GDP growth rate has been strongly negative for the first years after the crisis. The same reasoning can be applied also to France, Italy and Spain: they all present a decrease in the PD because of their decrease in the GDP.

A different situation is the one of Germany: it has always been characterized by a positive GDP growth rate, and for this reason its default probability increases after having included $Z_{t,2}$ in the derivation of the PD.

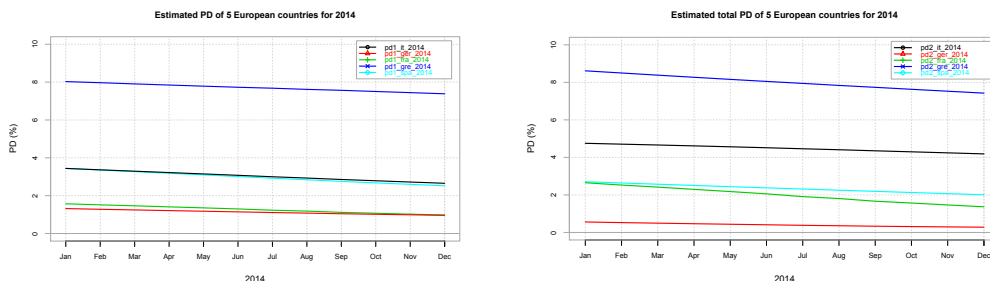Figure 5, instead, shows the two estimated probabilities of default for 2014.

Figure 5: Estimated default probabilities for 2014: $PD_{t,1}$ (left) and $PD_{t,2}$ (right).

Figure 5 shows a decrease in the probabilities of default for all the countries during 2014. This is explained by the radical drop in the interest rates of the 10-years bonds during the last period. Moreover, by comparing the curves, it is clear that Spain and Germany have $PD_{2014,2} < PD_{2014,1}$: this is consistent with their increase (very strong for Spain) in the GDP during the last year. Italy and France, on the contrary, present $PD_{2014,2} > PD_{2014,1}$, and again this is due to the actual negative values of their GDP growth rates.

It is important to remark that Figure 4 and 5 show a change in the default probabilities after having included the GDP growth rates in the analyses. This is a clear evidence of the importance of including an economic perspective, together with a financial one, in the analysis of the evolution of a country.

Finally, we can calculate the correlation matrices between the two default probabilities, and see how they change considering only the financial viewpoint ($PD_{t,1}$) or both a financial and an economic perspective ($PD_{t,2}$).

By comparing Table 11 and 12 it is clear that France and Italy are the most interesting situations: their default probabilities are, respectively, positively and negatively related to those of Germany on the liability side, but if we implement also the GDP growth rate, such

	France	Germany	Greece	Italy	Spain
France	1.000				
Germany	0.703	1.000			
Greece	0.014	-0.508	1.000		
Italy	0.410	-0.257	0.785	1.000	
Spain	0.299	-0.250	0.798	0.843	1.000

Table 11: Correlation coefficients between the default probabilities PD_1^i.

	France	Germany	Greece	Italy	Spain
France	1.000				
Germany	-0.594	1.000			
Greece	-0.059	-0.211	1.000		
Italy	-0.448	0.278	0.686	1.000	
Spain	0.316	-0.280	0.781	0.517	1.000

Table 12: Correlation coefficients between the default probabilities PD_2^i.

correlations become opposite in sign. This behaviour is consistent with the previous observations, and also with Table 7, where can be noticed that France/ Italy and Germany have an opposite relationship with the Eurozone GDP growth rate.

From this Section an important conclusion emerges: correlations between idiosyncratic processes are significant, but it is even more important to consider correlations between each country-specific parameter and the overall european level of the same quantity. This final remark justifies our choice of including an european level within our model, looking at the spread between an idiosyncratic variable in each country and its mean value in the Eurozone.

4 Future research

We have demonstrated that correlated stochastic processes, coupled with network models, can be very useful in the joint modelling of the dynamic of debt sustainability of an economy, as measured by the Debt/Gdp ratio.

Future research involves extending what suggested in Acharya et al. (2014) and Grey et al. (2013) who explicitly models the correlations and the associated systemic loops between the different sectors of an economy: the government, the financial sector and the financial sector.

To this aim, our model should be extended to both the financial and non financial corporate aggregate sectors and, possibly, to model the dynamic of individual firms.

5   Acknowledgements

References

ACHARYA, V.V., DRECHSLER, I., & SCHNABL, P. 2014. A Pyrrhic Victory? Bank Bailouts and Sovereign Credit Risk. *Journal of Finance*, **69**(6), 2689 – 2739.

ANG, A., & LONGSTAFF, F.A. 2012. *Systemic sovereign credit risk: lessons from the U.S. and Europe.* Tech. rept. National Bureau of Economic Research.

BILLIO, M., CAPORIN, M., PANZICA, R., & PELIZZON, L. 2015. *Network Connectivity and Systematic Risk.* Tech. rept. University Ca' Foscari.

BROWNLEES, C., HANS, C., & NUALART, E. 2014. Bank Credit Risk Networks: Evidence from the Eurozone Crisis. *Technical Report.*

CHONG, B.S., LIU, M., & SHRESTHA, K. 2006. Monetary transmission via the administered interest rates channel. *Journal of Banking and Finance*, **30**(5), 1467–1484.

COX, J.C., INGERSOLL, J.E.JR, & ROSS, S.A. 1985. A Theory of Term Structure of Interest Rates. *Econometrica*, **53**(2), 385–408.

GIUDICI, P., & SPELTA, A. 2015. Graphical Network Models for International Financial Flows. *Journal of Business and Economic Statistics*, to appear.

KALOGEROPOULOS, K., DELLAPORTAS, P., & ROBERTS, G.O. 2011. Likelihood-based inference for correlated diffusions. *The Canadian Journal of Statistics*, **39**(1), 52–72.

OSTRY, J.D., GHOSH, A.R., & ESPINOZA, R. 2015. *When Should Public Debt Be Reduced?* discussion paper. International Monetary Fund.

RAMSAY, B.A., & SARLIN, P. 2015. *Ending over-lending: assessing systemic risk with debt to cash flow.* working paper. European Central Bank.

SCHWAAB, B., KOOPMAN, S.J., & LUCAS, A. 2015. *Global Credit Risk: World, Country and Industry Factors.* discussion paper. Duisenberg School of Finance - Tinbergen Institute.