## Bank business models at zero interest rates

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The views expressed in this presentation are those of the authors and they do not necessarily reflect the views or policies of the European Central Bank.

## Motivation

In November 2014, the ECB became the single supervisor for a large number of significant banks in the euro area.


Source: 'SSM SREP Methodology Booklet' by ECB Banking Supervision

## Motivation

Banks are highly heterogeneous, differing widely in terms of size, complexity, activities, organization, funding, and geographical reach.

Dynamic econometric modeling permits insight into diversity of business models, to

- form relevant peer groups of banks for effective micro-prudential supervision;
- study risks originating from and acting upon the financial sector;
- assess the impact of newly proposed financial regulations, as well as unconventional monetary policies.


## Econometric contribution

- We introduce a new model for clustering multivariate panel data on bank characteristics and apply it to European bank data: Moderate $T$, large $N$, potentially many indicators $D$, and an unknown number of clusters J .
- Component means and covariance matrices can be time-varying.
- Our approach builds on static finite mixture models, and augments them with outlier-robust score-driven parameter dynamics. Estimation via a suitable Expectation-Maximization (EM) algorithm.
- Monte Carlo experiments suggest that our modeling framework works reliably regarding both classification and parameter tracking in a variety of settings.


## Main empirical findings

- European banks can be divided into approximately six peer groups: (A) Large universal banks, (B) corporate/wholesale lenders, (C) fee-focused banks/asset managers, D) small diversified lenders, (E) domestic retail lenders, and (F) mutual/co-operative banks.
- Banks with different business models reacted differently to the financial crisis 2008-09, and also the sovereign debt crisis 2010-12. Small domestic lenders and retail banks were relatively less affected.
- Low long-term interest rates are potentially problematic from a financial stability perspective. The largest and the smallest lenders respond the most to falling rates.


## Related literature

1. Identifying bank business models using static clustering methods: Ayadi \& De Groen $(2011,2014,2015)$, Roengpitya, Tarashev \& Tsatsaronis (2014), Farne \& Vouldis (2016).
2. Dynamic finite mixture models for panel data: Catania (2016).
3. Linking banks' business models and their riskiness: Demirguc-Kunt \& Huizinga (2010), Beltratti \& Stulz (2012), Laeven, Ratnovski \& Tong (2015).

## Outline

- Introduction
- Dynamic clustering model
- Simulations
- Bank business models at zero interest rates
- Conclusion


## Dynamic finite mixture model for panel data

- Let $\mathbf{y}_{i t}$ denote a $D$-vector of observations for unit $i$ at time $t$ and $\mathbf{Y}_{i}=\left(\mathbf{y}_{i 1}^{\prime}, \ldots, \mathbf{y}_{i T}^{\prime}\right)^{\prime}$.
- $\mathbf{y}_{i t}$ are assumed to be independent draws from a common parametric mixture density with $J$ components,

$$
\begin{equation*}
f\left(\mathbf{Y}_{i} ; \boldsymbol{\Psi}\right)=\sum_{j=1}^{J} \pi_{j} f_{j}\left(\mathbf{Y}_{i} ; \boldsymbol{\theta}_{j}\right) \tag{1}
\end{equation*}
$$

with parameter vector $\boldsymbol{\Psi}=\left(\pi_{1}, \ldots, \pi_{J-1}, \boldsymbol{\theta}_{1}^{\prime}, \ldots, \boldsymbol{\theta}_{J}^{\prime}\right)^{\prime}$, where $\pi_{j}$ is the mixture probability of component density $f_{j}$.

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- If (unknown) cluster indicators $z_{i j}$ were known, the likelihood function would be

$$
\begin{equation*}
\log L_{c}(\boldsymbol{\Psi})=\sum_{i=1}^{N} \sum_{j=1}^{J} z_{i j}\left[T \log \pi_{j}+\log f_{j}\left(\mathbf{Y}_{i} ; \boldsymbol{\theta}_{j}\right)\right] \tag{2}
\end{equation*}
$$

## EM algorithm

Idea: Given the observed data and some previously determined value $\boldsymbol{\Psi}^{(k-1)}$ for $\boldsymbol{\Psi}$, the conditionally expected likelihood

$$
\begin{aligned}
& Q\left(\boldsymbol{\Psi} ; \boldsymbol{\Psi}^{(k-1)}\right)=\sum_{j=1}^{J} \sum_{i=1}^{N} \mathbb{P}\left[z_{i j}=1 \mid \mathbf{Y}_{1}, \ldots, \mathbf{Y}_{n} ; \boldsymbol{\Psi}^{(k-1)}\right] \\
& \times\left[T \log \pi_{j}+\log f_{j}\left(\mathbf{Y}_{i} ; \boldsymbol{\theta}_{j}\right)\right]
\end{aligned}
$$

is optimized by alternately updating the component probabilities ('E-Step') and maximizing the remainder of the function ('M-Step'); see Dempster, Laird \& Rubin (1977).

E-Step The conditional component probabilities are updated using

$$
\begin{align*}
\tau_{i j}^{(k)} & :=\mathbb{P}\left[z_{i j}=1 \mid \mathbf{Y}_{1}, \ldots, \mathbf{Y}_{n}, \boldsymbol{\Psi}=\boldsymbol{\Psi}^{(k-1)}\right] \\
& =\frac{\pi_{j}^{(k-1)} f_{j}\left(\mathbf{Y}_{i} ; \boldsymbol{\theta}_{j}^{(k-1)}\right)}{\sum_{h=1}^{J} \pi_{h}^{(k-1)} f_{h}\left(\mathbf{Y}_{i} ; \boldsymbol{\theta}_{h}^{(k-1)}\right)}, \tag{3}
\end{align*}
$$

with $f_{j}\left(\mathbf{Y}_{i} ; \boldsymbol{\theta}_{j}^{(k-1)}\right)=\prod_{t=1}^{T} f_{j}\left(\mathbf{y}_{i t} ; \boldsymbol{\theta}_{j}^{(k-1)}\right)$.
M-Step Given $\tau_{i j}^{(k)}, i=1, \ldots, N, j=1, \ldots, J$, estimates of mixture probabilities are obtained:

$$
\pi_{j}^{(k)}=\frac{1}{N} \sum_{i=1}^{N} \tau_{i j}^{(k)}
$$

and the parameters $\boldsymbol{\theta}_{1}, \ldots, \boldsymbol{\theta}_{J}$ are estimated by maximizing the remaining part of the likelihood function.

## Score-driven finite mixture model

Extension to time-varying cluster parameters via score dynamics; see Creal, Koopman \& Lucas (2013), Harvey (2013), Creal, Schwaab, Koopman \& Lucas (2014), and Lucas \& Zhang (2015):

$$
\boldsymbol{\theta}_{j, t+1}=A_{j} s_{\boldsymbol{\theta}_{j t}}+\boldsymbol{\theta}_{j t}
$$

where

- $A_{j}=a_{j} \cdot I_{D}$ is a diagonal matrix to be estimated, and
$\Rightarrow s_{\boldsymbol{\theta}_{j t}}=S_{\boldsymbol{\theta}_{j t}} \nabla_{\boldsymbol{\theta}_{j t}}$ is the scaled first derivative of the conditionally expected likelihood function, with

$$
\nabla_{\boldsymbol{\theta}_{j t}}^{(k)}=\frac{\partial Q\left(\boldsymbol{\Psi} ; \boldsymbol{\Psi}^{(k-1)}\right)}{\partial \boldsymbol{\theta}_{j t}} \text { and } S_{\boldsymbol{\theta}_{j t}}^{(k)}=-\mathbb{E}\left(\frac{\partial Q\left(\boldsymbol{\Psi} ; \boldsymbol{\Psi}^{(k-1)}\right)}{\partial \boldsymbol{\theta}_{j t} \boldsymbol{\theta}_{j t}^{\prime}}\right)^{-1}
$$

## Score-driven finite mixture model

Simple benchmark model: A mixture of Gaussian densities with time-varying means, static covariance matrices, and a common smoothing parameter, so that

- $\nabla_{\mu_{j t}}^{(k)}=\Omega_{j}^{-1} \sum_{i=1}^{N} \tau_{i j}^{(k)}\left(\mathbf{y}_{i t}-\mu_{j t}\right), \quad S_{\mu_{j t}}^{(k)}=\Omega_{j} / \sum_{i=1}^{N} \tau_{i j}^{(k)}$
- Score-driven mean: $\mu_{j, t+1}^{(k)}=a \cdot \frac{\sum_{i=1}^{N} \tau_{i j}^{(k)}\left(\mathbf{y}_{i t}-\mu_{j t}\right)}{\sum_{i=1}^{N} \tau_{i j}^{k}}+\mu_{j t}$,
- Parameter vector: $\boldsymbol{\Psi}=\left(\pi_{1}, \ldots, \pi_{J-1}, a, \mu_{1,0}, \ldots, \mu_{J, 0}, \boldsymbol{\xi}_{1}^{\prime}, \ldots, \boldsymbol{\xi}_{J}^{\prime}\right)^{\prime}$, where $\boldsymbol{\xi}_{j}$ contains the distinct entries in the $j$ th cluster-specific covariance matrix $\Omega_{j}$.


## Score-driven finite mixture model

- Assuming normal mixture components may not be appropriate for fat-tailed accounting data.
- EM algorithm can easily be adapted to include outlier-robust parameter dynamics by considering mixtures of $t$-distributions, yielding

$$
\begin{aligned}
\nabla_{\mu j t}^{(k)} & =\Omega_{j t}^{-1} \sum_{i=1}^{N} \tau_{i j}^{(k)} w_{i j t} \cdot\left(\mathbf{y}_{i t}-\mu_{j t}\right), \text { with } \\
w_{i j t} & =\left(1+\nu_{j}^{-1} D\right) /\left(1+\nu_{j}^{-1}\left(\mathbf{y}_{i t}-\mu_{j t}\right)^{\prime} \Omega_{j t}^{-1}\left(\mathbf{y}_{i t}-\mu_{j t}\right)\right)
\end{aligned}
$$

- Further extensions (in the paper):
$\triangleright$ score-driven component covariance matrices $\Omega_{j t}$,
$\triangleright$ additional explanatory variables to model $\mu_{j t}$.


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## Simulation: Classification and tracking

- Simulation setting: $T=\{10,30\}, N=\{100,400\}$.
- Bivariate sinusoid mean functions and disturbance terms with identity covariance matrix. Data are either Gaussian or $t$-distributed with $\nu=5$ or $\nu=3$.
- We alter the characteristics of the moving circles to check under which circumstances our method
$\triangleright$ correctly classifies a data into distinct components and
$\triangleright$ enables the accurate tracking of the dynamic cluster means over time.


## Simulation: Classification and tracking

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| $N=400$ |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| misspecification 1 |  |  |  |  |  |  |  |
|  |  | $\mathrm{T}=10$ |  |  | $\mathrm{T}=30$ |  |  |
| rad. | dist. | MSE | \% C1 | \% C2 | MSE | \% C1 | \% C2 |
| 4 | 8 | 0.32 | 100 | 100 | 0.35 | 100 | 100 |
| 4 | 0 | 0.32 | 100 | 100 | 0.35 | 100 | 100 |
| 1 | 8 | 0.03 | 100 | 100 | 0.03 | 100 | 100 |
| 1 | 0 | 0.06 | 94.16 | 91.68 | 0.03 | 99.71 | 99.61 |
| misspecification 2 |  |  |  |  |  |  |  |
|  |  |  | $\mathrm{T}=10$ |  |  | $\mathrm{T}=30$ |  |
| rad. | dist. | MSE | \% C1 | \% C2 | MSE | \% C1 | \% C2 |
| 4 | 8 | 0.41 | 100 | 100 | 0.44 | 100 | 100 |
| 4 | 0 | 0.41 | 100 | 100 | 0.44 | 100 | 100 |
| 1 | 8 | 0.03 | 100 | 100 | 0.04 | 100 | 100 |
| 1 | 0 | 0.05 | 95.03 | 95.18 | 0.06 | 97.74 | 97.78 |

## Simulation: Choice of cluster numbers

We consider three sets of model selection criteria in our simulation settings with true $J=2$, but estimation assuming one, two, and three components, respectively:

- Likelihood-based (AIC, BIC): Systematic over-estimation of cluster number.
- Distance-based (within-cluster SSE + penalty): Overall better than likelihood-based, but not ideal in all settings.
- Cluster validation indices (Davies-Bouldin, Calinki-Harabasz): Most robust, DBI outperforms all other considered criteria.


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## Dataset

- Quarterly accounting data from SNL Financial. Mostly public data.
- $N=208$ banks between 2008,Q1 - 2015, Q4 ( $T=32$ ).
- Unbalanced panel. Missing values, e.g. due to different reporting frequencies. Substitute the most recently available observation.
- Dimensions for distinguishing bank business models: size, complexity, activities, geographical reach, funding structure, ownership. $D=13$ indicators are selected as clustering variables.


## Indicator variables

| Category | Variable | Transformation |
| :---: | :---: | :---: |
| Size | 1. Total assets | $\ln$ (Total Assets) |
|  | 2. Leverage w.r.t. CET1 capital | $\ln \left(\frac{\text { Total Assets }}{\text { CET1 capital }}\right)$ |
| Complexity/ | 3. Net loans to assets | $\Phi^{-1}\left(\frac{\text { Loans }}{\text { Assets }}\right)$ |
| Non-traditional | 4. Risk mix | $\ln \left(\frac{\text { Market Risk+Operational Risk }}{\text { Credit Risk }}\right)$ |
|  | 5. Assets held for trading | $\frac{\text { Assets in trading portfolios }}{\text { Total Assets }}$ |
|  | 6. Derivatives held for trading | Derivatives held fors trading |
| Activities | 7. Share of net interest income | Total Assets Net interest income |
|  | 7. Share of net interest income | Operating revenue |
|  | 8. Share of net fees \& commission income | $\frac{\text { Net fees and commissions }}{\text { Operating income }}$ |
|  | 9. Share of trading income | $\frac{\text { Trading income }}{\text { Operating income }}$ |
|  | 10. Retail loans | Retail loans |
|  | 10. Retail loans | $\overline{\text { Retail and corporate loans }}$ |
| Geography | 11. Domestic loans ratio | $\Phi^{-1}\left(\frac{\text { Domestic loans }}{\text { Total loans }}\right)$ |
| Funding | 12. Loan-to-deposits ratio | Total loans <br> Total deposits |
| Ownership | 13. Ownership index | categorial, plus noise |

## Model specification with $J=6$

| Density | $\nu$ | value | $A_{1}$ | $\Sigma_{j} ; \Sigma_{j t}$ | loglik | $\Delta$ loglik |
| :--- | :--- | :---: | :--- | :---: | ---: | ---: |
| N | - | $\infty$ | scalar | static | $9,913.1$ |  |
| t | fixed | 5 | scalar | static | $12,910.8$ | $2,997.7$ |
| t | fixed | 5 | vector | static | $12,921.3$ | 10.6 |
| t | est | 8.5 | scalar | static | $12,928.7$ | 7.3 |
| t | est | 8.5 | vector | static | $12,939.0$ | 10.3 |
| N | - | $\infty$ | scalar | dynamic | $13,411.0$ | 472.0 |
| t | fixed | 10 | scalar | dynamic | $19,146.9$ | $5,735.9$ |
| t | fixed | $\mathbf{5}$ | scalar | dynamic | $\mathbf{1 9 , 5 7 5 . 4}$ | $\mathbf{4 2 8 . 5}$ |
| t | est | 5.1 | scalar | dynamic | $19,575.6$ | 0.2 |

## Cluster labels

(A) Large universal banks (10.6\% of firms; comprising e.g. Barclays plc, Banco Santander SA, Deutsche Bank AG.)
(B) Corporate/wholesale lenders (7.7 \% of firms; comprising e.g. Bayerische Landesbank, HSH Nordbank, RBC Holdings plc.)
(C) Fee-focused bank/asset managers (21.2 \% of firms; comprising e.g. Julius Bär Group, DEKA Bank, Banco Comercial Portugues, Credit Lyonais SA.)
(D) Small diversified lenders (21.6 \% of firms; comprising e.g. Aareal Bank AG, Piraeus Bank SA, SEB AG.)
(E) Domestic retail lenders (26.4\% of firms; comprising e.g. Newcastle Building Society, ProCredit Holding AG \& Co. KGaA, Skandiabanken ASA.)
(F) Mutual/co-operative banks (12.5\% of firms; comprising e.g. Banco Mare Nostrum, Berner Kantonalbank AG, Helgeland Sparebank.)

## Time-varying component means



## Time-varying component means

- Cluster means differ from each other, for each indicator.
- Financial crisis 2008-2009 and sovereign debt crisis 2011-2012 had different impacts on bank business models: Small diversified lenders (D) and domestic retail lenders (E) were relatively more stable than wholesale/corporate lenders (B) and large universal banks (A).
- Visible de-leveraging effect for all groups but small mutual/cooperative banks and domestic retail lenders, possibly due to introduction of Basel 3 rules.
- Large universal banks stand out in terms of size, inter-nationality, volume of derivative positions, sources of income, and risk mix.


## Term structure factors as explanatory variables




- Since 2007: downshift and flattening of yield curve; 'zero lower bound' phenomenon.
- Impact of monetary policy on European banks may depend on their respective business model.


## Term structure factors as explanatory variables

An extended model allows us to quantify how the interest rate environment contributes to explaining banks' business models:

$$
\tilde{\mu}_{j, t+1}=\tilde{\mu}_{j t}+A_{1} \cdot \frac{\sum_{i=1}^{N} \tau_{i j}^{(k)} w_{i j t}\left(\mathbf{y}_{i t}-B_{j} \cdot W_{t}-\widetilde{\mu}_{j t}\right)}{\sum_{i=1}^{N} \tau_{i j}^{(k)}}
$$

where $W_{t}$ contains the first, or first two, yield curve factors extracted from euro area AAA-government bonds based on a Svensson (1994) model. Yield factors are public data (ECB homepage).

## Results: Term structure factors as explanatory variables

As long-term interest rates decline and the slope becomes flatter, on average

- banks grow larger,
- banks tend to take on more leverage,
- relative derivative positions do not change much.

|  | GAS-X: levels |  | GAS-X: first differences |  |
| :--- | :---: | :---: | :---: | :---: |
|  | $I_{t}$ | $s_{t}$ | $\Delta I_{t}$ | $\Delta s_{t}$ |
| $\ln ($ TA $)$ | $-4.495^{* * *}$ | $-0.295^{* * *}$ | $-5.241^{* * *}$ | -0.330 |
|  | $(0.636)$ | $(0.076)$ | $(1.108)$ | $(0.243)$ |
| $\ln ($ Lev $)$ | $-1.299^{* * *}$ | -0.033 | $-0.792^{*}$ | $-0.151^{*}$ |
|  | $(0.243)$ | $(0.028)$ | $(0.453)$ | $(0.081)$ |
| TL/TA | -0.049 | 0.007 | $-0.257^{* *}$ | $0.048^{* *}$ |
|  | $(0.059)$ | $(0.007)$ | $(0.108)$ | $(0.019)$ |
| AHFT/TA | -0.001 | 0.000 | 0.001 | 0.001 |
|  | $(0.002)$ | $(0.000)$ | $(0.004)$ | $(0.001)$ |

## Results: Term structure factors as explanatory variables

Results from disaggregated panel regressions: As the level of long-term interest rates declines,

- the positive size effect is particularly large (and significant) for the large banks (clusters A, B) and the smallest banks (cluster F);
- the positive effect on leverage ratios is largest for mutual/coop. banks (cluster F);
- large banks (clusters A, B) tend to increase their trading positions, smaller ones don't;
- the largest banks (cluster A) become more international;
- there is no significant effect on net interest income (except for cluster B): 'stealth recapitalization' (Brunnermeier/Sannikov, 2015).


## Conclusion

- Robust clustering model for bank panel data.
- Works well on simulated data, and in practice.
- European banks can be divided into different groups with heterogeneous dynamic parameters.
- These groups respond differently to declining long-term interest rates.
- Low long-term interest rates are potentially problematic from a financial stability perspective.

Thank you.

## Simulation results: Choice of $J$

| radius $=4$, distance $=8$ | correct spec. |  |  |  |  |  |  |  |  |  |  | misspec. 1 |  |  | misspec. 2 |  |  |
| :--- | :--- | ---: | ---: | :--- | ---: | ---: | ---: | ---: | ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |  |  |  |  |  |  |  |  |
| no. clusters | 0 | 65 | 35 | 0 | 66 | 34 | 0 | 54 | 46 |  |  |  |  |  |  |  |  |
| AICc | 0 | 70 | 30 | 0 | 71 | 29 | 0 | 57 | 43 |  |  |  |  |  |  |  |  |
| BIC | 0 | 69 | 31 | 0 | 75 | 25 | 0 | 56 | 44 |  |  |  |  |  |  |  |  |
| SSE | 0 | 100 | 0 | 0 | 100 | 0 | 0 | 94 | 6 |  |  |  |  |  |  |  |  |
| AICk | 0 | 100 | 0 | 0 | 100 | 0 | 0 | 85 | 15 |  |  |  |  |  |  |  |  |
| BNG1 | 0 | 100 | 0 | 0 | 100 | 0 | 0 | 86 | 14 |  |  |  |  |  |  |  |  |
| BNG2 | 0 | 100 | 0 | 0 | 99 | 1 | 0 | 84 | 16 |  |  |  |  |  |  |  |  |
| BNG3 | 0 | 100 | 0 | 0 | 100 | 0 | 0 | 100 | 0 |  |  |  |  |  |  |  |  |
| CHI | 0 | 85 | 15 | 0 | 89 | 11 | 0 | 75 | 25 |  |  |  |  |  |  |  |  |
| SI | 0 | 100 | 0 | 0 | 100 | 0 | 0 | 100 | 0 |  |  |  |  |  |  |  |  |
| DBI |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |

## Simulation results: Choice of $J$

| radius $=1$, distance $=0$ |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | correct spec. |  |  | misspec. 1 |  |  | misspec. 2 |  |  |
| no. clusters | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| AICc | 0 | 53 | 47 | 1 | 55 | 44 | 0 | 59 | 41 |
| BIC | 0 | 55 | 45 | 2 | 56 | 42 | 0 | 64 | 36 |
| SSE | 0 | 64 | 36 | 0 | 67 | 33 | 14 | 46 | 40 |
| AICk | 100 | 0 | 0 | 100 | 0 | 0 | 94 | 6 | 0 |
| BNG1 | 1 | 99 | 0 | 61 | 39 | 0 | 67 | 28 | 5 |
| BNG2 | 4 | 96 | 0 | 66 | 34 | 0 | 70 | 25 | 5 |
| BNG3 | 0 | 100 | 0 | 45 | 55 | 0 | 58 | 35 | 7 |
| CHI | 0 | 100 | 0 | 0 | 98 | 2 | 0 | 100 | 0 |
| Silhouette | 0 | 100 | 0 | 0 | 100 | 0 | 0 | 99 | 1 |
| DBI | 0 | 100 | 0 | 0 | 100 | 0 | 0 | 100 | 0 |

## Model selection: Number of clusters

| $\Sigma_{j t}$ dynamic, $\nu=5$ |  |  |  |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| J | loglik | AICc | BIC | AICk | BaiNg2 | CHI | DBI | SSE |
| 2 | $1,114.9$ | $-1,791.9$ | -363.6 | $\mathbf{2 , 4 1 1 . 3}$ | $\mathbf{- 0 . 2 8 8}$ | 19.56 | 3.25 | $1,579.3$ |
| 3 | $9,057.1$ | $-17,448.6$ | $-15,323.7$ | $\mathbf{2 , 6 9 6 . 6}$ | $\mathbf{- 0 . 2 4 9}$ | 13.59 | $\mathbf{3 . 1 5}$ | $1,448.6$ |
| 4 | $13,542.2$ | $-26,183.0$ | $-23,369.3$ | $\mathbf{3 , 1 2 6 . 3}$ | $\mathbf{- 0 . 1 1 5}$ | 15.67 | 3.34 | $1,442.3$ |
| 5 | $16,014.2$ | $-30,883.7$ | $-27,389.2$ | $3,493.0$ | -0.024 | 15.89 | 3.33 | $1,413.0$ |
| 6 | $18,053.8$ | $-34,710.8$ | $-30,544.0$ | $3,884.7$ | 0.083 | $\mathbf{2 8 . 1 9}$ | 3.19 | $\mathbf{1 , 3 8 8 . 7}$ |
| 7 | $20,431.7$ | $\mathbf{- 3 9 , 2 0 5 . 6}$ | $\mathbf{- 3 4 , 3 7 5 . 4}$ | $4,308.2$ | 0.214 | $\mathbf{3 3 . 5 0}$ | 3.28 | $\mathbf{1 , 3 9 6 . 2}$ |
| 8 | $\mathbf{2 3 , 8 3 1 . 2}$ | $\mathbf{- 4 5 , 7 3 4 . 2}$ | $\mathbf{- 4 0 , 2 5 0 . 1}$ | $4,733.3$ | 0.345 | 20.10 | 3.34 | $\mathbf{1 , 4 0 5 . 3}$ |
| 9 | $\mathbf{2 3 , 7 7 2 . 0}$ | $\mathbf{- 4 5 , 3 3 9 . 2}$ | $\mathbf{- 3 9 , 2 1 1 . 0}$ | $5,177.0$ | 0.490 | $\mathbf{2 4 . 8 8}$ | $\mathbf{2 . 8 6}$ | $1,433.0$ |
| 10 | $\mathbf{2 5 , 8 3 2 . 7}$ | $\mathbf{- 4 9 , 1 6 5 . 9}$ | $\mathbf{- 4 2 , 4 0 4 . 3}$ | $5,587.1$ | 0.611 | 5.41 | $\mathbf{3 . 1 3}$ | $\mathbf{1 , 4 2 7 . 1}$ |

