Systemic Risk and Sovereign Default in the Euro Area*

Deyan Radev†

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Abstract

We devise a new and intuitive measure of systemic risk contributions based on the information content in a default of a sovereign in an interdependent financial system. We apply it to estimate the effect of sovereign default on the European financial system. The sovereign contributions increase after Lehman and especially during the sovereign debt crisis, with a considerable potential for cascade effects among Eurozone sovereigns. The banking systems vulnerability to sovereign default is driven by size, riskiness, asset quality, funding and liquidity constraints. The new measure can further help to assess the impact of macro- and microprudential policies.

Keywords: Sovereign debt, Sovereign default, Financial distress, Systemic risk, Contagion, Banking stability, Tail risk

JEL-Classification: C16, C61, G01, G21.

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†Research Center SAFE, House of Finance, Goethe University Frankfurt, Theodor-W.-Adorno-Platz 3 , 60323 Frankfurt am Main, email: radev@safe.uni-frankfurt.de, work phone: +49 (69) 798 30039.

I. Introduction

The potential consequences of a default of a sovereign are a recurring issue in the current policy debates in the euro area. This question has grown in importance in the last several years and the fear of sovereign default has led to a number of sovereign and bank bailouts and has affected interest rates, capital flows, trade, and economic growth in the euro area (EA). The necessity for consistent and timely macro- and microprudential policies puts forth the need for an in-depth analysis of the level of sovereign risk and how it affects the broader financial system. The financial stability literature has focused primarily on measuring the systemic consequences of a default of financial institutions (see, e.g., Acharya et al. (2009), Acharya et al. (2012), Adrian and Brunnermeier (2016), Tarashev, Borio, and Tsatsaronis (2010)) using stock market and balance sheet data. Such data are rarely available for sovereigns, therefore these measures cannot be directly applied to measure the systemic risk arising from a sovereign default. Furthermore, the default of EA sovereigns is a very rare event that is difficult to analyze based on country characteristics alone. This paper describes an innovative approach to analyze systemic risk and contagion among sovereigns, and between sovereigns and banks, based on market expectations.

There are several critical issues when we attempt to measure sovereign default risk in a systemic context. First, we need to derive *joint* probabilities of default. Joint probabilities of default provide more information about the risk in the euro area than the individual probabilities of default of each sovereign, because in addition to the individual country risk, they capture the complex dependence patterns and interactions between euro area countries. Second, we should be able to analyze how various hypothetical scenarios about the default of one or several governments affect the systemic risk in the euro area. Hence, we need to derive *conditional* multivariate probabilities of default. Third, in order to analyze spill-over and cascade effects between euro area sovereigns and between these sovereigns and the European Union banking system, we need to consider measures of systemic importance/contribution that examine the effect of interdependence on the system's vulnerability to default.

To address these critical points, we introduce a new measure of systemic risk, the change in the conditional joint probability of default ($\Delta CoJPoD$), that represents the contribution of the interdependence of a sovereign with the financial system to the overall default risk of the system.[1] We define the measure as the difference between the empirical probability that the euro area system defaults jointly given a sovereign defaults, and a hypothetical joint conditional probability, where the sovereign in question is assumed to be independent from the rest of the system. This difference constitutes the *new* information about the default vulnerability of the euro area financial system stemming from a default of a sovereign.

Our probability-based procedure to estimate sovereign systemic risk involves three steps. First, we derive *individual* probabilities of default from each entity's credit default swap (CDS) spread series, using a comprehensive procedure that follows Hull and White (2000). Second, since *joint* default risk is not traded, we need to impose a flexible structure on the interdependence between the individual entities under investigation. We apply the recently developed Consistent Information Multivariate Density Optimizing (CIMDO) methodology (Segoviano (2006), Segoviano and Goodhart (2009)) to recover the EA multivariate probability distribution. Third, we calculate the new systemic risk measure, the change in the Conditional Joint Probability of Default ($\Delta CoJPoD$) using the derived multivariate density and investigate its properties. In the subsequent empirical analysis, we use our measure to assess the systemic importance of euro area (EA) sovereigns, cascade effects among the sovereigns, as well as spillover effect from EA sovereigns to the European Union (EU) banking system.

Our results show that *joint* sovereign distress risk has increased since the end of 2009, parallel to a decoupling of investors' perceptions about *individual* sovereign default risk. We find that Germany and the Netherlands have the highest systemic importance in the euro

[1]Following Lehar (2005), Adrian and Brunnermeier (2016) and Tarashev, Borio, and Tsatsaronis (2010), we assume that the financial system is a portfolio of European institutions – governments and banks.

area, while the effect of a default of Greece is marginal at best.

With respect to the potential cascade effects within the euro area, we concentrate on a particular scenario: we investigate how perceptions about a default of a relatively small EA sovereign (Portugal) affect default expectations of another small sovereign (Ireland), and how the expectations about their *joint* default impact the default perceptions about a larger sovereign (Spain). We find a high probability of distress spillover between the small sovereigns and between the latter and the large sovereign. This effect rises substantially after Lehman Brothers' bankruptcy and during the sovereign debt crisis. Therefore, we argue that possible default cascade effects of reasonable size should be taken into account in political decision-making.

Concerning the effect of sovereign default on the EU banking system, we find that large banks are more vulnerable to sovereign risk, compared to medium-sized and small banks. This might indicate that these banks are considered by investors to be "too big to save" (Hellwig (1998), Hüpkes (2005), Demirgüç-Kunt and Huizinga (2013), Völz and Wedow (2011), Barth and Schnabel (2013)). Regarding financial gearing, we are not able to confirm a relationship between an increase in leverage and default vulnerability. However, we find that higher-performing banks are expected to be more vulnerable to sovereign default, which might be explained with market perceptions that the higher returns are an indication of riskier activities. We also find that banks with poorer asset quality and banks that are funding- and liquidity-constrained tend to be considered more vulnerable to sovereign default.

Our approach is related to several strands of literature. In developing $\Delta CoJPoD$, we draw from the theory on information contagion in the banking literature, developed by Acharya and Yorulmazer (2008). This theory postulates that a poor performance of a particular bank may contain valuable information about the performance of the other banks in the system and that healthy banks can also be negatively affected if they are considered too similar to the ailing bank (Acharya and Yorulmazer (2008)). The connection to $\Delta CoJPoD$ is based on the observation that the conditional probability of default has two ingredients

- a potentially informative and an uninformative part. The informative portion provides additional information about the vulnerability of the euro area system, conditional on the default of a sovereign, and it needs to be disentangled from the uninformative portion, where the default of the sovereign has no impact on the system. The $\Delta CoJPoD$ focuses on the informative portion of conditional default probabilities and therefore represents the *information content* in a sovereign default. In Section III we show that focusing solely on conditional probabilities without a consideration about the actual additional information that they provide could lead to spurious conclusions and overestimation of the actual systemic risk. This feature of conditional default probabilities has not yet been analyzed in the financial stability literature.

Apart from the aforementioned information contagion theory, our measure is conceptually related to other measures of systemic risk like the CoVaR (Adrian and Brunnermeier (2016)), the Shapley value (Tarashev, Borio, and Tsatsaronis (2010)) and the Marginal Expected Shortfall (Acharya et al. (2009), Acharya et al. (2012)), which view systemic risk contributions as the difference in the *value at risk* (VaR) of the system when an entity defaults, compared to the case when no default occurs in the system. The main difference to these three concepts is that while they focus on conditional *value at risk* (the CoVaR) and conditional *expected shortfall* (the Shapley value and the Marginal Expected Shortfall), the objects of our analysis are conditional *probabilities of default*. In concentrating on probabilities of default, our approach has several important advantages over the CoVaR and its related measures. First, the previous measures rely on a restrictive definition of default: an institution is considered under distress if its returns drop to the 5-percent or the 1-percent region of its return distribution. In contrast, our approach does not take a stand on what a default actually means, but rather relies on market expectations about the likelihood of a default to occur, based on an unobservable latent process. The main benefit of such an approach is that it allows us to derive systemic risk measures in a *sovereign* context without explicitly defining what sovereign assets are. Since systemic risk is usually defined as the risk of the collapse of the financial system if one (or more) of its participants defaults, focusing on probabilities of default allows us to address the definition more directly, and, in addition, to avoid the problem with the measurement of sovereign assets.

Another drawback of the CoVaR, the Shapley value and the MES is that they rely on historical stock market data. Giglio (2014) points out that reduced-form approaches, recovering return distributions from historical data, suffer from the low number of extreme events in market data. Even if such events existed for euro area sovereigns, their low frequency would at best yield static estimates of probabilities of default. Hence, we would not be able to capture the changes in systemic risk through time, and especially after regulatory interventions. In contrast, our approach tries to circumvent this issue by recovering forward-looking default probabilities from derivatives which are more sensitive to default risk, such as CDS contracts. The further benefits of probabilities of default derived from CDS data are that they also reflect market expectations about sovereign default. Monitoring financial market expectations is of significant importance for policy decision-making in the current sovereign debt crisis in the euro area, since the governments in the eurozone rely on financial markets to finance their short and long term liquidity needs. Moreover, the reactions of the financial markets are an important indicator of how viable and credible policy measures are.

The third strand of literature that our paper relates to is the literature on estimating joint probabilities of default in the financial system. Segoviano and Goodhart (2009) and Radev (2014) have developed a number of measures to analyze the joint default risk of banks using CDS data. Regarding the estimation of sovereign default risk, Gray, Bodie, and Merton (2007) and Gray (2011) develop the so called Sovereign Contingent Claims Analysis where the authors try to sort the capital structure of a sovereign in a particular way depending on its maturity, in order to fit it to a Merton model's framework (Merton (1974)). This requires the assets and liabilities to be assigned to a category at every given point in time, making the method relatively cumbersome. In the current paper, we argue that relying on market perceptions about default, embedded in CDS premia, appears to be an attractive alternative to the Sovereign Contingent Claims Analysis, because we avoid the sorting procedure by focusing directly on probabilities of default derived from market data. Notwithstanding, we still rely on the intuition of the Merton model that an entity (in our case – a bank or a sovereign) defaults on its debt, once its assets process crosses a certain default threshold.

The $\Delta CoJPoD$ measure seems to be most related to the "spillover component" by Zhang, Schwaab, and Lucas (2014), who analyze the difference of the probability of default of Portugal, conditional on the default of Greece and the probability of default of Portugal conditional on Greece not defaulting. Apart from being a bivariate measure (hence, not a systemic measure), the main difference of the "spillover component" to the multivariate $\Delta CoJPoD$ is in the definition of the counterfactual conditional probability, and therefore the measure cannot be interpreted as the *information content* of sovereign default. Section II contains a further discussion regarding this measure, as well as the remaining measures related to the $\Delta CoJPoD$.

We contribute to the existing literature in a number of ways. First, we introduce a new and intuitive systemic risk measure that evaluates the contribution of the system's interdependence to systemic default risk. Similar to the Shapley value, an important feature of our measure is the flexibility in the choice of the "coalitions" of defaulting and non-defaulting entities (sovereign and banks). This means that we could develop a large number of $\Delta CoJPoD$ measures for a single sample of sovereigns or banks, conditioning on the default of one, two or more governments or financial institutions. Therefore, $\Delta CoJPoD$ could be seen as a *family* of measures of systemic risk, rather than as a single measure. We illustrate this flexibility of the $\Delta CoJPoD$ when we investigate cascade effects between euro area sovereigns in Section III. Second, we contribute to the theory on measuring information contagion and extend it to sovereigns. Third, we extend the relatively sparse literature on sovereign default in a multivariate setting by investigating not only the contribution of individual sovereigns to the systemic default risk, but also possible cascade effects within the euro area. Fourth, our study contributes to the financial stability literature that analyzes the feedback effects between sovereigns and the banking system (see, for instance, Demirgüç-Kunt and Huizinga (2013), Barth and Schnabel (2013), and Gorea and Radev (2013)). Fifth, at the methodological level, we are the first to analytically prove and among the first to explicitly address some of the limitations of the original CIMDO approach with regard to multivariate dependence. Sixth, we propose a procedure that alleviates the "curse of dimensionality" inherent in multivariate distribution modeling, based on sorting by bank financial characteristics.

The paper is organized as follows. In Section II, we introduce the $\Delta CoJPoD$ measure and propose a procedure to derive it. Section III presents our results with regard to the systemic importance of euro area sovereigns. We also present an empirical comparison with other existing probability measures and discuss the possible cascade effects among sovereigns. In Section IV, we present our solution to the "curse of dimensionality" and explore how $\Delta CoJPoD$ could be applied to assess the spillover effects between euro area sovereigns and the European Union banking system. Section V concludes by providing the main policy implications of our analysis and suggesting possible applications of $\Delta CoJPoD$ in regulatory decision-making.

II. Conditional Joint Probability of Default

A Definition

At the basis of our conditional probability measure is the joint (unconditional) probability of default ($JPoD$) of the system, which can be interpreted as the *system's overall vulnerability to default events.*

Let the system be described by an n-dimensional joint distribution, $P(x_1, x_2, ..., x_n)$, with density $p(x_1, x_2, ..., x_n)$, where x_1, x_2, ...,x_n are the logarithmic assets of the respective sovereign X_1, X_2, ..., X_n. Following Segoviano and Goodhart (2009), we then define the Joint Probability of Default ($JPoD$) as:

$$JPoD_{x_1,x_2,...,x_n} = JPoD_{system} = \int\limits_{\overline{\mathbf{x}}_1}^{+\infty} \int\limits_{\overline{\mathbf{x}}_2}^{+\infty} ... \int\limits_{\overline{\mathbf{x}}_n}^{+\infty} p(x_1, x_2, ..., x_n) dx_1, dx_2 ... dx_n \tag{1}$$

where $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$, ..., $\overline{\mathbf{x}}_n$ are the individual default thresholds[2] of the respective sovereigns.[3]

Then, applying Bayes rule, we derive the Conditional Joint Probability of Default of the system of n sovereigns, conditional on sovereign k defaulting:

$$\begin{aligned} CoJPoD_{system_{-k}|x_k > \overline{\mathbf{x}}_k} &= JPoD_{x_1,x_2,...,x_{k-1},x_{k+1},...,x_n|x_k > \overline{\mathbf{x}}_k} \\ &= \frac{JPoD_{x_1,x_2,...,x_n}}{PoD^k} \\ &= \frac{JPoD_{system}}{PoD^k}, \end{aligned} \tag{2}$$

where PoD^k is the individual default probability of sovereign k.[4] Therefore, the conditional

[2]The default thresholds are defined in the sense of the classical structural model by Merton (1974): an entity defaults if its assets drops below a specific value. Note that as in Segoviano (2006) and Segoviano and Goodhart (2009), the default region is in the right tail of the distribution. This does not affect our results, due to the assumption of a symmetrical prior distribution, but simplifies our estimation procedure.

[3]As in Segoviano and Goodhart (2009), we define the default threshold to be:

$$\overline{\mathbf{x}}_k = \Phi^{-1}(1 - \overline{PoD^k}),$$

where $\Phi^{-1}(\cdot)$ is the inverse of the cumulative distribution function of a standard normal distribution and $\overline{PoD^k}$ is the through-the-period average of the CDS-derived individual probabilities of default of sovereign k. More details regarding the assumptions and the estimation procedure can be found in Appendix A: Solutions and Proofs.

[4]The individual probability of default of sovereign k is defined as

$$\int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \cdots \int\limits_{-\infty}^{+\infty} p(x_1, x_2, ..., x_n) \mathbf{I}_{[\overline{\mathbf{x}}_1, \infty)} dx_1 \cdots dx_{n-1} dx_n = PoD^k$$

,

probability of default is a ratio between the general vulnerability of the system and the individual default vulnerability of a sovereign.

Note that, by definition, Bayes rule gives us the conditional probability of the remaining (non-defaulting) sovereigns in the system. It does not, however, convey information about what their probability would have been without the shock due to sovereign k's default. We seek to compare the conditional probability of default of the surviving sovereigns with their *unconditional* probability of default, which may also be considered their general vulnerability during "tranquil" times. Therefore, to calculate the contribution of sovereign k's default on the system's default risk, we subtract from $CoJPoD_{system_{-k}|x_k>\overline{\mathbf{x}}_k}$ the unconditional $JPoD$ of the system constituents *excluding* the sovereign in question. Our $\Delta CoJPoD$ measure is then

$$(3) \qquad \Delta CoJPoD_{system_{-k}|x_k>\overline{\mathbf{x}}_k} = CoJPoD_{system_{-k}|x_k>\overline{\mathbf{x}}_k} - JPoD_{system_{-k}}.$$

In essence, we compare the risk of the system when sovereign k is included and defaults, to the situation in which sovereign k is excluded, or otherwise said - independent from the system. So defined, $\Delta CoJPoD$ is the probabilistic alternative to the CoVaR (Adrian and Brunnermeier (2016)).

Next, let us define $JPoD'_{system}$ as the joint probability of default of the system if sovereign k is independent from the rest of the system, all other things equal. The prime indicates that the only difference between $JPoD_{system}$ and $JPoD'_{system}$ is that in $JPoD'_{system}$ we assume independence between entity k and every other entity in the system. Both probability measures are identical in all other aspects of their underlying dependence structure. Applying

where $\mathbf{I}_{[\overline{\mathbf{x}}_1,\infty)}$ is an indicator variable that takes the value of 1 if the latent asset process of sovereign k crosses the sovereign-specific default threshold and 0 otherwise. In practice, it is estimated empirically from CDS spreads using a bootstrapping procedure. The estimation procedure is outlined in Section "Marginal Probability of Default".

Bayes rule, we can reformulate $JPoD'_{system}$ as

$$
\begin{aligned}
JPoD'_{system} &= JPoD'_{x_1,x_2,\ldots,x_{k-1},x_k,x_{k+1},\ldots,x_n} \\
&= JPoD'_{x_1,x_2,\ldots,x_{k-1},x_{k+1},\ldots,x_n|x_k>\overline{\mathbf{x}}_k} \cdot PoD^k \\
&= JPoD'_{x_1,x_2,\ldots,x_{k-1},x_{k+1},\ldots,x_n} \cdot PoD^k \\
&= JPoD_{system_{-k}} \cdot PoD^k.
\end{aligned} \tag{4}
$$

Then, $JPoD_{system_{-k}}$ can also be represented in the following way:

$$
\begin{aligned}
JPoD_{system_{-k}} &= \frac{JPoD'_{x_1,x_2,\ldots,x_n}}{PoD^k} \\
&= CoJPoD'_{system_{-k}|x_k>\overline{\mathbf{x}}_k},
\end{aligned} \tag{5}
$$

where $CoJPoD'_{system_{-k}|x_k>\overline{\mathbf{x}}_k}$ is the conditional counterpart of $JPoD'_{system}$ with respect to entity k. Thus, our systemic risk contribution from equation 3, $\Delta CoJPoD_{system_{-k}|x_k>\overline{\mathbf{x}}_k}$, transforms to

$$
\Delta CoJPoD_{system_{-k}|x_k>\overline{\mathbf{x}}_k} = CoJPoD_{system_{-k}|x_k>\overline{\mathbf{x}}_k} - CoJPoD'_{system_{-k}|x_k>\overline{\mathbf{x}}_k}. \tag{6}
$$

The measure can be viewed then as the difference between the (potentially) *informative* conditional probability $CoJPoD_{system_{-k}}$ and the *uninformative* (due to the independence assumption) conditional probability $CoJPoD'_{system_{-k}}$. Thus, $\Delta CoJPoD_{system_{-k}}$ measures the *information content* of a default of sovereign k, which could also be viewed as the contribution to the systemic default risk due to the system's interdependence with sovereign k.

We should note that in our approach interdependence does not mean interconnection. Sovereigns with no or minor direct financial and trade linkages could still be interdependent if the markets perceive them to be similar in any way. This perceived similarity causes a co-movement of their individual CDS series, which reflects a co-movement of market perceptions about the default of the individual sovereigns. Therefore, analogously to the CoVaR, our measure does not reflect a causal relationship. Nevertheless, we could witness a directionality in our conditional indicator: the *CoJPoD* of sovereign A defaulting given that sovereign B defaults could be different from the *CoJPoD* of sovereign B defaulting given sovereign A defaults.

There are numerous ways to calculate the individual and joint probabilities of default to derive $\Delta CoJPoD_{system_{-k}|x_k}$. To calculate individual probabilities of default (PoD), we choose a bootstrapping procedure that incorporates all available CDS contracts of an entity up to a 5-year horizon. Then we transform the *individual PoD*s into *multivariate JPoD*s using the CIMDO procedure introduced by Segoviano (2006).

B Comparison with Other Measures

Bisias et al. (2012) provide an extensive overview and classification of most of the existing (at the time) systemic risk measures. By definition, the $\Delta CoJPoD$ is related to two main types of measures of systemic risk: on the one hand, to general systemic risk contributions measures, like the CoVaR and the Shapley value, and on the other, to measures of joint default risk of financial institutions, like the unconditional probability of at least two banks to default simultaneously (Avesani, Li, and Pascual (2006) and Radev (2014)) and the conditional probability of at least one bank to default if another bank defaults (Segoviano and Goodhart (2009) and Radev (2014)). This section compares the $\Delta CoJPoD$ to these related measures.

Since the $\Delta CoJPoD$ is a measure of default *risk contributions*, the first candidate for a comparison is the $\Delta CoVaR$ (Adrian and Brunnermeier (2016)), which compares the value-at-risk of the financial system if an institution's return is in the extreme negative tail of its distribution (crisis period) to the value-at-risk of the system if the aforementioned institution is at the median of its return distribution. Gramlich and Oet (2011) outline several properties that a successful systemic risk measure should possess: consistency, flexibility, a forward-looking focus, correspondence with empirical data, suitability for the need of financial regulators. In that respect, as mentioned earlier, the CoVaR suffers from the limited number of extreme returns in market data, which precludes it from a consistent forward-looking estimation of systemic risk. As we base our measure on CDS data, it is exclusively forward-looking and is consistent with the default expectations of market participants within a 5-year horizon. Another limitation of the CoVaR is that it is unable to tackle multivariate interactions, as it models either interactions between two institutions, or between an institution and an aggregated index of the financial system. Therefore, it neglects the underlying dependence between the institutions in the system's "portfolio". In contrast, in our approach we employ a "true" multivariate setting for our $\Delta CoJPoD$ measure by explicitly modeling the dependence structure among all entities in the system. Furthermore, our empirical applications in the next subsections will show how flexible the measure is in capturing not only individual default contributions, but also cascade effects conditioning on a default of several countries, and sovereign default spillover effects to the banking system.

The Shapley value (SV) may be considered as an umbrella term for all systemic risk measures, since it can use any systemic risk contribution measure as a basis and calculates an average value of all possible permutations of the difference in risk between coalitions within the financial system that include a particular bank and coalitions that exclude it. The notion of the SV comes from the game theory literature, where it is used to measure the fair allocation of gains within a coalition. The SV has a number of desired properties: the total gain of the coalition is distributed (Pareto efficiency); players with the same contribution to the gains have the same SV (symmetry); the individual contributions add up to the total gain (additivity); a player with no marginal contribution has a SV of zero (zero player). The $\Delta CoJPoD$ shares many similarities to the SV, since the choice of the "correct" sub-system is at the basis of the derivation of any of the measures in the extended $\Delta CoJPoD$ family. The difference here is that the focus is on a particular permutation of the set of players/countries/financial institutions and not on an averaged value of all possible permutations. The $\Delta CoJPoD$ estimation procedure can easily be extended to meet the general requirements for a Shapley value measure.

With respect to the estimation of multivariate (or joint) probability of default, the financial stability literature puts an emphasis mainly on measuring distress risk of banks (see, for example, Lehar (2005), Avesani, Li, and Pascual (2006), Segoviano and Goodhart (2009), and Giglio (2014)). Of particular interest to supervisors is the probability of at least two banks defaulting (see Avesani, Li, and Pascual (2006), and Radev (2014), for two approaches to derive this measure). Gorea and Radev (2013) and Radev (2014) propose procedures to calculate these measures in a sovereign context. The latter application is rarely explored in the literature, due to the different nature of sovereign and corporate assets. The aforementioned papers do not try to define sovereign assets (contrary to the approach of, for instance, Gray (2011)), but rather rely on the judgement of market participants about the probability of a latent sovereign assets process to cross a given sovereign default threshold. The latter probability is derived from CDS spreads. Such a general view allows researchers to extend the definition of sovereign assets to anything that would affect a government's decision to default, be it public revenues or willingness to pay. Furthermore, the focus on market perceptions about sovereign default risk has important implications for regulators during the current sovereign debt crisis, as euro area governments rely predominantly on the international financial markets to cover their liquidity needs.

Most of the proposed probability measures in the cited studies are unconditional by definition and when authors derive conditional measures, they do not compare them to counterfactual events. As we show in Section III, this can lead to spurious estimation of spillover effects, since no consideration is given to the actual additional information provided by these measures. A notable exception is the "spillover component" or "contagion effect" measure, derived by Zhang, Schwaab, and Lucas (2014). In contrast to the approach underlying the $\Delta CoJPoD$, however, the authors choose to take the difference between the conditional probability of default of a sovereign (Portugal in their case), given another sovereign (Greece) defaults and the conditional probability given the latter does not default. The latter term is the authors' definition of the "tranquil" state of the world. The authors keep the same dependence structure in both components of their measure, which precludes them to interpret it as a contribution to systemic default risk due to interdependence. It also might lead regulators to interpret the "spillover component" as measure of the risk of the system if a sovereign is allowed to default, relative to the case when it is bailed out. This is at odds with the general modeling of bailout probability in the financial stability literature, where bailout expectations affect directly the expected default risk and hence, they reduce CDS spreads. Since CDS spreads are used for the calculation of both the conditional probability and its counterfactual, the effect of bailout expectations remains unclear, even after estimating the "spillover component". As outlined in the previous section, the $\Delta CoJPoD$ has an intuitive interpretation and we suggest applying this measure for regulatory purposes, after explicitly adjusting CDS spreads for bailout expectations in the first step of the procedure. This extension of the procedure remains beyond the scope of the current paper.

C Marginal Probability of Default

We use a *refined* way of estimating probabilities of default (PoD),[5] the CDS bootstrapping, outlined in Appendix C. The procedure follows Hull and White (2000) and is based on

[5] A common (and imprecise) method for estimating probabilities of default from CDS spreads is to use the most liquid contracts in the market, 5-year CDS spreads, to estimate one-year probabilities of default, applying the simple formula

$$PoD_t = \frac{CDS_t * 0.0001}{1 - Recovery\,Rate}, \tag{7}$$

where CDS_t is the 5-year CDS spread at time t, PoD_t is the resulting probability of default estimate and $Recovery\,Rate$ is an assumed recovery rate of the face value of the underlying bond in case of default.

a simple cumulative probability model, which incorporates recovery rates, refinancing rates and cumulative compounding. The model uses CDS contracts of different maturities to calibrate hazard rates of particular time horizons in order to estimate cumulative probabilities of default. This method can be used for both sovereign and corporate probability of default estimation. The resulting risk measures are risk-neutral probabilities of default and satisfy the no-arbitrage condition in financial markets.

We propose using all available maturities from 1 to 5 years of CDS spreads to derive the PoD of an entity. The CDS contracts have quarterly premium payments as a general rule, so we adjust the procedure accordingly. We also correct for accrual interest, as suggested by Adelson, Bemmelen, and Whetten (2004). As refinancing rates, required as inputs, we use all available maturities of AAA Euro Area bond yields from 1 to 5 years. The recovery rate is uniformly set at 40 %, both for banks and sovereigns, as this is the prevailing assumption in the literature and in practice.[6] The resulting series are 5-year cumulative probabilities of default, which we annualize in order to accommodate the one-year horizon of interest to policy makers, using the formula:

$$PoD_t^{annual} = 1 - (1 - PoD_t^{cum})^{\frac{1}{T}}, \tag{8}$$

where T is the respective time horizon (T=5 for 5-year PoD) and PoD_t^{annual} is the annualized

[6] Sturzenegger and Zettelmeyer (2008) find that the historical sovereign recovery rates are usually between 30 and 70%. Zhang, Schwaab, and Lucas (2014) use those results as motivation to choose a 50% recovery rate for their default estimations. We decide to be more conservative with regard to the loss given default assumption, as the recent negotiations for the Private Sector Involvement (PSI) in the Greek bailout packages suggest haircuts between 50 and 70%. As non-institutional investors are the main participants in the CDS markets, we argue that their expectations of default risk are what the CDS spreads reflect, thus we follow the usual recovery rate convention in financial literature. For a discussion on how different recovery rates affect the PoD estimates, please refer to the robustness checks section in Gorea and Radev (2013).

version of the cumulative PoD_t^{cum}.

Figure 1 presents the results from the naïve and the bootstrapping procedures for a distressed sovereign, namely Greece, for the period 01/01/2008 to 12/31/2011. We notice the main drawback of the simple calculation method. While the series generally overlap in tranquil times, they diverge during the distress period starting in May 2010. The margin increases rapidly with the rise of CDS spreads, leading to results higher than unity at the end of the period, which we truncate at 1 to match the definition of probability. The bootstrapped probabilities, on the other hand, have fairly reasonable annualized values in the distress period, peaking at 45 to 50%. The reason for this misalignment is that the naïve Formula 7 can be seen as a linear approximation of the more elaborate bootstrapping procedure, and does not account for all its caveats. The formula performs well for entities with low levels of CDS spreads (Germany, France, Deutsche Bank), but fails for distressed sovereigns or corporates (Greece, Dexia).

It can be argued that the risk neutral probabilities recovered from market CDS data are downward-biased because of the euro area sovereign bailout packages and the government guarantees for the banking sector. Therefore, we can interpret the individual probabilities of default in our analysis (as well as the joint probabilities based on them) as lower bounds for the risk neutral probabilities for the case in which no bailout guarantee is available.

D Multivariate Probability Density

Since joint credit events are rarely traded in the default insurance market, we need to impose a certain structure on the system's joint probability density, in order to transform individual to joint probabilities of default. Our structure of choice is the CIMDO distribution, a result of the CIMDO method introduced by Segoviano (2006). This method builds on the minimum cross-entropy procedure by Kullback (1959) and consists in recovering an unknown multivariate asset distribution using empirical information about its constituting marginal distributions. As Segoviano and Goodhart (2009) point out, the CIMDO approach is related to the structural credit model by Merton (1974), where an entity defaults if its asset value crosses a predefined default threshold. The CIMDO model differs from the structural model in the fact that in the former the threshold is fixed, while in the latter it is allowed to vary. With the default threshold fixed, the CIMDO approach changes the probability mass in the tails of an *ex ante* (or *prior*) joint asset distribution according to the market expectations about the probability of default of each individual entity. The subsequent *posterior* joint distribution, or CIMDO distribution, has two main properties: first, it reflects the market consensus views about the default region of the unobserved asset distribution of the system, and second, it possesses fat tails, even if our starting assumption is joint normality. The latter property reflects the well-documented fact that financial markets are characterized by a higher number of crashes than predicted by the normal distribution. Furthermore, regardless of the *ex ante* joint distribution assumption (a joint normal or a fatter-tailed distribution) the posterior CIMDO distribution is consistent with the observed data.

The CIMDO method therefore has an advantage over many Merton-based methods, most prominently the Contingent Claims Analysis (CCA) by Gray, Bodie, and Merton (2007) and the approach of Lehar (2005), due to its departure from normality and the intrinsically dynamic dependence structure, represented by the CIMDO copula. The CIMDO approach has also been shown to perform exceptionally well in the default region of the system's joint distribution, compared to standard and mixture distributions that are usually used to model market co-movement. [7]

Segoviano (2006) and Gorea and Radev (2013) and Radev (2014) analyze the robustness of the CIMDO approach with respect to some of its main underlying parameters, namely the prior distribution and dependence structure assumptions, paying special attention to the performance in the default region of the posterior joint distribution. Their results show that assuming a multivariate standard normal distribution as a prior provides very similar results to employing a fatter-tail distribution. The resulting posterior distribution is sufficient to

[7] See Segoviano (2006) and Segoviano and Goodhart (2009) for further information and discussions.

explain the behavior in the default region of the distribution of sovereign assets. Assuming a more complex prior distribution does not provide a significant improvement. Therefore, we decide to use a multivariate joint normal distribution for our prior, as in the original work by Segoviano (2006). In Appendix C, we provide a formal definition of the CIMDO approach, as well as a solution of the minimum cross entropy procedure.

A commonly overlooked property of the CIMDO model is that if independence is assumed for the prior distribution (e. g. by assuming a zero-correlation structure for the prior distribution, as in Segoviano (2006)),[8] this transfers to the posterior distribution as well. Appendix C provides a multivariate proof of this caveat when multivariate joint normal distribution is assumed as a prior. In a recent study, Peña and Rodriguez-Moreno (2013) compare the predictions of several systemic risk models, including the CIMDO-derived Banking Stability Index (BSI), but assume a zero-correlation structure for the CIMDO's initial distribution. If this assumption proves wrong, which will likely be the case for the bank assets investigated in the mentioned study, there will be a significant underestimation of the joint default risk between the considered entities. What is more, due to the independence of the posterior distribution, any conditional measures derived using it will be identical to their unconditional counterparts. The latter fact has a pronounced effect on our $\Delta CoJPoD$ measure, as it is exactly the difference between the conditional JPoD and its unconditional alternative. It can be shown that this measure will be exactly 0 at any point of time, despite any dynamics in the individual PoDs. Empirical evidence for this analytical result is provided in Section III.

[8]In general, zero correlation does not imply independence and simple analytical examples are readily available. However, if zero correlation is assumed for a joint normal asset distribution, the resulting joint probabilities of default are a product of the individual entity probabilities of default. Hence, any systemic probability measure that conditions on particular entities defaulting will be equal to the product of the PoDs of the remaining entities. Otherwise said, we do not obtain additional information about the default of the remaining entities, apart from the one already contained in their individual probabilities of default, by conditioning on an entity defaulting. The latter fact exactly complies with the probabilistic definition of independence.

Since the initial correlation structure assumption is crucial for the CIMDO approach, we rely on market estimates to explicitly allow it to differ from the identity matrix. This distress correlation structure is proxied by the empirical correlation between changes in the 5-year CDS spreads of the sovereigns and banks in our sample.

III. Euro Area Sovereign Default Risk

Our empirical analysis is organized as follows. First, we investigate the default risk contributions among 10 euro area sovereigns. With the sovereign debt crisis at its peak in the end of 2011, it is important to analyze the dynamics of our systemic risk measure and identify possible trends, as well as major regulatory interventions and their effects. Second, we focus on the influence of sovereign default risk on the European banking system. We select both euro area and non-euro area EU banks for our analysis, as recent events have shown that the high interconnectedness of the EU banking system facilitates spillover effects from the distressed euro area sovereigns. The empirical financial stability literature that concentrates on CDS markets usually incorporates a very limited number of European banks. Therefore, our set of thirty-six banks makes the current analysis a representative study of the systemic fragility of the European Union banking system.

A Data and Descriptive Statistics

We estimate marginal probabilities of default using CDS premia for contracts with maturities from 1 to 5 years for the period 01/01/2008 and 12/31/2011. The employed procedure (for details, see Hull and White (2000), Gorea and Radev (2013), and Appendix C) requires as additional inputs refinancing interest rates, which we choose to be the AAA euro area government bond yields for maturities from 1 to 5 years. The CDS spreads and the government bond yields are at daily frequency, which is also the frequency of the resulting probabilities of default. Our analysis covers 10 euro area (EA) sovereigns (Austria, Belgium, France, Germany, Greece, Ireland, Italy, the Netherlands, Portugal and Spain). For consistency, the CDS contracts are denominated in euro.[9] The source for the sovereign CDS spreads and the government bond yields is Datastream, while the exchange rate quotes are downloaded from Bloomberg.

Table 1 presents the descriptive statistics of the 5-year CDS spreads of the ten countries in our sample. The average 5-year CDS spread in the cross-section ranges from 29 basis points for Germany to 774 basis points for Greece. We also notice a substantial increase of CDS premia even for the safest country, Germany, from 3 to 91 basis points. However, this does not compare to the dynamics of the price for protection against the default of Greece, which starts at 15 basis points at the beginning of the period and reaches 11034 basis points on December 16, 2011. We also see that the safest countries, Germany and the Netherlands, exhibit the lowest volatility of the price for protection against default.

B Marginal Probabilities of Default

Figure 2 depicts the CDS-implied annualized probabilities of default for the 10 euro area sovereigns in our analysis. We observe very similar values in the beginning of our sample period, pointing towards investors' confidence in the individual EA members' ability to service their debt. We observe a peak in the individual PoDs during the global recession after Lehman Brother's collapse, but the individual default risk gradually falls throughout 2009. A major decoupling occurs in November 2009, after the announcement of the newly-elected

[9] In order to arrive at compareable CDS-derived probabilities of default, all components in the calculation should be under a common currency measure. For many of the sovereigns both euro and US-dollar-denominated CDS contracts are traded. In an unreported analysis, we came to the conclusion that the difference in the absolute levels of the series cannot be explained solely by the exchange rate dynamics. As CDS contracts are usually traded over the counter, it is difficult to find information on the exact volumes traded of each type. After additional talks with professionals, we were assured that in the case of sovereigns, the US-dollar-denominated contracts are more liquid. For this reason, where available, these were chosen in our analysis and the data was transformed using euro-dollar exchange rates, downloaded from Bloomberg.

Greek government that the previously reported data on the government deficit was strongly misleading. The following divergence of market expectations about individual sovereign default risk might be due not only to doubts about the *individual* governments' ability to service their debt, but also about the potential of the euro area as a whole to support its members in need. We also notice that the PoD level of Greece rises throughout the whole period, while the default risk perceptions with regard to the remaining distressed countries - Ireland, Italy, Portugal and Spain - seem to stabilize in the second half of 2011.

C Conditional Joint Probabilities of Default

In this section, we present the $\Delta CoJPoD$ results for our set of euro area sovereigns.[10] Let us first investigate the ingredients of the $\Delta CoJPoD$ measure. Figure 3 shows the results for the correction term $JPoD_{system_{-k}}$ in Equation 3. The general vulnerability of the reduced system rises throughout the period and reaches 0.25% by the end of 2011. What might seem surprising at first glance, is that excluding Greece apparently increases the vulnerability of the rest of the system. This result can be explained after a closer examination of Table 2. Due to the already mentioned decoupling in investors' perceptions about individual sovereign risk, especially with regard to Greece, Greek assets seem to be less correlated with the rest of the system. Hence, if Greece is included in $JPoD_{system_{-k}}$ (all 9 cases where Greece is not the entity k), and another, more highly correlated sovereign, is excluded (that is – assumed to be independent from the rest of the system), this intuitively reduces the $JPoD_{system_{-k}}$. Conversely, if Greece is the particular entity k, the correlation between the remaining entities in $system_{-k}$ is higher, leading to a higher probability of them to default jointly (purple line).

Figure 4 provides the results for the conditional joint probability of default of the system, given the default of a particular sovereign. We notice that the ordering is now inverted, compared to the individual PoDs depiction. The highest CoJPoD is achieved by a default of Germany, narrowly traced by that of the Netherlands. This is intuitive, since these two

[10] In Table 2, we report the dependence structure that we employ in our euro area sovereign analysis.

countries are perceived to be the safest in the euro area system. Therefore, their default should have a significant effect on the perceptions about the default risk of the remaining, riskier countries.

In Figure 5, we present the $\Delta CoJPoD$ results for the 10 euro area sovereigns. As expected from the analysis of $CoJPoD$, Germany and the Netherlands have the highest perceived contribution to the euro area default risk, given their own default. We observe that before Lehman Brothers' bankruptcy in September 2008 the perceptions of the systemic risk contribution of a country's default were practically non-existent. This derives directly from the fact that a *joint* sovereign default within the euro area was perceived as a highly unlikely event. The contribution rises during the turmoil period after Lehman's default, and peaks between January and April 2009, gradually subsiding afterwards. The $\Delta CoJPoD$ measure starts rising again after the announcement of the Greek government budget problems in November 2009 and peaking at nearly 10 percentage points for Germany at the end of November 2011.

A more elaborate interpretation of the $\Delta CoJPoD$ is that its first part, the $CoJPoD$, reflects the relative dynamics of systemic fragility ($JPoD_{system}$) and the individual default risk of a sovereign (PoD_k). For the case of Germany, although its individual risk increases slightly but steadily throughout the sample period, systemic fragility rises at a faster (or falls at a slower) pace. At the other end of the spectrum is Greece, where the individual risk dynamics outpace those of the system, in terms of both growth and magnitude, resulting in a lower risk contribution due to interdependence. A positive result for the risk contribution $\Delta CoJPoD$ means that, due to the interconnectedness of the respective sovereign with the rest of the euro area, the fragility of the system increases more than if the country default is an independent event. Overall, the results for $\Delta CoJPoD$ imply that this difference is much higher for Germany than for the default of any other country.

The reader should notice that there is a second effect contributing to the final results, apart from pure dependence, namely the level effect of the systemic and individual default risks. As the individual level of default risk of Greece is high compared to the systemic default risk level, $CoJPoD$ will be low, leading to low results for $\Delta CoJPoD$, as well. The benefit of our model is that it takes into account the interaction of both effects when evaluating the effects of interdependence on systemic default risk.

We do not detect large differences when comparing the results for $CoJPoD$ and $\Delta CoJPoD$, especially not for Germany and the Netherlands. This stems from the relatively low magnitude of the unconditional adjustment term $JPoD_{system_{-k}}$ for these countries. The lower $CoJPoD$, the higher the relative contribution of the adjustment term to $\Delta CoJPoD$. We see this in the results for Greece, where, after the adjustment, the relative contribution to systemic default risk is practically wiped out. We can relate this fact to our observation that $JPoD_{system_{-k}}$ for Greece is higher than for any other country, due to its low correlation with the rest of the system.

D Comparison to Probability Measures by Segoviano and Goodhart (2009)

As mentioned in Section II, the existing probability measures might offer misleading information about contagion and spillover effects, since authors focus primarily on conditional probabilities and not on contributions to default risk that assess the information content in the default of a sovereign or a bank. In this section, we provide empirical comparison of the $\Delta CoJPoD$ to the probability of at least one bank to default, given a certain bank defaults, or PAO, introduced by Segoviano and Goodhart (2009). Radev (2014) shows that a generalized version of this measure, the probability of all banks to default, given a bank defaults is exactly the CoJPoD in Equation 2 and hence this version of the PAO is directly comparable to the measure that we introduce. For the purpose of the comparison, we translate the measure to sovereigns. Segoviano and Goodhart (2009) assume zero correlation between the entities under investigation and interpret the dynamics of the resulting series as a probability of spillover effects. As we show in Appendix C, assuming zero correlation in a CIMDO setting effectively means that we assume the underlying entities to be independent. Also, as argued in Section II and shown in Appendix C, in this case the default contribution of any of the sovereigns is not different from 0, due to the fact that under independence, the conditional and unconditional JPoD are the same. These theoretical considerations are illustrated in Figure 6, where the $\Delta CoJPoD$ for all countries is practically zero.[11] [12] Therefore, the analysis of $CoJPoD$ alone, as would be suggested by Segoviano and Goodhart (2009), cannot provide any additional information stemming from the default of a particular sovereign, as no contagion effects can be captured.

E Cascade Effects between Euro Area Sovereigns

When analyzing contagion in the financial system, one has to take into account its spacial dimension: a default event for a particular sovereign might affect perceptions about the default of another and their potential joint default might spread to others, in many cases safer sovereign borrowers, and may therefore ultimately affect the default expectations with regard to the entire sovereign financial system. In this section, we show how the $\Delta CoJPoD$ could be used to investigate such "cascade" effects by considering a particular path through which a default of one sovereign might spread through the system. Figure 7 represents this hypothetical path: we assume that the default cascade starts from Portugal and in a first stage, we analyze how it affects Ireland. In a second stage, we examine how a joint default of these two countries could affect the solvency of Spain. In a third and final stage, we study the systemic risk contribution of a joint default of Portugal, Ireland and Spain on the default vulnerability of the EA system of sovereigns.

[Place Figure 7 about here.]

We start by examining the change in the conditional probability of default of Ireland if Portugal defaults. With a minor abuse of notation for the sake of parsimony, the general

[11] The dynamics of the $CoJPoD$ without correlation is similar to the one presented in Figure 4

[12] The spikes in certain periods are due to minor rounding errors.

$\Delta CoJPoD$ definition in Equation 3 transforms to:

$$\Delta CoJPoD_{Ir|Pt} = CoJPoD_{Ir|Pt} - PoD_{Ir}, \tag{9}$$

where $CoJPoD_{Ir|Pt}$ is the conditional probability of default of Ireland (Ir) given the assets of Portugal (Pt) cross its default threshold and PoD_{Ir} is the marginal (empirical) probability of default of Ireland. Compared to Equation 3, the counterfactual joint probability of default of the surviving entities in the system narrows down to a single dimension.

In Figure 8, we present the results for each of the components on the right-hand side of Equation 9. We witness a significant gap between the conditional probability of Ireland defaulting given Portugal defaults (blue) and its unconditional counterpart (red). Observing the individual probabilities of default, or even the raw individual CDS data, does not provide a comprehensive perspective of the complex interactions underlying investors' perceptions about sovereign default.

Figure 9 displays the dynamics of $\Delta CoJPoD_{Ir|Pt}$. We observe that the contribution of a Portuguese default to the distress vulnerability of Ireland rises from relatively modest 10 percentage points to almost 60 percentage points at the peak of the global recession after Lehman Brothers' bankruptcy. Although our measure falls subsequently, it hardly drops below 30 percentage points. We document a new rise from the beginning of 2010 onwards, reaching 50 percentage points in early 2011. The contribution slowly declines thereafter and stabilizes at around 40 percentage points by the end of 2012. Overall, we find a strong effect of a potential Portuguese default on the default expectations about Ireland.

The results in Figure 9 show that, as far as market perceptions are concerned, a default of Portugal is expected to substantially affect the default likelihood of Ireland. Yet, since both countries are relatively small, their difficulties could be fully addressed by the financial stabilization facilities, organized to prevent the spread of the sovereign debt crisis.[13] A recurring theme in the debates between policymakers and regulators is whether a default of these smaller sovereigns could spread to the bigger EA periphery economies, that is, to Spain and Italy.[14] The amount of public debt of these two countries exceeds the size of the aforementioned funding facilities. Therefore, as a next step, we investigate how the joint default of Ireland and Portugal could affect the default perceptions with regard to Spain. To address this issue, we reformulate our Equation 3 to take the following form:

$$\Delta CoJPoD_{Sp|Ir,Pt} = CoJPoD_{Sp|Ir,Pt} - PoD_{Sp}, \tag{10}$$

with $CoJPoD_{Sp|Ir,Pt}$ being the probability of Spain defaulting, given Ireland and Portugal default.

The results for $\Delta CoJPoD_{Sp|Ir,Pt}$ are presented in Figure 10. Although the time series is less volatile, we observe similar level and time paths of default risk contribution to those in Figure 9. The peak is in mid-2010 (55 percentage points), followed by a gradual reduction of the perceived default risk contribution up until July 2011 when the contribution rises again, stabilizing at the relatively high level of 40 percentage points. Therefore, we find that a joint default of Ireland and Portugal substantially increases the perceived default risk of Spain.

The final step of our cascade effects analysis is an examination of how a default of all

[13]The European Financial Stability Facility (EFSF) and the European Financial Stabilization Mechanism (EFSM) were introduced in May 2010 with an initial mandate to borrow up to 500 billion euro to maintain the financial stability of the eurozone. An additional 250 billion euro could be borrowed by the International Monetary Fund within this initial agreement. As of December 2011, the sizes of public debt of Ireland and Portugal are 169 billion euro and 184 billion euro, respectively (Eurostat (2012)).

[14]Digressing to the general financial stability literature, Zhou (2010) points out that when assessing the systemic role of a financial institution, we should consider whether its distress co-occurs with distress of other institutions - the so called "too-many-to-fail" problem, investigated by Acharya and Yorulmazer (2007). Zhou (2010) argues that this effect is more relevant for financial crises than the popular "too-big-to-fail" argument.

three entities affects the perceived default vulnerability of the EA system. This leads to the following $\Delta CoJPoD$ definition:

$$(11)\quad \Delta CoJPoD_{system-Sp,Ir,Pt|Sp,Ir,Pt} = CoJPoD_{system-Sp,Ir,Pt|Sp,Ir,Pt} - JPoD_{system-Sp,Ir,Pt},$$

where $CoJPoD_{system-Sp,Ir,Pt|Sp,Ir,Pt}$ is the perceived probability of default of the surviving EA sovereigns, given a joint default of Spain, Ireland and Portugal.

In Figure 11, we present the results for $\Delta CoJPoD_{system-Sp,Ir,Pt|Sp,Ir,Pt}$ in the sample period (blue), compared to the respective $\Delta CoJPoD$ results when only Spain defaults in the EA system taken from Figure 5 (red). We notice that the systemic default risk contribution triples during the post-Lehman global recession and doubles during the current sovereign debt crisis, compared to the case of a standalone default of Spain. Therefore, we argue that possible default cascade effects of reasonable size should be taken into account in policy decision-making.

IV. Sovereign Default and the European Union Banking System

In this section, we shift the focus of our investigation from the pure sovereign debt perspective and study the perceived effects of sovereign default on the EU banking system. The topic of whether and how a sovereign default could affect the EU financial system is of major concern for regulators, as EU banks hold most of the debt generated by euro area countries and this debt accounts for a sizable part of the banks' assets portfolios.

A Bank Data

In the analysis of spillover effects of sovereign default to the banking system, we use 36 European Union (EU) banks, out of which 28 are euro area banks. The lists of banks in our analysis are presented in Tables 3 and 4. The sources of the CDS data are Datastream and Bloomberg.

To reduce the effect of the curse of dimensionality, we choose to form equal-size portfolios within our banking sample. The latter construction choice makes our results comparable across portfolios. The allocation of each bank in a portfolio is governed by the position of the bank in the distribution of a set of financial characteristics. We select ten financial statement indicators, singled out in the financial literature as important systemic risk factors.[15] Those factors form five broad groups: size, financial gearing, asset quality, performance, and liquidity and funding.

Size. We measure the bank's size by the amount of its total assets (TA). Adrian and Brunnermeier (2016) identify size as a major driver of systemic risk, according to the theory of the "margin spiral" (Brunnermeier and Pedersen (2009)). The authors provide evidence that banks adjust their assets, such that leverage is high in upturns and low in downturns of the economic cycle, making leverage a procyclical characteristic. Sorting by size should provide us with insights whether bigger banks were exposed to higher default risk stemming from sovereign difficulties in the indicated period.

Leverage. Adrian and Brunnermeier (2016) propose the assets-to-equity (AE) ratio as a measure of financial gearing. Our hypothesis is that banks with higher leverage should be more susceptible to adverse credit events in the financial markets. Moreover, many large European Union banks invested heavily in EA sovereign bonds before and during the sovereign debt crisis (EBA (2011), IMF (2011)) and could become insolvent in case of a

[15]The raw data for the individual bank characteristics for the analyzed period are provided by Bloomberg and Bankscope.

sudden drop in the value of their assets.

Asset quality. We use two measures for asset quality. The first is the ratio of loan loss provisions to net interest income. This indicator reflects whether the lending risk undertaken by the banks is appropriately remunerated by higher interest margins. Hence, this measure should be as low as possible. Our second measure for the quality of a bank's portfolio of assets is the ratio of non-performing loans to total loans and is sometimes referred to in practice as the "doubtful loans" (DL) ratio. An increase in this measure should make banks more vulnerable to credit events that further impair their loan quality.

Performance. We use four indicators to measure a bank's performance. The first indicator is the return on equity (ROE), which is a standard measure of corporate efficiency. The main benefit of this measure is that it shows the profitability of the funds invested or reinvested in the company's equity. The main drawback comes from the fact that high-leverage companies could have artificially high ROE ratios, which might reflect the company's excessive risk-taking, rather than its growth potential. The second indicator is the return on assets (ROA), which is the profit from every euro of assets that the bank controls. A potential weakness of this accounting measure is that the balance sheet value of assets may differ from the market value of assets, making it difficult to draw comparisons across industries. Within the banking industry this is less of an issue, due to the relatively regular marking to market of assets. Our third measure is the net interest margin (NIM). The NIM is calculated as interest income minus interest expenses over average earning assets. It indicates how successful the bank's investment decisions were in comparison to the interest-bearing assets. A negative value could indicate a non-optimal banking credit policy or a fast deterioration in the quality of assets. Our last measure is the bank's efficiency ratio (ER). This ratio compares the overhead costs of running the bank to the revenues from the bank's business. The higher the ratio, the less efficient the bank's operations are.

Liquidity and Funding. Our last category includes two indicators. The first is the deposits-to-funding (DF) ratio, which is calculated by dividing total deposits by total funding (sum of total deposits, short- and long-term borrowing and repurchase agreements). This measure reflects the share of stable funding (deposits) to the total amount of a bank's funding. The less a bank relies on wholesale funding, the less exposed it is to global volatility and credit crunches during global crises. The higher this ratio, the better protected a bank is against global market fluctuations.[16] The second liquidity measure is net loans to total assets. This liquidity indicator reflects the share of loans less loan loss provisions to total assets. An increase in this ratio may signal liquidity shortages, as the company has an increasing proportion of illiquid assets (loans).

The frequency of the financial characteristics is quarterly for Bloomberg and annual for Bankscope data. In Table 5, we present the ranking of banks according to the ten factors.

B Estimation Strategy

We choose a particular sovereign, Spain, to be the trigger of default risk in the banking system. As previously argued, due to their small relative size, it is safe to assume that Greece, Ireland and Portugal could be bailed out if needed and hence the resulting default risk within the EU banking system could be relatively easily defused. That leaves Spain and Italy as the main concern among the GIIPS (Greece, Ireland, Italy, Portugal and Spain). The debt level of these two countries might make it infeasible to prevent a default event if they meet difficulties to service their payments (e. g. due to short-term illiquidity issues). For this reason, the ECB has continuously intervened in the debt market once Spain and Italy announced that they would issue new debt to cover their short-term funding needs. To relate to the example from the section on cascade effects, in this section we analyze how a default of Spain could affect the market perceptions about risk in the banking system.

We start by sorting our set of banks by the time average of each of the financial character-

[16] Of course, this measure is only meaningful when there are no bank runs. Since bank runs will affect not only the deposits, but also the general funding availability, the information content of this liquidity measure is reduced during such periods.

istics in the previous section. To address the "curse of dimensionality," we then divide the 36 banks into 9 subsets, resulting in four banks per (sub)portfolio. According to our definition of the financial system, the system could be represented by a portfolio of a set of entities, and it is independent from the remaining entities, which are not included in the system. Therefore, in order to examine the effect of a sovereign default on the banking system in this framework, we need to explicitly include a sovereign (Spain) in each subportfolio, which will act as a trigger for sovereign default in the respective portfolio. Figure 12 demonstrates the construction of each portfolio. Using this strategy, we reduce the joint density modeling to a 5-dimensional problem. For each portfolio within each characteristic, we consider the $\Delta CoJPoD$ in case of Spain's default, resulting in 90 time series for further analysis. For the ease of exposition, we present averages of the results for three subgroups: portfolios 1 to 3, 4 to 6 and 7 to 9.

[Place Figure 12 about here.]

The main hypothesis in the analysis in this section is that if a financial characteristic is important for international investors, we should notice a particular ordering of $\Delta CoJPoD$ across the portfolios. For example, if leverage is an important characteristic for international investors in forming their perceptions about banking susceptibility to sovereign default, we would expect that higher-leveraged banks react more strongly to such an event. Therefore, in solving the "curse of dimensionality," we also manage to provide additional economic intuition behind our results. This is, of course, by no means a ceteris paribus analysis, but nonetheless, it could provide useful policy implications and important insights for further research.

C Spillover Effects from Sovereigns to Banks

Figure 13 depicts the $\Delta CoJPoD$ results[17] given a default of Spain for 9 portfolios sorted by *size*. The results show a clear ordering – we notice a split of our portfolios in two groups, with the biggest banks in our sample reacting much more strongly to increases in Spanish default risk. Several spikes occur throughout 2008 up to the end of the global recession in mid-2009. After relatively stable 9 months, the conditional sovereign default contribution to the fragility of the biggest banks rises again in March-April 2010; and in mid-2011 it surpasses the levels during Lehman Brothers' turmoil. The higher level after July 2011 could be attributed to increased attention of markets to the problems of Italy and Spain. Our results could be explained not only by the sizeable EA sovereign debt holdings on the balance sheets of the biggest banks, but also by the uncertainty about the economic conditions in the European Union during the sample period. The high susceptibility of big banks to sovereign default risk might be related to "too-big-to-save" (Hellwig (1998)) considerations by international investors. The recent experience with the prolonged political process of bailout-package ratifications might explain why investors could be skeptical about multilateral government cooperation to support these international conglomerates.

With regard to *leverage*, Figure 14 provides a rather mixed picture. There are significant peaks during the sample period, especially in the second half of 2011, but the most vulnerable bank groups turn out to be those with relatively modest levels of leverage. This indicates that financial gearing might not be a good indicator for the reaction of banks to sovereign debt problems. An argument why leverage can provide misleading results is the fact that during crises financial institutions tend to procyclically reduce their leverage level, sometimes at a high cost, which makes them highly vulnerable to financial market volatility.

Interestingly enough, the sorting by return on equity (Figure 15) reveals that the market

[17]We present and interpret the results for several financial characteristics only. The rest of the results are available upon request.

perceptions of the default risk of the *highest-performing* banks tend to react more intensively to sovereign default risk. The top three portfolios appear to have four to six times higher $\Delta CoJPoD$ than the second subgroup, especially in the periods around the Bear Stearns episode, the bankruptcy of Lehman Brothers and the following global recession, as well as during the more recent events, related to the sovereign debt crisis. A possible explanation might be that in international investors' view the higher performance might signal that the banks in question are involved in activities that are too risky.

We now turn our attention to the sorting by *asset quality*, measured by the doubtful loans ratio. Figure 16 provides some evidence that international investors do take asset quality into account when assessing default risk. The middle set of portfolios is consistently above the top and bottom set, but since April 2010 the banks with the highest doubtful loans ratio gradually reduce the differential. Therefore, it could be the case that in the second half of our sample period this factor gains increasing importance for international investors.

The ordering by total deposits to total funding (Figure 17) seems to follow our expectations that banks with lower values for this indicator (hence more reliant on funding from financial markets) are more vulnerable to sovereign default risk. The "correct" ordering of the portfolios is especially evident after the outbreak of the sovereign debt crisis.

We conclude that according to our analysis, the most important factors that affect the transmission of default risk from EA sovereigns to EU banks are bank size, riskiness of operations, asset quality and liquidity and funding. The effect of leverage is difficult to interpret, due to the endogeneity of this measure.

V. Conclusion

This paper introduces a new way of thinking about systemic risk contributions – not in terms of losses to the system as in the traditional financial stability literature, but rather in terms of conditional probabilities of default. Thus, we bridge the gap between two strands of literature: the literature on systemic risk contributions (Adrian and Brunnermeier (2016), Acharya et al. (2009), Acharya et al. (2012), Tarashev, Borio, and Tsatsaronis (2010)) and the literature on multivariate default risk measurement (Gray, Bodie, and Merton (2007), Zhang, Schwaab, and Lucas (2014), Gorea and Radev (2013), Radev (2014)).

The new systemic risk measure that we propose, the $\Delta CoJPoD$, assesses the information content of a sovereign default and the effect of interdependence on the general default risk of the financial system. The measure is related to the CoVaR, the Shapley value and the Marginal Expected Shortfall and captures the relationship between the overall systemic fragility and the individual default risk. We apply our measure to three cases: first, we estimate individual default contributions of euro area sovereigns to the systemic default risk in the euro area. Second, we investigate cascade effects among euro area sovereigns. Last, we analyze sovereign default spillover effects to the European Union banking system.

Our results suggest that interdependence plays a major role in investors' perceptions about systemic risk in the European financial system. We find that countries with a relatively small size, such as the Netherlands, might have a significant systemic risk contribution if investors perceive them to be interdependent with others in the euro area. Another important conclusion from our analysis is that investors expect it to be difficult to prevent a sovereign default from spreading, once it has been triggered. The joint default of two relatively small sovereigns, such as Ireland and Portugal, increases the probability of default of Spain by up to 55 percentage points. This effect has been persistent since Lehman's collapse. Therefore, our results provide support for the determined and often costly efforts of the European regulators and policy-makers to prevent a sovereign default. With regard to the analysis of spillover effects from EA sovereigns to the EU banking system, investors seem to perceive bank size, balance sheet composition and risk, and asset quality as important systemic vulnerability indicators. Therefore, regulators should allocate more resources to supervising the operations of the largest EU banks, in order to prevent the collapse of the European banking system. Surprisingly, financial gearing seems to be less informative in this respect.

This might indicate that international investors consider the European banking system to be already highly leveraged and is an interesting issue for further research.

With regard to policy decision-making, there are heated debates whether euro area sovereigns should be allowed to default. The proponents of this view claim that the economic costs of a sovereign default would be lower than the size of the bailout packages to keep the sovereign solvent. The *CoJPoD* could be used to estimate the market expectations about the size of those alternative regulatory measures. To come up with meaningful regulatory suggestions pro and con a bailout package, the *CoJPoD* should be coupled with an estimate of the losses to the system given the respective sovereign defaults. The resulting expected loss estimate should be used to determine the size of the considered bailout package. This expected size should be then compared to the welfare costs of alternative instruments in the regulatory toolkit. Note that even if a sovereign bailout package turns out to be optimal in order to minimize social costs, it might not be feasible even with the broadest possible international cooperation. Policy makers should then resort to their remaining tools to address the consequences of a sovereign default. In either case, we hope that *CoJPoD* becomes a useful ingredient in the decision-making process of regulators and policy makers.

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Appendix A. Figures

Figure 1: 5-year annualized CDS-implied probabilities of default of Greece, using the simple formula 7 (GR(simple)) and the bootstrapping procedure (GR). The 5-year annualized CDS-implied bootstrapped probabilities of default are derived from the respective cumulative ones using formula 8. Euro-denominated CDS spreads are used. Period: 01/01/2008 – 12/31/2011. Source: own calculations.

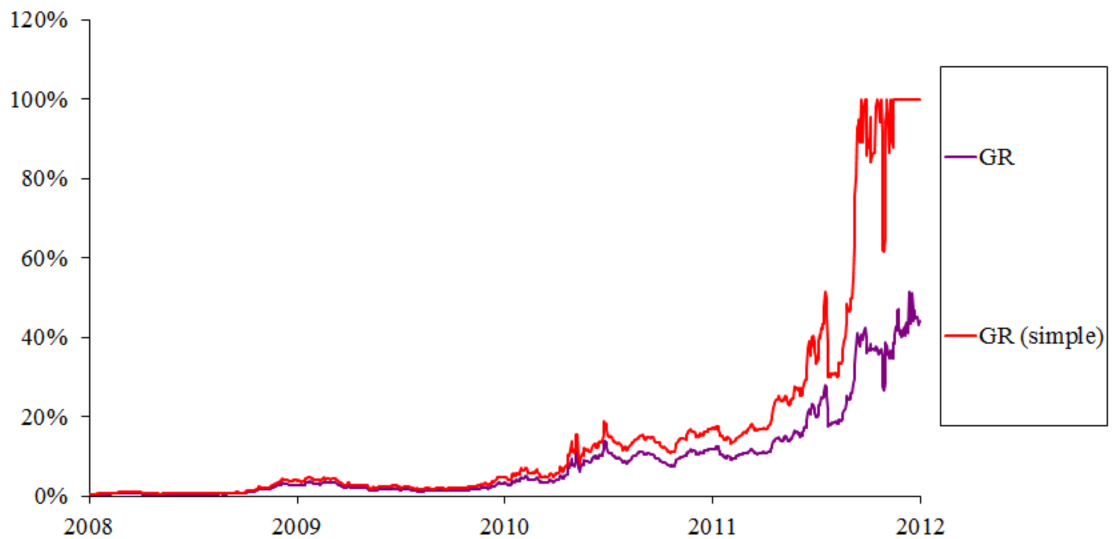


Figure 2: 5-year annualized CDS-implied bootstrapped probabilities of default for 10 sovereigns: Austria (AT), Belgium (BE), France (FR), Germany (GE), Greece (GR), Ireland (IE), Italy (IT), the Netherlands (NL), Portugal (PT), Spain (ES). Euro-denominated CDS spreads are used. Period: 01/01/2008 – 12/31/2011. Source: own calculations.

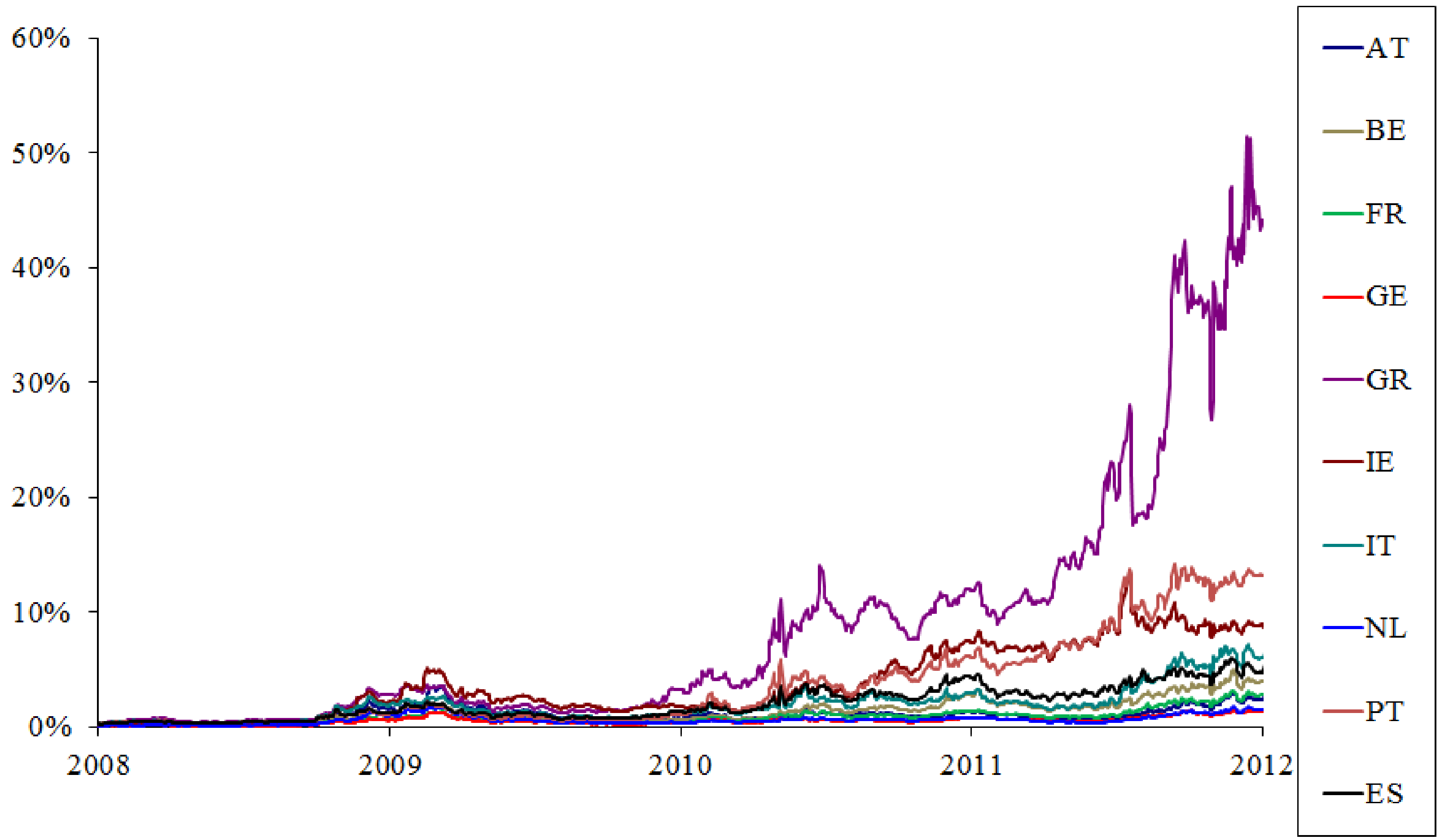


Figure 3: Joint probabilities of default (JPoD) involving 10 sovereigns for the period 01/01/2008 – 12/31/2011. "System" is the full set of sovereigns, listed in Figure 2 and the text. For each series plotted, a particular sovereign is not included in the calculation, reducing the problem to 9 dimensions. E. g. "System-AT" means that Austria is excluded from the calculations. The abbreviations of the sovereigns are analogous to those in Figure 2. Non-zero correlation structure is used in defining the prior distribution. The actual correlation matrix used in deriving the current figure excludes the respective sovereign, listed in the legend. The full system correlation matrix is presented in Table 2. The correlation structure calculation is explained in the text and in Table 2.

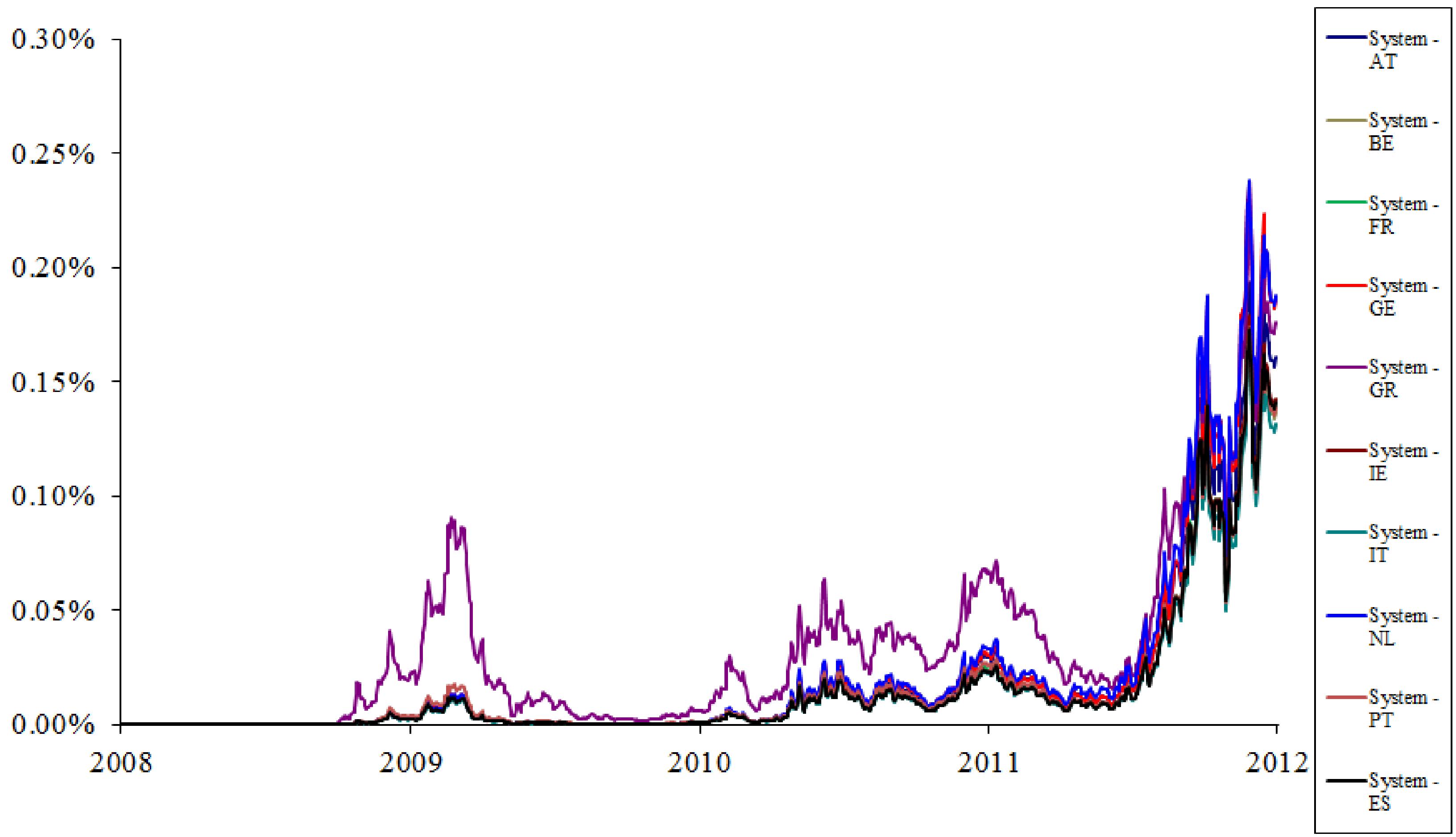


Figure 4: Conditional joint probabilities of default (CoJPoD) involving 10 sovereigns for the period 01/01/2008 – 12/31/2011. Non-zero correlation structure is used in defining the prior distribution. The correlation matrix is presented in Table 2. The correlation structure calculation is explained in the text. The legend details the country that we condition our 10-entity- system joint probability of default on to derive each series. For explanation of the abbreviations, see Figure 2. **Note:** In contrast to Figure 3, here we use 10-dimensional joint probabilities of default.

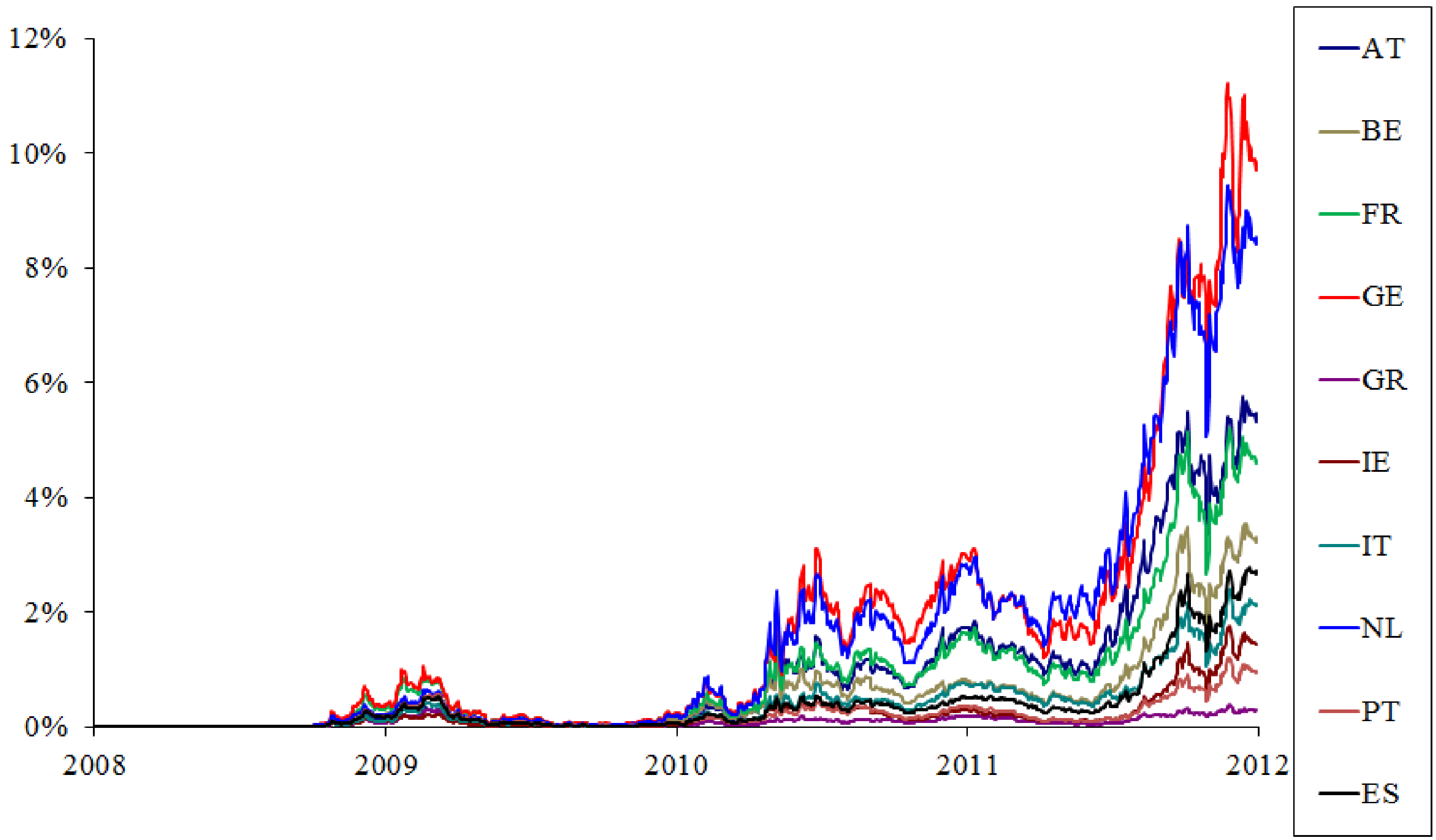


Figure 5: Change in the conditional joint probabilities of default ($\Delta CoJPoD$) involving 10 sovereigns for the period 01/01/2008 – 12/31/2011. Non-zero correlation structure is used in defining the prior distribution. The correlation matrix is presented in Table 2. The correlation structure calculation is explained in the text and in Table 2. For explanation of the abbreviations, see Figure 2. $\Delta CoJPoD$ is the difference between the respective series presented in Figures 4 and 3.

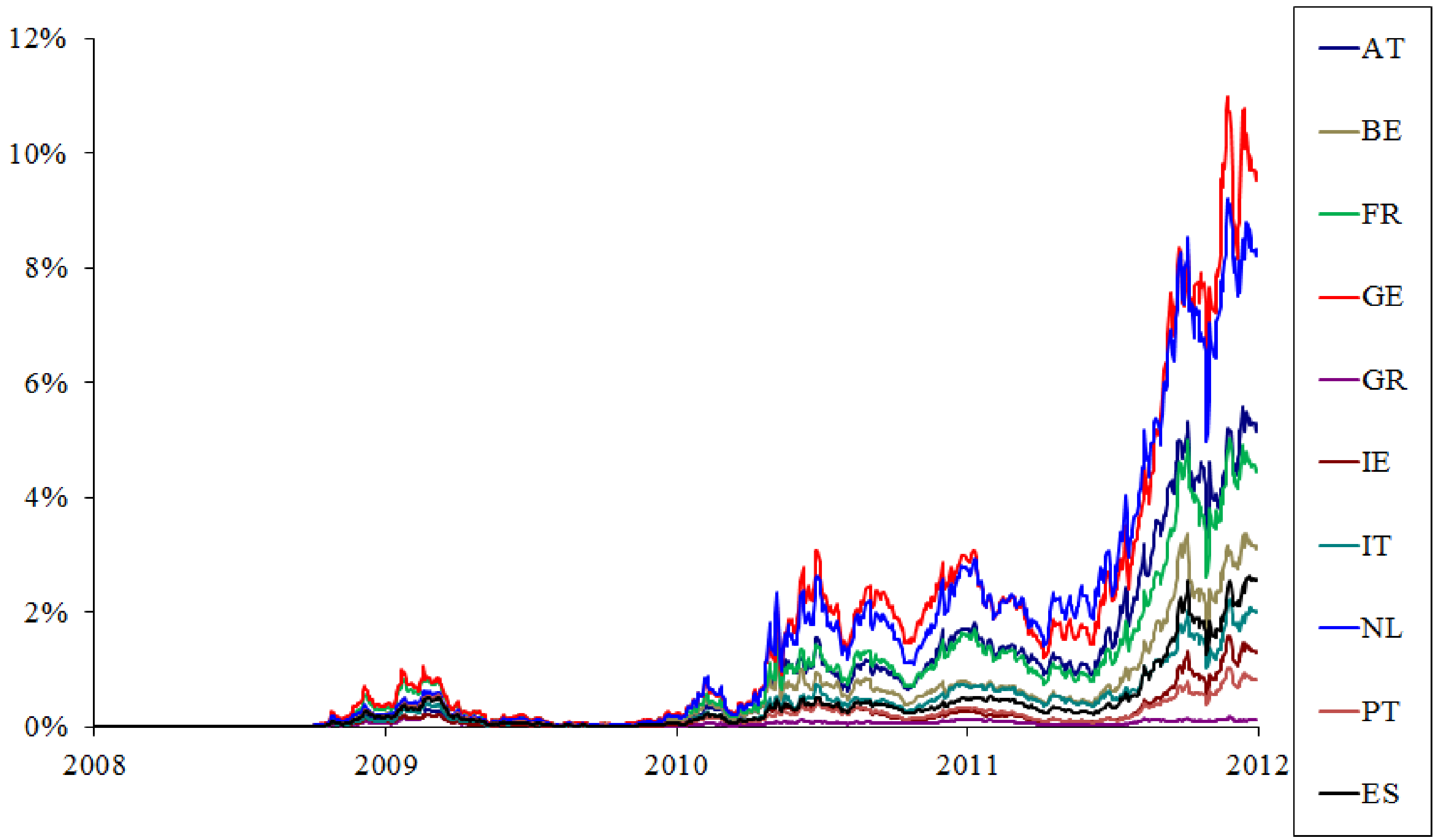


Figure 6: Change in the conditional joint probabilities of default ($\Delta CoJPoD$) involving 10 sovereigns for the period 01/01/2008 – 12/31/2011. Zero correlation between all sovereigns involved is assumed in defining our prior distribution. $\Delta CoJPoD$ is the difference between zero-correlation analogues of the respective series presented in Figures 4 and 3. This plot confirms the analytical result that if independence is assumed for the prior distribution, it transfers to the CIMDO posterior distribution, yielding $\Delta CoJPoD = 0$. For explanation of the abbreviations, see Figure 2.

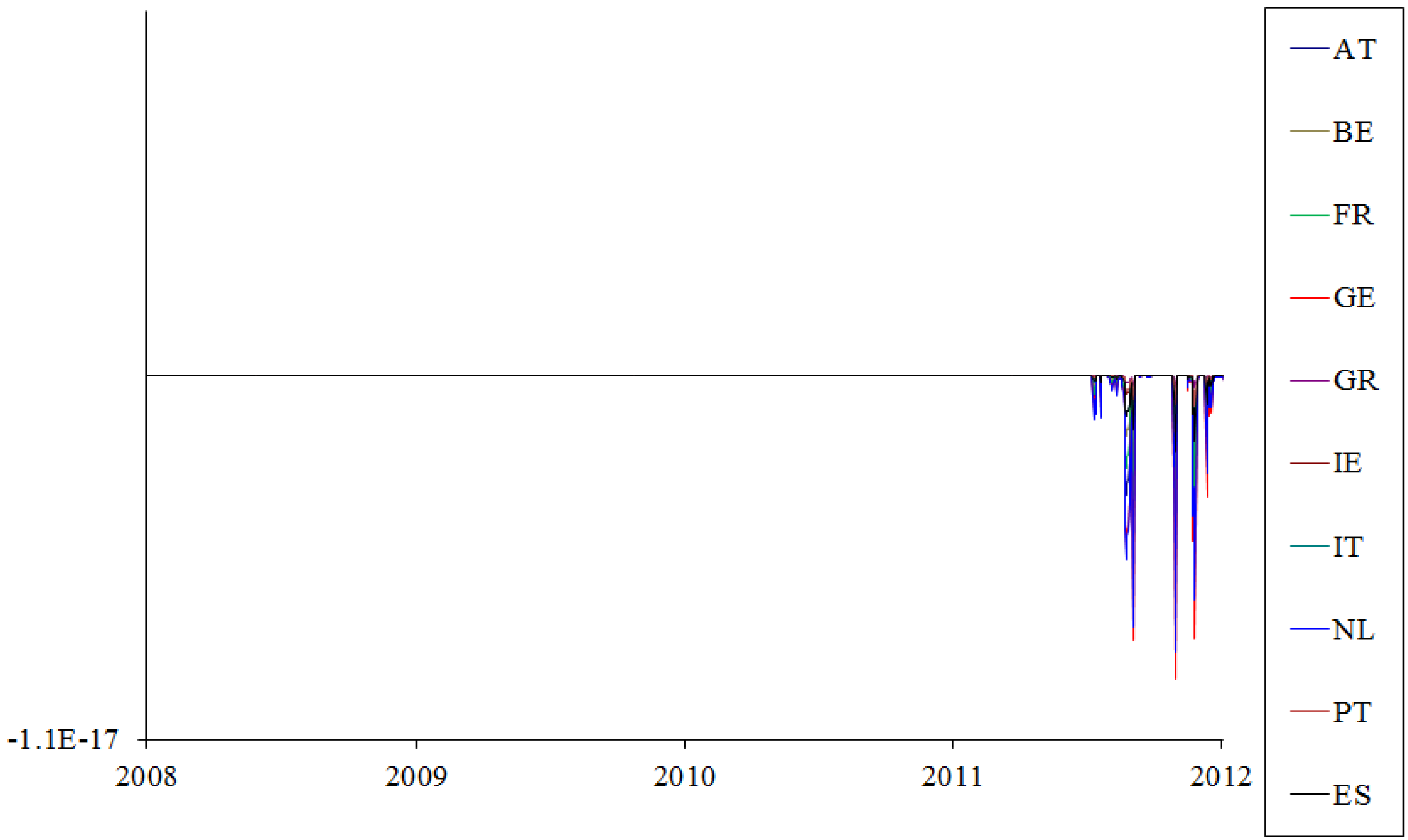


Figure 7: Scheme for cascade effects in the euro area system of sovereigns. We assume that the default cascade starts from Portugal and in a first stage, we analyze how it affects Ireland. In a second stage, we examine how a joint default of these two countries could affect the solvency of Spain. In a third and final stage, we study the systemic risk contribution of a joint default of Portugal, Ireland and Spain on the default vulnerability of the EA system of sovereigns.

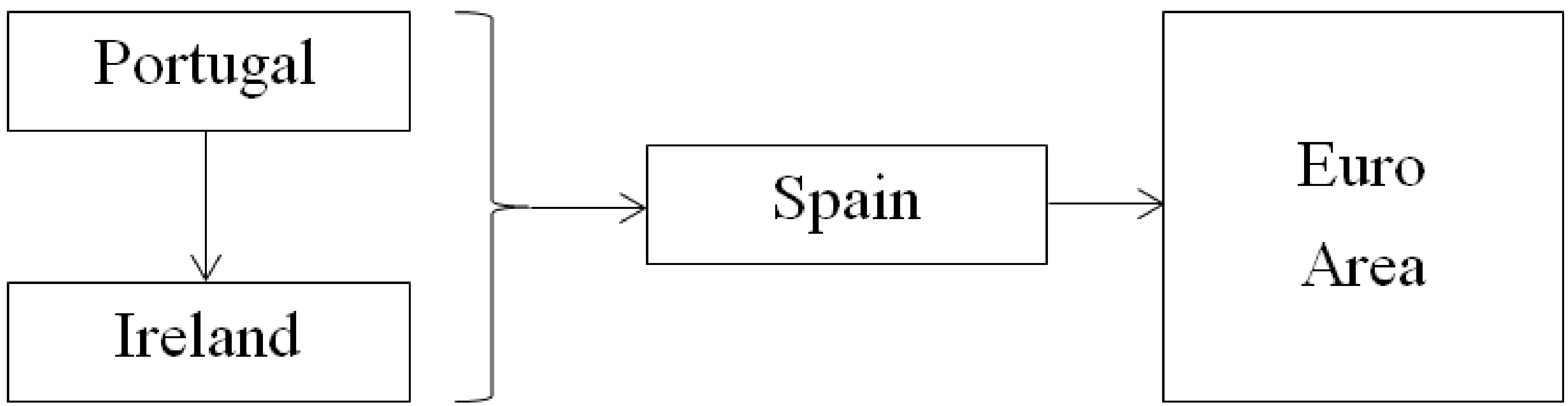


Figure 8: Conditional probability of default ($\Delta CoJPoD$) of Ireland given Portugal defaults (blue) and unconditional probability of Ireland (red) for the period 01/01/2008 – 12/31/2011. Non-zero correlation structure is used in defining the prior distribution. The respective correlation coefficient is presented in Table 2. The correlation structure calculation is explained in the text and in Table 2. For explanation of the abbreviations, see Figure 2.

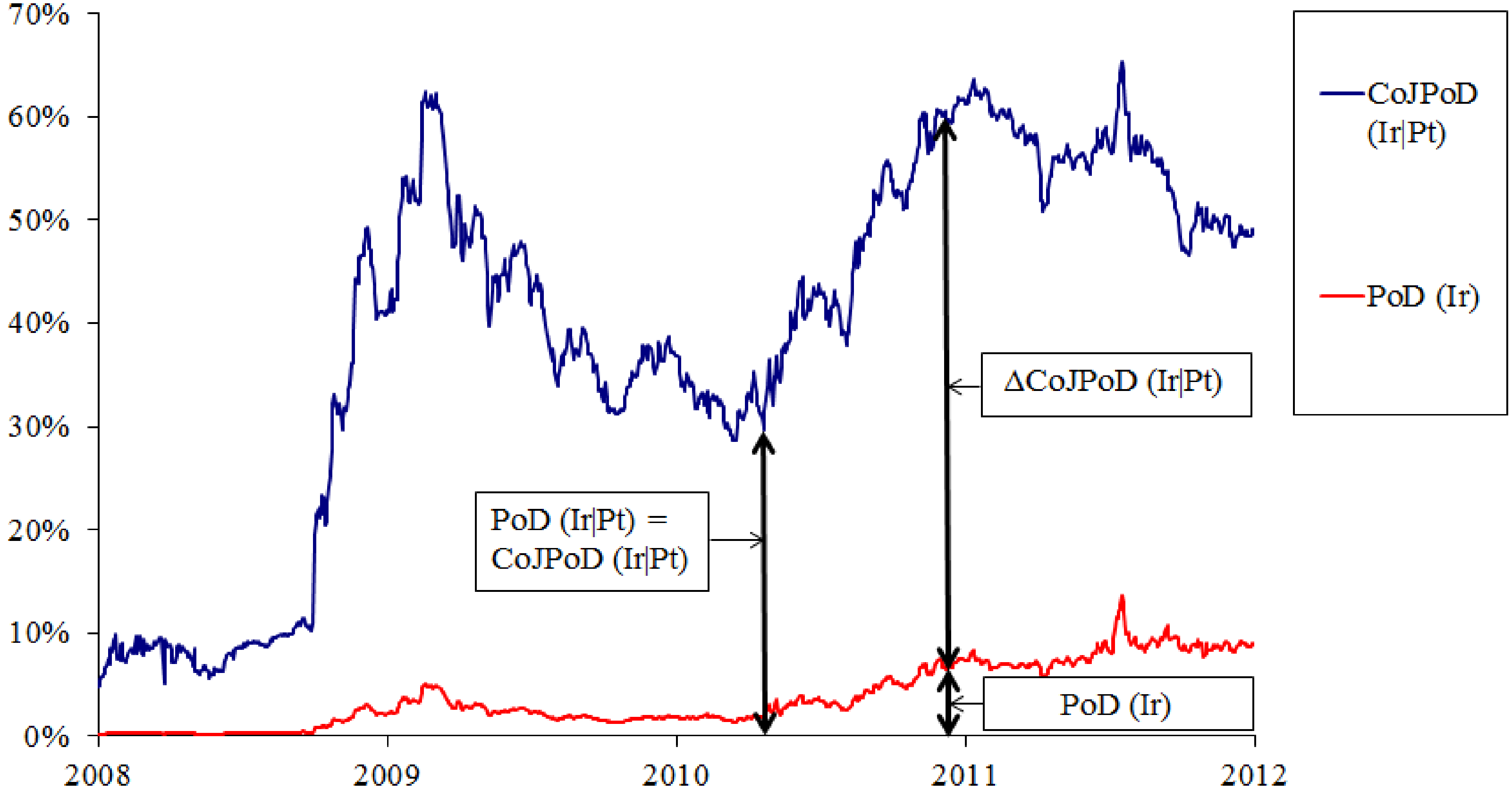


Figure 9: Change in the conditional probability of default ($\Delta CoJPoD$) of Ireland given Portugal defaults for the period 01/01/2008 – 12/31/2011. Non-zero correlation structure is used in defining the prior distribution. The respective correlation coefficient is presented in Table 2. The correlation structure calculation is explained in the text and in Table 2. For explanation of the abbreviations, see Figure 2.

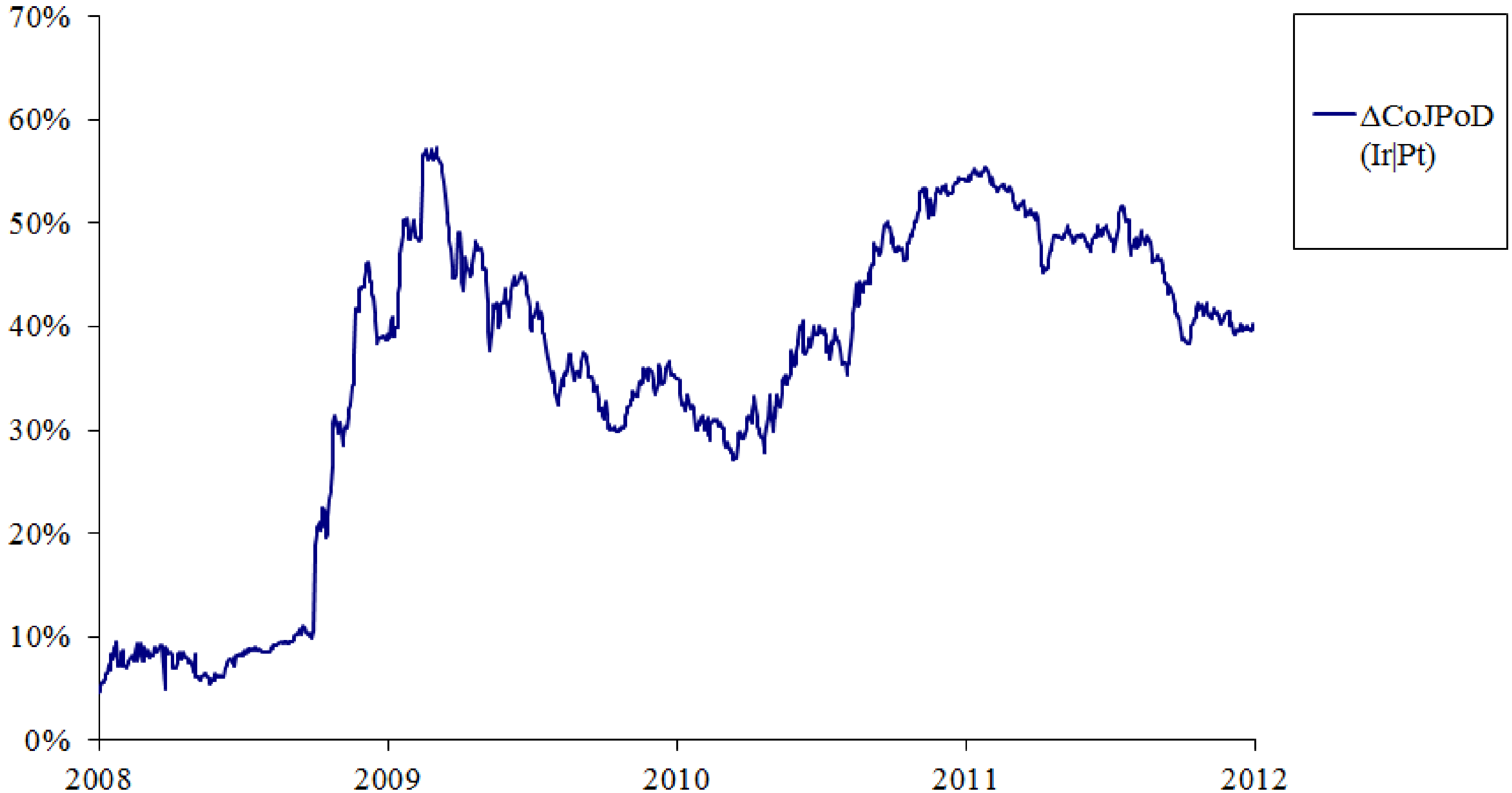


Figure 10: Change in the conditional probability of default ($\Delta CoJPoD$) of Spain given Ireland and Portugal jointly default for the period 01/01/2008 – 12/31/2011. Non-zero correlation structure is used in defining the prior distribution. The respective correlation coefficients are presented in Table 2. The correlation structure calculation is explained in the text and in Table 2. For explanation of the abbreviations, see Figure 2.

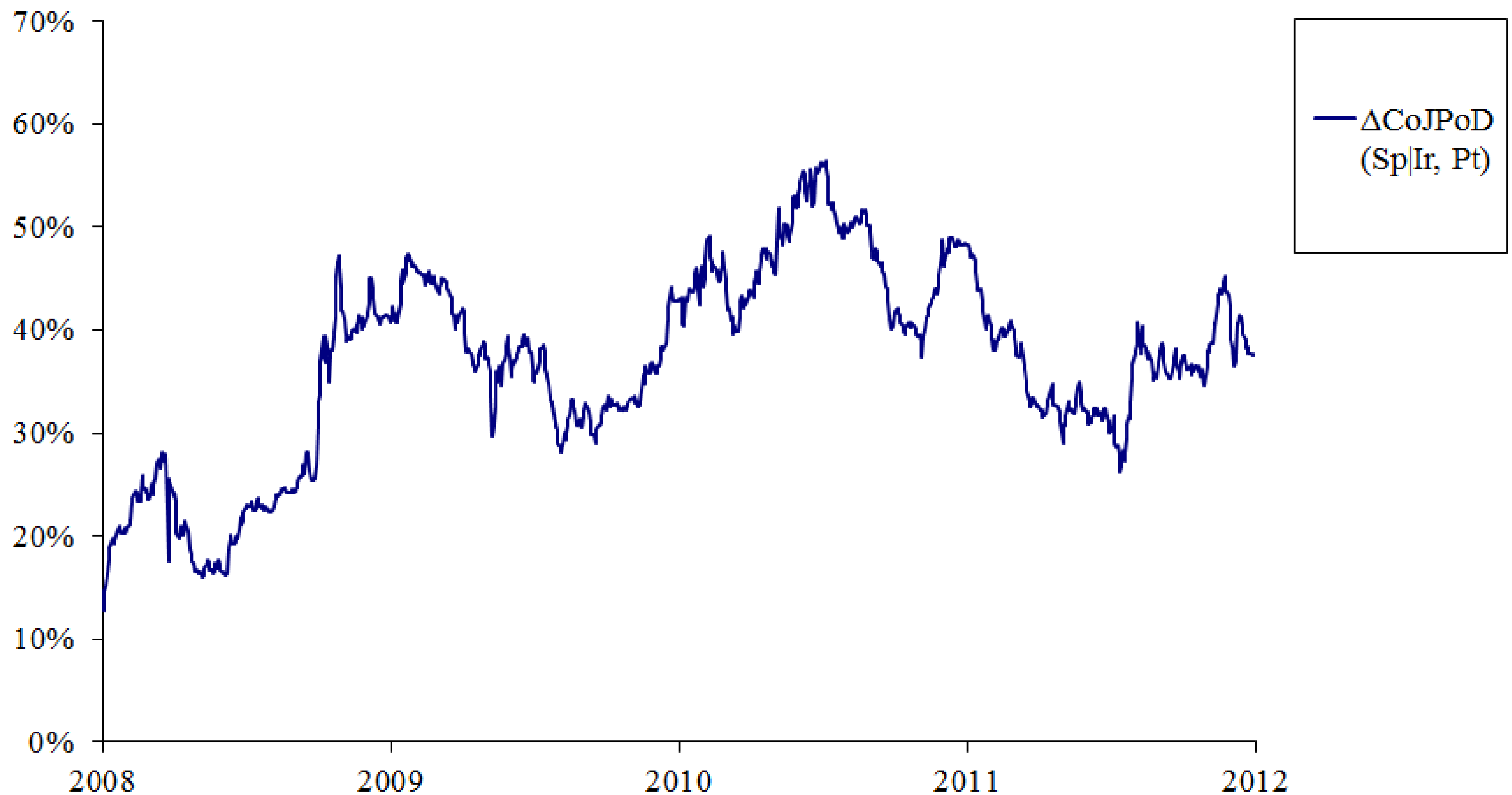


Figure 11: Change in the conditional joint probabilities of default ($\Delta CoJPoD$) for the period 01/01/2008 – 12/31/2011. $\Delta CoJPoD(Sp, Ir, Pt)$ (red) is the conditional probability given Spain, Ireland and Portugal jointly default, while $\Delta CoJPoD(Sp)$ (black) is the respective conditional probability given a default of Spain, depicted in Figure 5. Non-zero correlation structure is used in defining the prior distribution. The correlation matrix is presented in Table 2. The correlation structure calculation is explained in the text and in Table 2.

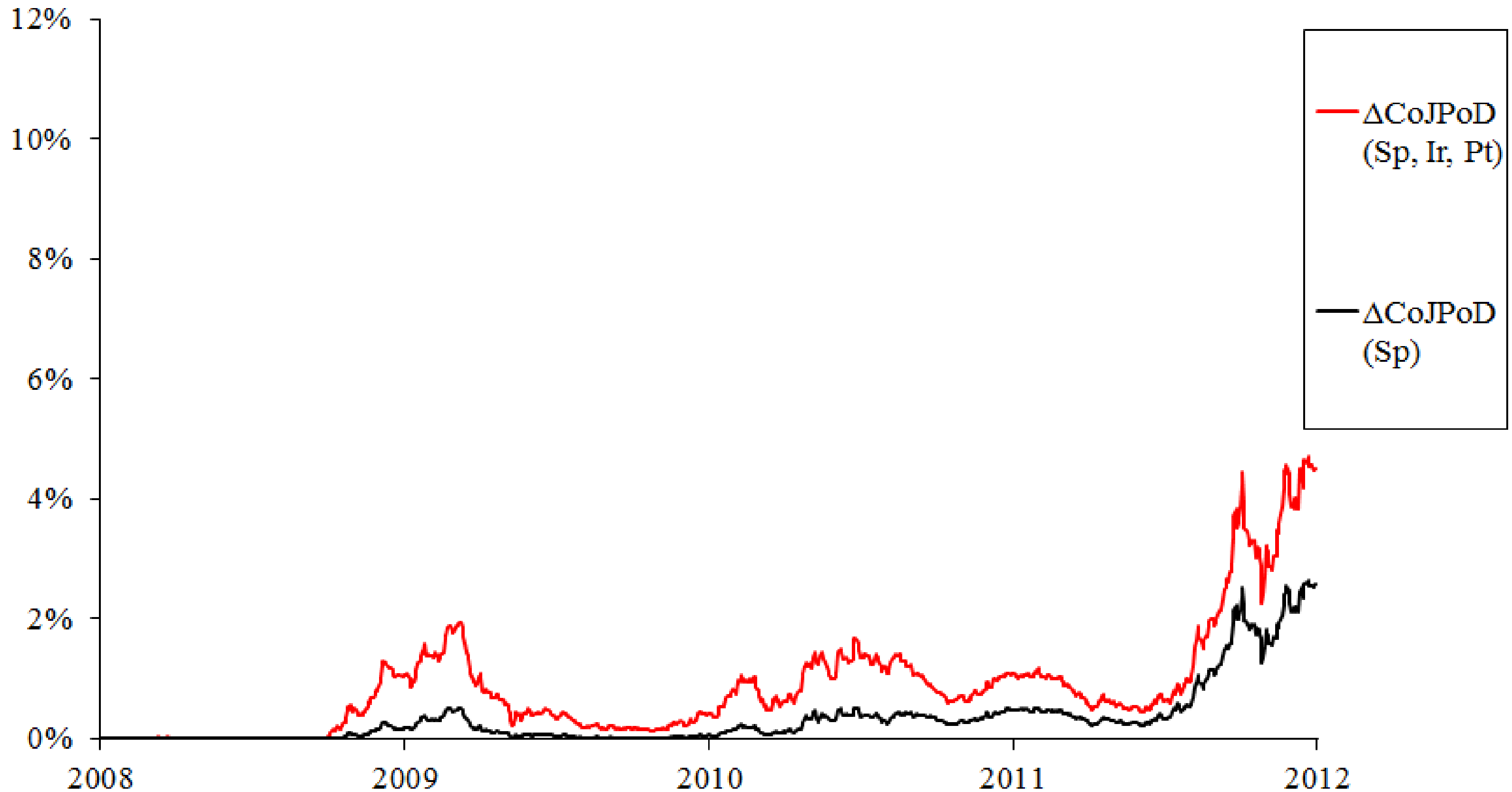


Figure 12: Scheme for construction of portfolios to examine spillover effects of sovereign default to the banking system. We start by sorting our set of 36 banks by the time average of each of the financial characteristics in Section IV. The banks are listed in Tables 3 and 4 and the results from the sorting are presented in Table 5. To address the "curse of dimensionality," we divide the 36 banks into 9 subsets, resulting in 4 banks per (sub)portfolio. In order to examine the effect of a sovereign default on the banking system in this framework, we need to explicitly include a sovereign (Spain) in each subportfolio, which will act as a trigger for sovereign default in the respective portfolio. Using this strategy, we reduce the joint density modeling to a 5-dimensional problem. For each portfolio within each characteristic, we consider the $\Delta CoJPoD$ in case of Spain's default.

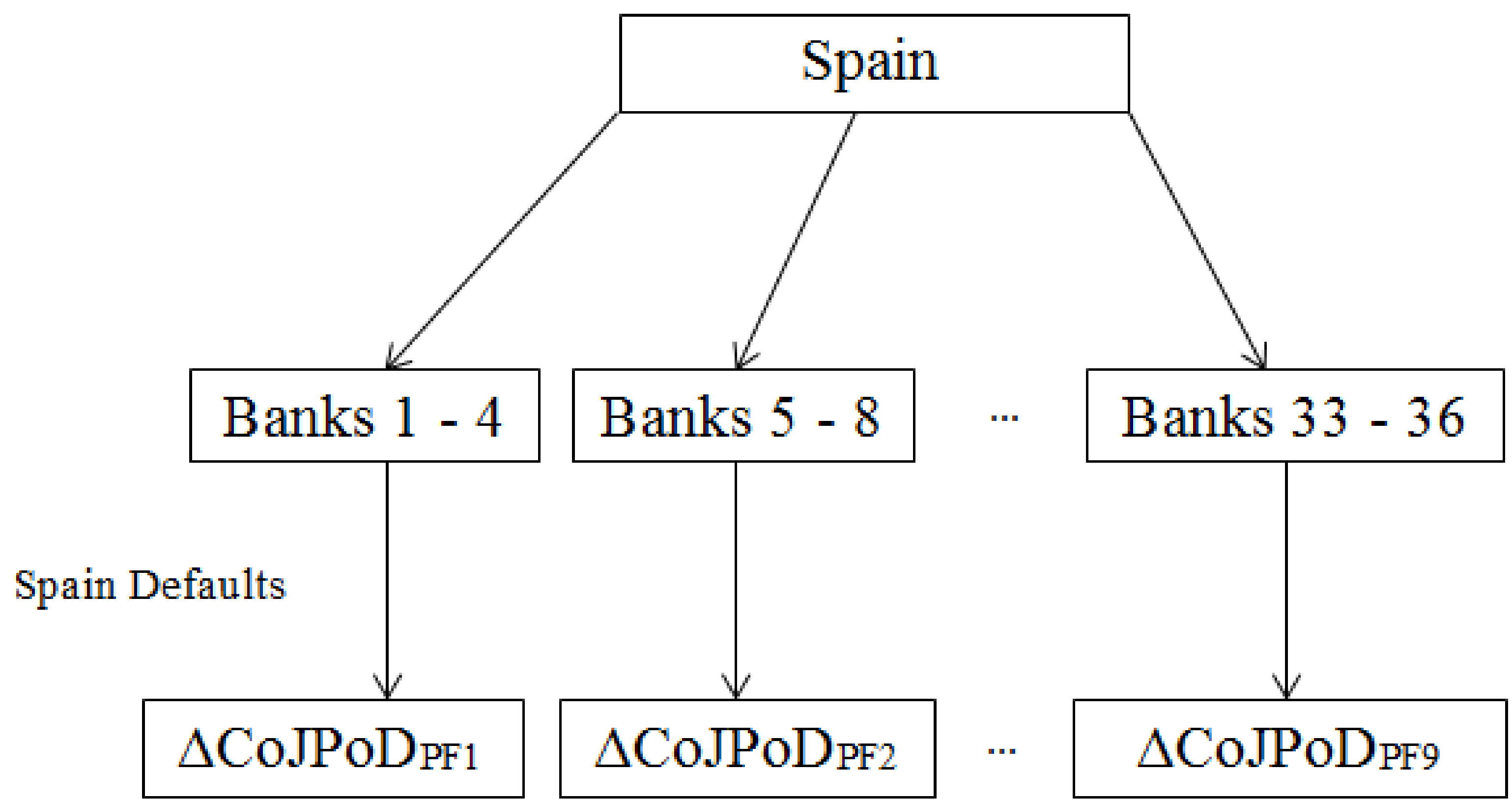


Figure 13: Change in the conditional joint probabilities of default ($\Delta CoJPoD$) involving 9 portfolios of 36 banks and 1 sovereign for the period 01/01/2008 – 12/31/2011. Each portfolio includes 4 banks and 1 sovereign. Spain is considered uniformly as a default-risk-triggering sovereign. Only the series conditioning on default of Spain are presented for each portfolio. The assignment of a bank to a portfolio is governed by its **size**, compared to the rest, sorted in descending order. E. g. Portfolio 1 (PF 1) includes the four banks with the highest value of *total assets*. For presentation purposes, the **average results of the top, middle and bottom 3 portfolios** are shown in the figure. The set of banks is presented in Tables 3 and 4. The complete portfolio assignment is presented in Table 5, column TA. Non-zero correlation structure is used in defining the prior distribution. The correlations are calculated between changes in the 5-year CDS spreads of the respective banks and sovereigns in each portfolio. $\Delta CoJPoD$ is calculated analogously to the sovereign case. The total assets are denominated in euro.

Figure 14: Change in the conditional joint probabilities of default ($\Delta CoJPoD$) involving 9 portfolios of 36 banks and 1 sovereign for the period 01/01/2008 – 12/31/2011. Each portfolio includes 4 banks and 1 sovereign. Spain is considered uniformly as a default-risk-triggering sovereign. Only the series conditioning on default of Spain are presented for each portfolio. The assignment of a bank to a portfolio is governed by its **leverage**, compared to the rest, sorted in descending order. E. g. Portfolio 1 (PF 1) includes the four banks with the highest value of *assets to equity*. For presentation purposes, the **average results of the top, middle and bottom 3 portfolios** are shown in the figure. The set of banks is presented in Tables 3 and 4. The complete portfolio assignment is presented in Table 5, column AE. Non-zero correlation structure is used in defining the prior distribution. The correlations are calculated between changes in the 5-year CDS spreads of the respective banks and sovereigns in each portfolio. $\Delta CoJPoD$ is calculated analogously to the sovereign case. The financial data used to calculate the leverage are denominated in euro.

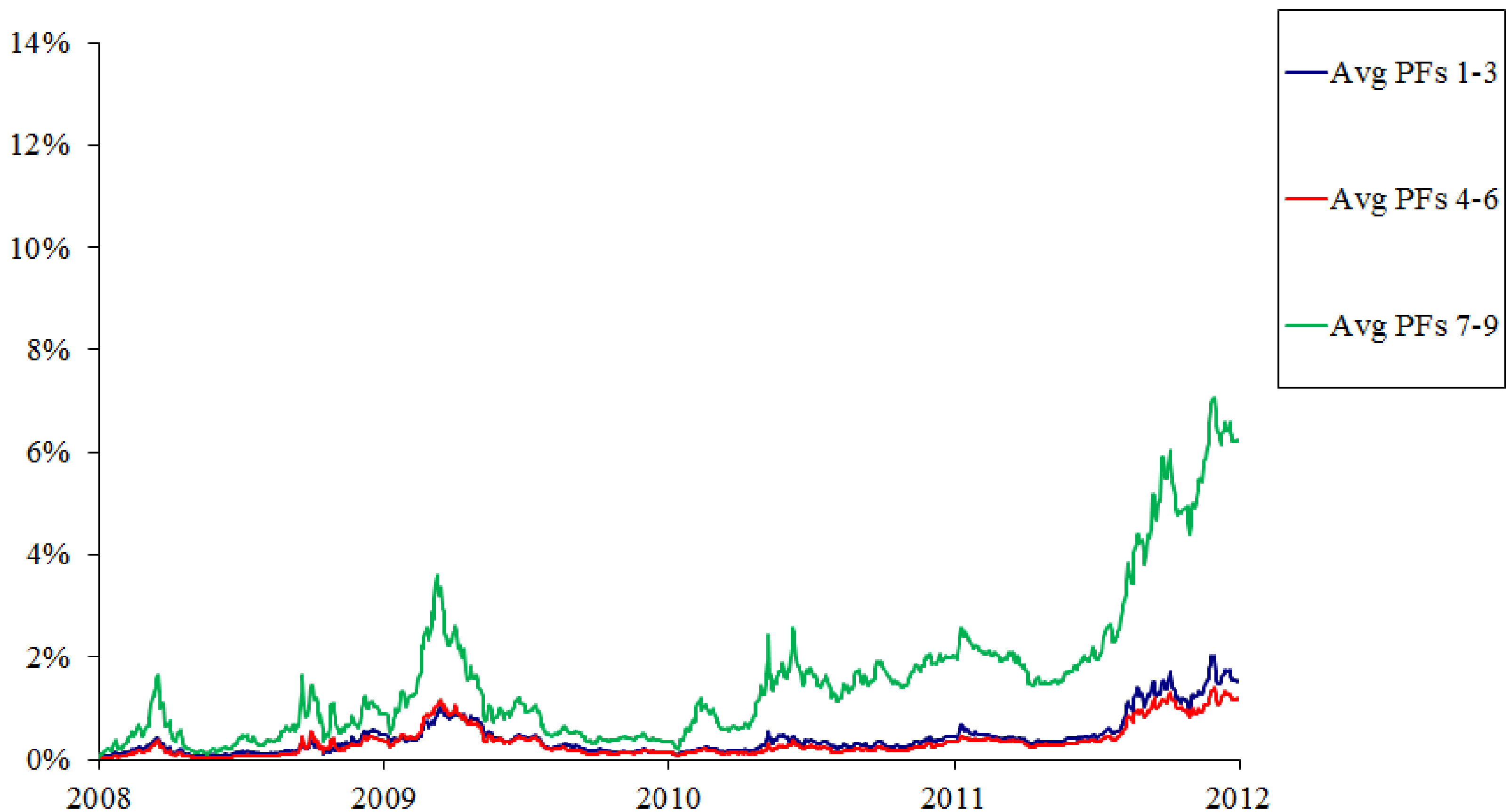


Figure 15: Change in the conditional joint probabilities of default ($\Delta CoJPoD$) involving 9 portfolios of 36 banks and 1 sovereign for the period 01/01/2008 – 12/31/2011. Each portfolio includes 4 banks and 1 sovereign. Spain is considered uniformly as a default-risk-triggering sovereign. Only the series conditioning on default of Spain are presented for each portfolio. The assignment of a bank to a portfolio is governed by its **performance**, compared to the rest, sorted in descending order. E. g. Portfolio 1 (PF 1) includes the four banks with the highest value of *return on equity*. For presentation purposes, the **average results of the top, middle and bottom 3 portfolios** are shown in the figure. The set of banks is presented in Tables 3 and 4. The complete portfolio assignment is presented in Table 5, column ROE. Non-zero correlation structure is used in defining the prior distribution. The correlations are calculated between changes in the 5-year CDS spreads of the respective banks and sovereigns in each portfolio. $\Delta CoJPoD$ is calculated analogously to the sovereign case. The financial data used to calculate the leverage are denominated in euro.

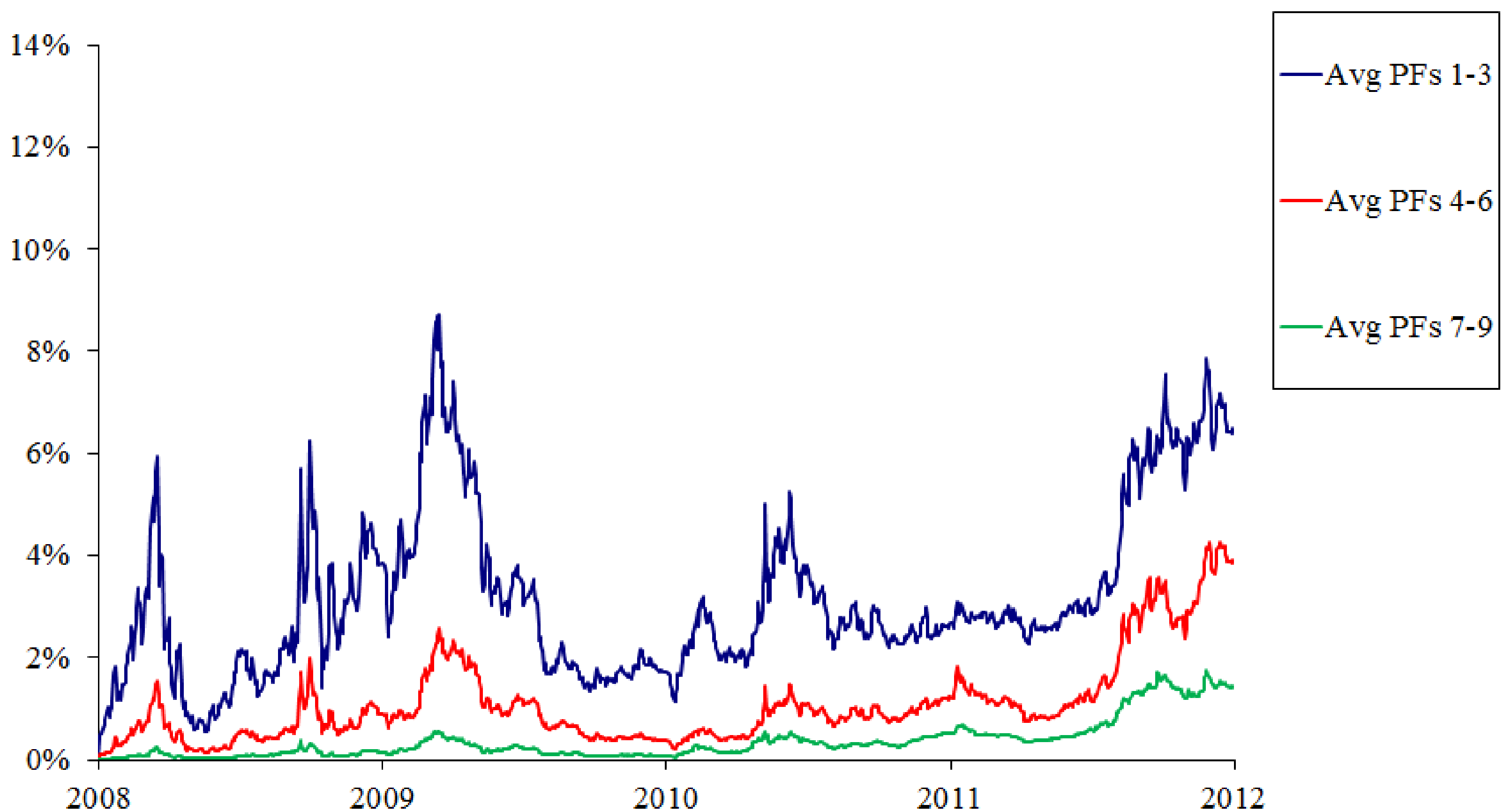


Figure 16: Change in the conditional joint probabilities of default ($\Delta CoJPoD$) involving 9 portfolios of 36 banks and 1 sovereign for the period 01/01/2008 – 12/31/2011. Each portfolio includes 4 banks and 1 sovereign. Spain is considered uniformly as a default-risk-triggering sovereign. Only the series conditioning on default of Spain are presented for each portfolio. The assignment of a bank to a portfolio is governed by its **asset quality**, compared to the rest, sorted in descending order. E. g. Portfolio 1 (PF 1) includes the four banks with the highest value of *doubtful loans*. For presentation purposes, the **average results of the top, middle and bottom 3 portfolios** are shown in the figure. The set of banks is presented in Tables 3 and 4. The complete portfolio assignment is presented in Table 5, column DL. Non-zero correlation structure is used in defining the prior distribution. The correlations are calculated between changes in the 5-year CDS spreads of the respective banks and sovereigns in each portfolio. $\Delta CoJPoD$ is calculated analogously to the sovereign case. The financial data used to calculate the doubtful loans ratio are denominated in euro.

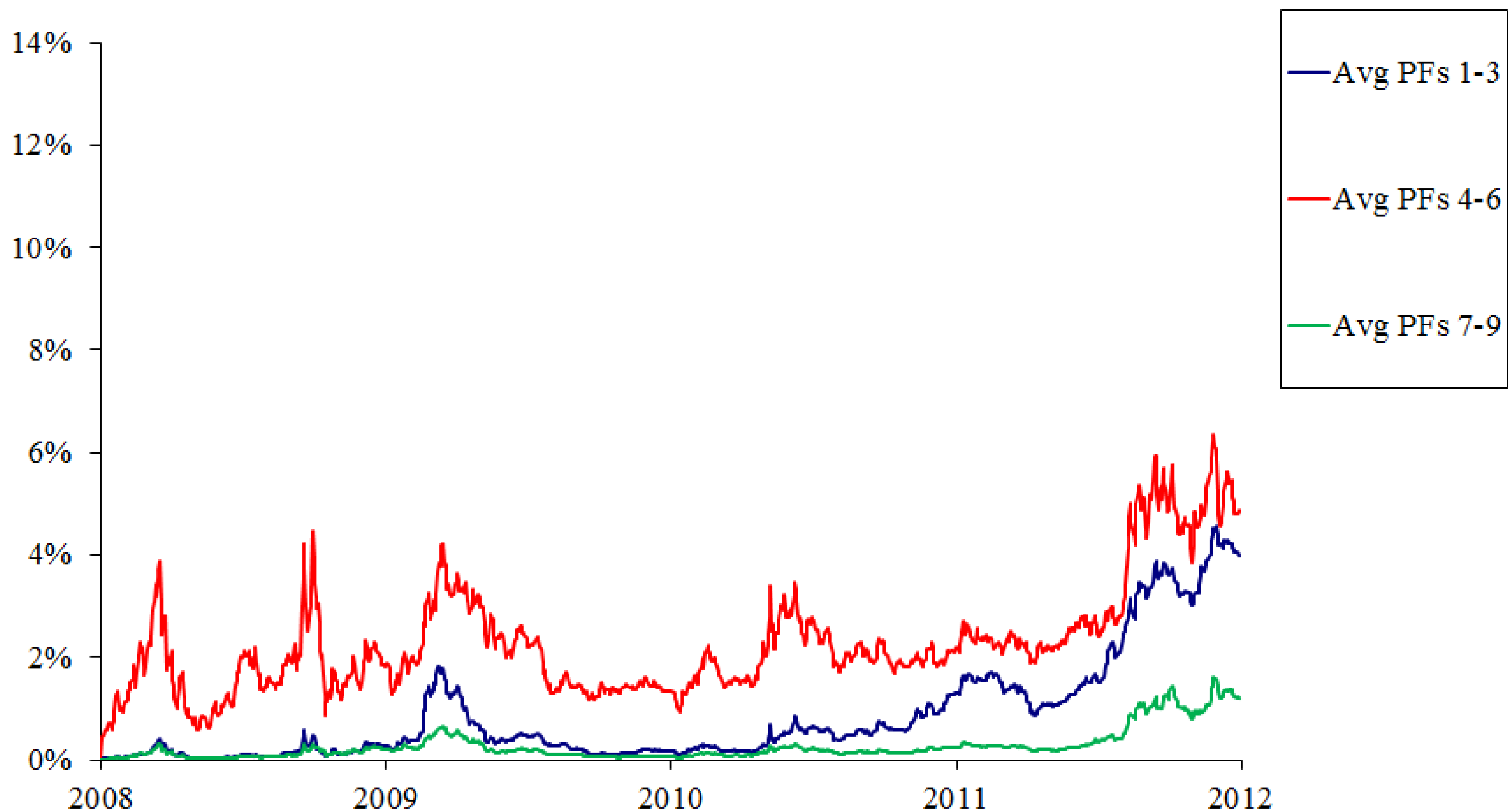
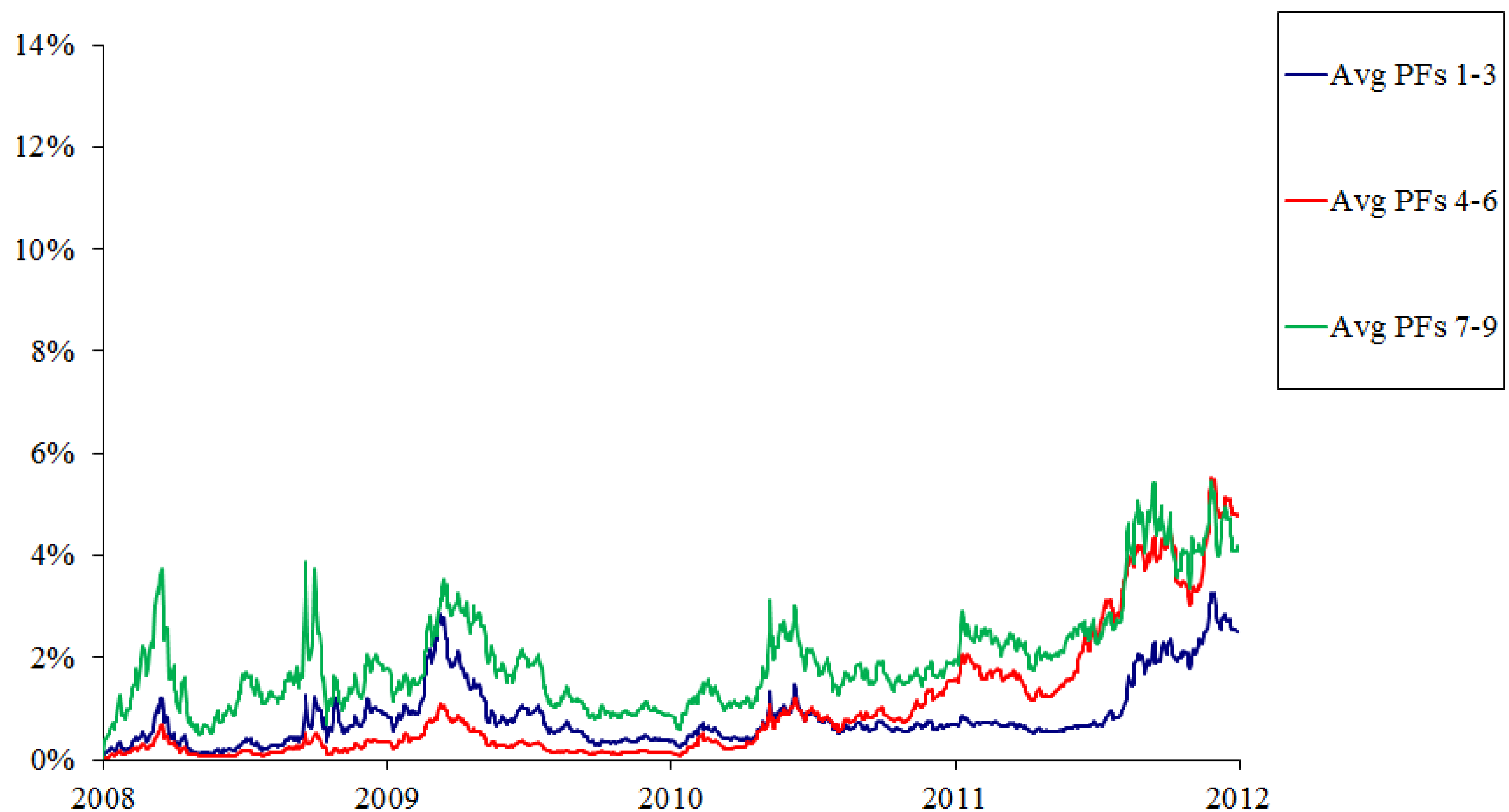


Figure 17: Change in the conditional joint probabilities of default ($\Delta CoJPoD$) involving 9 portfolios of 36 banks and 1 sovereign for the period 01/01/2008 – 12/31/2011. Each portfolio includes 4 banks and 1 sovereign. Spain is considered uniformly as a default-risk-triggering sovereign. Only the series conditioning on default of Spain are presented for each portfolio. The assignment of a bank to a portfolio is governed by its **liquidity and funding**, compared to the rest, sorted in descending order. E. g. Portfolio 1 (PF 1) includes the four banks with the highest value of *total deposits to total funding*. The set of banks is presented in Tables 3 and 4. For presentation purposes, the **average results of the top, middle and bottom 3 portfolios** are shown in the figure. The complete portfolio assignment is presented in Table 5, column DF. Non-zero correlation structure is used in defining the prior distribution. The correlations are calculated between changes in the 5-year CDS spreads of the respective banks and sovereigns in each portfolio. $\Delta CoJPoD$ is calculated analogously to the sovereign case. The financial data used to calculate the doubtful loans ratio are denominated in euro.



Appendix B. Tables

Table 1: Descriptive statistics of the 5-year CDS spread series of Austria (AT), Belgium (BE), France (FR), Germany (GE), Greece (GR), Ireland (IE), Italy (IT), the Netherlands (NL), Portugal (PT), Spain (ES). The data are in basis points. Period: 01/01/2008 – 12/31/2011. Number of observations: 1044.

	AT	BE	FR	GE	GR
Minimum	4.08	7.25	4.33	2.92	15.08
Mean	60.69	77.16	47.53	29.10	773.93
Maximum	215.77	303.72	186.21	90.82	11033.74
Standard deviation	40.73	61.26	38.36	18.95	1444.67
Number of observations	1044	1044	1044	1044	1044
	IE	IT	NL	PT	ES
Minimum	9.10	13.88	4.31	12.11	11.61
Mean	234.99	116.18	34.81	231.00	121.61
Maximum	917.55	445.26	103.63	961.47	367.64
Standard deviation	199.37	91.20	23.39	259.73	87.88
Number of observations	1044	1044	1044	1044	1044

Table 2: Correlation structure between 10 sovereigns: Austria (AT), Belgium (BE), France (FR), Germany (GE), Greece (GR), Ireland (IE), Italy (IT), the Netherlands (NL), Portugal (PT), Spain (ES). Period: 01/01/2008 – 12/31/2011. The correlations are calculated between changes in the 5-year CDS spreads of the sovereigns in the respective column and row.

	AT	BE	FR	GE	GR	IE	IT	NL	PT	ES
AT	1.00	0.70	0.73	0.75	0.13	0.54	0.66	0.79	0.46	0.62
BE		1.00	0.82	0.74	0.16	0.69	0.83	0.72	0.65	0.81
FR			1.00	0.82	0.22	0.63	0.81	0.73	0.60	0.76
GE				1.00	0.19	0.59	0.72	0.75	0.56	0.69
GR					1.00	0.19	0.21	0.16	0.17	0.17
IE						1.00	0.71	0.55	0.77	0.74
IT							1.00	0.68	0.71	0.90
NL								1.00	0.49	0.63
PT									1.00	0.73
ES										1.00

Table 3: List of euro area banks used in our analysis.

Euro Area Banks		
	Country code	Name
1	AT	Erste Group Bank AG
2	AT	Raiffeisen Bank International Austria
3	BE	Dexia SA
4	BE	KBC Groep NV
5	DE	Bayerische Landesbank
6	DE	Commerzbank AG
7	DE	Deutsche Bank AG
8	DE	Landesbank Berlin Holding AG
9	ES	Banco Bilbao Vizcaya Argentaria
10	ES	Banco de Sabadell SA
11	ES	Banco Santander SA
12	FR	BNP Paribas
13	FR	Credit Agricole SA
14	FR	Natixis
15	FR	Societe Generale
16	IE	Allied Irish Banks PLC
17	IE	Governor & Co of the Bank of Ireland
18	IE	Irish Life and Permanent
19	IT	Banca Monte dei Paschi di Siena
20	IT	Banca Popolare di Milano
21	IT	Banco Popolare SC
22	IT	Intesa Sanpaolo SpA
23	IT	UniCredit SpA
24	NL	ING Groep NV
25	NL	Rabobank
26	NL	SNS Bank Netherlands
27	PT	Banco Comercial Portugues SA
28	PT	Espirito Santo Financial Group

Table 4: List of additional non-euro area European Union banks used in our analysis.

Other European Union Banks		
	Country code	Name
1	DK	Danske Bank A/S
2	GB	Barclays PLC
3	GB	HSBC Holdings PLC
4	GB	Lloyds Banking Group PLC
5	GB	Royal Bank of Scotland Group
6	GB	Standard Chartered PLC
7	SE	Nordea Bank AB
8	SE	Skandinaviska Enskilda Banken

Table 5: Ranking assignment in descending order of the 36 banks used in our analysis with respect to 10 financial characteristics. The numbers in the columns for the financial characteristics come from the ordering in Tables 3 and 4. The values from 29 to 36 are given to the non-euro area European Union banks in the same order as in Table 4. PF 1 to PF 9 list the 4 banks that are included in the final 5-entity portfolios for each financial characteristic. The abbreviations stand for total assets (TA), return on equity (ROE), return on assets (ROA), net interest margin (NIM), efficiency ratio (ER), deposits-to-funding (DF) ratio, assets-to-equity (AE) ratio, loan-loss-provisions-to-net-interest-income (LLP-to-NII) ratio, non-performing-loans-to-total-loans ("doubtful loans", DL) ratio, net-loans-to-total-assets (NL-to-TA) ratio.

		Financial Characteristics									
Ranking	Portfolios	TA	ROE	ROA	NIM	ER	DF	AE	LLP-to-NII	DL	NL-to-TA
1		33	11	2	2	4	31	3	16	16	26
2	PF 1	12	23	8	1	16	34	8	17	21	27
3		7	33	11	9	14	18	7	18	19	10
4		31	31	10	11	29	24	18	32	23	25
5		30	7	9	34	8	1	26	3	17	20
6	PF 2	13	9	35	20	6	7	24	26	22	17
7		24	34	22	19	20	23	6	33	1	21
8		11	30	19	31	32	19	13	5	32	19
9		15	32	1	22	21	4	29	27	2	16
10	PF 3	23	22	36	10	19	2	30	31	8	28
11		32	12	30	23	33	25	5	14	6	2
12		6	25	20	21	13	32	12	30	15	9
13		22	15	31	16	1	27	32	24	33	32
14	PF 4	25	6	34	4	22	11	17	21	30	1
15		3	4	27	27	24	9	14	6	12	11
16		9	1	12	28	18	20	36	15	13	23
17		35	35	24	32	15	10	35	1	10	22
18	PF 5	14	29	15	17	25	16	15	23	4	29
19		29	19	28	25	27	17	16	2	26	3
20		5	2	23	30	26	33	33	10	9	35
21		4	3	6	33	23	36	4	12	14	18
22	PF 6	34	13	21	35	28	21	31	11	29	24
23		36	24	7	29	12	26	9	9	27	36
24		19	21	25	26	36	35	25	28	11	5
25		1	36	29	12	30	6	10	19	31	34
26	PF 7	17	27	13	24	2	22	11	13	5	6
27		16	10	32	13	34	30	34	22	7	4
28		8	28	14	15	31	13	23	20	20	31
29		21	20	18	36	35	28	27	29	18	33
30	PF 8	2	8	5	6	10	29	1	36	34	15
31		27	14	26	7	5	15	19	4	25	8
32		10	18	17	3	9	8	28	35	24	12
33		26	26	4	8	11	5	20	25	28	13
34	PF 9	28	17	3	5	17	12	2	34	36	30
35		18	5	33	14	7	3	22	8	35	14
36		20	16	16	18	3	14	21	7	3	7

Appendix C. Solutions and Proofs (Online Appendix)

1. CDS Bootstrapping Procedure

If τ is the time to default, one can express the probability of default at market date t, PoD(t) as

$$PoD(t) = P[\tau \leq t] = 1 - P[\tau \geq t] = 1 - Q(t), \tag{12}$$

where $[\tau \leq t]$ and $P[\tau \geq t] = Q(t)$ are the probability of the time to default to be less than t and the survival probability, respectively.

We apply a standard survival probability model by expressing $Q(t)$ via a piecewise constant hazard rate $h(t)$. For instance, given that

$$h(t) = \begin{cases} \gamma_1 & \text{for } 0 \leq t \leq t_1, \\ \gamma_2 & \text{for } t_1 < t \leq t_2, \\ \gamma_3 & \text{for } t_2 < t, \end{cases} \tag{13}$$

the survival function is

$$Q(t) = \begin{cases} e^{\gamma_1 \cdot t} & \text{for } 0 \leq t \leq t_1, \\ e^{\gamma_1 \cdot t - \gamma_2 \cdot (t - t_1)} & \text{for } t_1 < t \leq t_2, \\ e^{\gamma_1 \cdot t_1 - \gamma_2 \cdot (t_2 2 - t_1) - \gamma_3 \cdot (t - t_2)} & \text{for } t_2 < t, \end{cases} \tag{14}$$

The CDS bootstrapping procedure then calibrates γ_1, γ_2 and γ_3 to the market CDS premia data S_1, S_2 and S_3, such that the present value of the payment in case of default (100% –

recovery rate), called also the "protection leg", equals the discounted premia flows in the CDS contract, or the "premium leg" at the given market dates t_1, t_2 and t_3. This equality relies on the no-arbitrage condition on financial markets. In practice this is an iterative procedure that starts with the shortest maturity contract to calculate the first hazard rate, γ_1, and works its way through to the longest maturity, making sure that the no-arbitrage condition holds at each step. In our calculation, we also account for the quarterly structure of the CDS contract and the accrued premium that should be paid, given the default is anywhere in between any two market dates. We then use Equation survpod to calculate our cumulative $PoD(t)$, with $t = 1, ..., T$ denoting the default *horizon* (T=5 years in our case). Effectively, we use $PoD(T)$ and annualize it using Formula 8.[18]

[18]Note the innocuous abuse of notation: in the text we use index t to denote each date in our sample, while here, we use it to denote a particular time horizon.

2. Consistent Information Multivariate Density Optimizing Approach

We proceed by defining the financial system as a portfolio of debt issuers.[19] We observe n issuers, namely the X_1, X_2 to X_n entities defined in Section II, with corresponding assets x_1, x_2, to x_n. We define our objective function as:

$$(15) \qquad \chi(p,q) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p(x_1,x_2,...,x_n)\ln\left[\frac{p(x_1,x_2,...,x_n)}{q(x_1,x_2,...,x_n)}\right]dx_1\cdots dx_{n-1}dx_n.$$

The function $q(x_1,x_2,...,x_n) \in \mathbb{R}^n$ stands for the multivariate *prior* density function,[20] while $p(x_1,x_2,...,x_n) \in \mathbb{R}^n$ is the corresponding *posterior* density. The primary objective of the minimum cross-entropy approach is to minimize the difference $\chi(p,q)$ between the *ex ante* joint distribution $q(\cdot)$ and the *ex post* joint distribution $p(\cdot)$, given that the latter fulfills a set of constraints on the tail mass of the underlying marginal distributions. This set of constraints should relate the *posterior* distribution to empirical data:

[19] In this section, we present the multivariate version of the CIMDO approach. For the bivariate and trivariate models, please refer to Gorea and Radev (2013) and Segoviano (2006).

[20] As in Segoviano and Goodhart (2009), we assume a standard multivariate normal distribution for our prior. The prior density q is then $\frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}}e^{\left(\frac{1}{2}\mathbf{x}'\Sigma^{-1}\mathbf{x}\right)}$, where $\mathbf{x}$ is an n-dimensional random vector, while Σ is an $n\times n$ variance-covariance matrix of standard-normally-distributed variables (mean zero and standard deviation equal to one). Segoviano and Goodhart (2009) assume an identity matrix for the correlation structure of the prior distribution. Since the initial correlation structure assumption is crucial for the CIMDO approach, we rely on market estimates to explicitly allow it to differ from the identity matrix. This distress correlation structure for the prior distribution is proxied by the empirical correlation between changes in the 5-year CDS spreads of the sovereigns and banks in our sample.

$$(16) \qquad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p(x_1, x_2, ..., x_n)\mathbf{I}_{[\overline{\mathbf{x}}_1,\infty)}dx_1\cdots dx_{n-1}dx_n = PoD_t^1$$

$$(17) \qquad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p(x_1, x_2, ..., x_n)\mathbf{I}_{[\overline{\mathbf{x}}_{x_2},\infty)}dx_1\cdots dx_{n-1}dx_n = PoD_t^2$$

$\cdots$

$$(18) \qquad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p(x_1, x_2, ..., x_n)\mathbf{I}_{[\overline{\mathbf{x}}_{x_n},\infty)}dx_1\cdots dx_{n-1}dx_n = PoD_t^n,$$

where PoD_t^1, PoD_t^2 to PoD_t^n stand for the CDS-derived expected probabilities of default of X_1, X_2, ..., X_n. The indicator functions $\mathbf{I}_{[\overline{\mathbf{x}}_1,\infty)}$, $\mathbf{I}_{[\overline{\mathbf{x}}_2,\infty)}$ to $\mathbf{I}_{[\overline{\mathbf{x}}_n,\infty)}$ incorporate the default thresholds $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ to $\overline{\mathbf{x}}_n$[21] of the respective institutions. The functions take the value of unity if the assets of the respective entities exceed their individual thresholds and zero when they are below it. As explained above, the moment consistency constraints should ensure that the region of default of the "posterior" distribution is consistent with the market consensus default expectations for each sovereign or bank. In addition, in order to qualify as a density, $p(\cdot)$ should conform to the additivity constraint $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p(x_1, x_2, ..., x_n)dx_1\cdots dx_{n-1}dx_n = 1$.

We then minimize the Lagrangian function:

[21]Each default threshold is derived by inverting a univariate standard normal cumulative distribution function at the sample average value of the individual entity's probabilities of default.

$$
\begin{aligned}
\mathcal{L}(p,q) = & \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p(x_1,x_2,...,x_n)\ln\left[\frac{p(x_1,x_2,...,x_n)}{q(x_1,x_2,...,x_n)}\right]dx_1\cdots dx_{n-1}dx_n \\
& + \lambda_1\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p(x_1,x_2,...,x_n)\mathbf{I}_{[\overline{\mathbf{x}}_1,\infty)}dx_1\cdots dx_{n-1}dx_n - PoD_t^1\right] \\
& + \lambda_2\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p(x_1,x_2,...,x_n)\mathbf{I}_{[\overline{\mathbf{x}}_2,\infty)}dx_1\cdots dx_{n-1}dx_n - PoD_t^2\right] \\
& + \cdots \\
& + \lambda_n\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p(x_1,x_2,...,x_n)\mathbf{I}_{[\overline{\mathbf{x}}_n,\infty)}dx_1\cdots dx_{n-1}dx_n - PoD_t^n\right] \\
& + \mu\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p(x_1,x_2,...,x_n)dx_1\cdots dx_{n-1}dx_n - 1\right],
\end{aligned} \tag{19}
$$

with μ, λ_1, λ_2 to λ_n being the Lagrange multipliers of the corresponding constraints. The optimal *ex post* distribution reads:[22]

$$
p^*(x_1,x_2,...,x_n) = q(x_1,x_2,...,x_n)exp\left\{-\left[1+\mu+\sum_{i=1}^{n}\lambda_i\mathbf{I}_{[\overline{\mathbf{x}}_i,\infty)}\right]\right\}. \tag{20}
$$

The posterior distribution has two important properties: first, regardless of the *prior* assumption, the *ex post* distribution allows for fat tails and second, due to the dynamic updating through the individual empirical information, the posterior joint distribution is time-varying by construction.

[22] Appendix A.3. contains a detailed solution of the minimum cross-entropy optimization problem in a CIMDO context.

3. Solution of Minimum Cross Entropy

The minimum cross-entropy procedure can be viewed as a part of an iterative algorithm to approximate a target probability density f, using empirical data describing its underlying unknown process.[23] In this procedure, an a-priori (or prior) density q is updated to a posterior density p, given the following **Cross-Entropy Postulate**:

1. Conditional on a prior density q of a set $\mathfrak{X} \subset \Re^d$,
2. we minimize the Csiszár Cross-Entropy measure [24]

$$\mathcal{D}(p \to q) = \int_{\mathfrak{X}} q(\mathbf{x}) \cdot \psi\left(\frac{p(\mathbf{x})}{q(\mathbf{x})}\right) d\mathbf{x} \tag{21}$$

   with respect to $p(\mathbf{x})$, where $\mathbf{x}$ is a column vector and $\mathbf{x} \in \Re^d$,

3. given the moment constraints

$$\mathbb{E}_p K_i(\mathbf{X}) = \int_{\mathfrak{X}} p(\mathbf{x}) \cdot K_i(\mathbf{x}) d\mathbf{x} = \hat{\kappa}_i, i = 0, ..., n, \tag{22}$$

   where $\{K_i(\mathbf{x})\}_{i=1}^n$ is a set of suitably chosen functions and $\hat{\kappa}_i$ is empirical information describing the behaviour of the system, $\mathbb{E}_f K_i(\mathbf{X})$.

The *Minimum Cross-Entropy Problem* is then defined as

[23]For further details on the cross-entropy method and its generalizations, please consult with e. g. Botev and Kroese (2011).

[24]The Csiszár Cross-Entropy measure is a measure of directed divergence between probability densities (Botev and Kroese (2011)).

$$\min_p \mathcal{D}(p \to q) \tag{23}$$

subject to the constraints

$$\int\limits_{\mathfrak{X}} p(\mathbf{x}) \cdot K_i(\mathbf{x}) d\mathbf{x} = \hat{\kappa}_i, i = 0, ..., n \tag{24}$$

and

$$\int p(\mathbf{x}) d\mathbf{x} = 1. \tag{25}$$

The corresponding Lagrangian is then

$$\begin{aligned}
&\mathcal{L}(p; \boldsymbol{\lambda}, \lambda_0) = \\
&\quad = \int q(\mathbf{x}) \cdot \psi\left(\frac{p(\mathbf{x})}{q(\mathbf{x})}\right) d\mathbf{x} + \lambda_0 \left(1 - \int p(\mathbf{x}) d\mathbf{x}\right) + \sum_{i=1}^{n} \lambda_i \left(\hat{\kappa}_i - \int p(\mathbf{x}) \cdot K_i(\mathbf{x}) d\mathbf{x}\right) \\
&\quad = \sum_{i=0}^{n} \lambda_i \cdot \hat{\kappa}_i + \int \left(q(\mathbf{x}) \cdot \psi\left(\frac{p(\mathbf{x})}{q(\mathbf{x})}\right) + p(\mathbf{x}) \cdot \sum_{i=0}^{n} \lambda_i \cdot K_i(\mathbf{x})\right) d\mathbf{x},
\end{aligned} \tag{26}$$

where $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, ..., \lambda_n]^T$, $\hat{\kappa}_0 = 1$, and $K_0(\cdot) = 1$.

Let us assume that $\{K_i(\mathbf{x})\}_{i=0}^n = \{I_i(\mathbf{x})\}_{i=0}^n$, where I_i, $i = 1, 2, ..., n$ are binary functions taking values of unity when the respective x_i satisfies some condition, and zero otherwise, and $I_0 = 1$. The first order condition with respect to $p(\mathbf{x})$ is then

$$\frac{\partial \left(q(\mathbf{x}) \cdot \psi\left(\frac{p(\mathbf{x})}{q(\mathbf{x})}\right) + p(\mathbf{x}) \cdot \sum_{i=0}^{n} \lambda_i \cdot \mathbf{I}_i\right)}{\partial p(\mathbf{x})} \stackrel{!}{=} 0. \tag{27}$$

The latter can be further simplified as follows:

$$q(\mathbf{x}) \cdot (q(\mathbf{x}))^{-1} \cdot \psi' \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \right) + \sum_{i=0}^{n} \lambda_i \cdot \mathbf{I}_i = 0 \tag{28}$$

$$\psi' \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \right) = - \sum_{i=0}^{n} \lambda_i \cdot \mathbf{I}_i. \tag{29}$$

Assume $\psi(\mathbf{x}) = \mathbf{x} \cdot \ln(\mathbf{x})$, which is referred to in the literature as the Kullback-Leibler distance[25]. The Csiszár Cross-Entropy measure can then me transformed as

$$\begin{aligned} \int q(\mathbf{x}) \cdot \psi \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \right) &= \int q(\mathbf{x}) \cdot \frac{p(\mathbf{x})}{q(\mathbf{x})} \cdot \ln \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \right) \\ &= \int p(\mathbf{x}) \cdot \ln \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \right), \end{aligned} \tag{30}$$

while our $\psi' \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \right)$ takes the form

$$\begin{aligned} \psi' \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \right) &= \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \cdot \ln \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \right) \right)' \\ &= \ln \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \right) + \frac{p(\mathbf{x})}{q(\mathbf{x})} \cdot \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \right)^{-1} \\ &= \ln \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \right) + 1. \end{aligned} \tag{31}$$

Substituting in our first order condition Equation (29) and simplifying further yields

[25]The Kullback-Leibler distance is a usual assumption that allows us to avoid setting additional constraints to secure the non-negativity of $p(\mathbf{x})$.

(32)
$$\ln\left(\frac{p(\mathbf{x})}{q(\mathbf{x})}\right)+1=-\sum_{i=0}^{n}\lambda_i\cdot\mathbf{I}_i$$

(33)
$$\ln\left(\frac{p(\mathbf{x})}{q(\mathbf{x})}\right)=-1-\sum_{i=0}^{n}\lambda_i\cdot\mathbf{I}_i.$$

The solution to the Minimum Cross-Entropy problem is then

(34)
$$p(\mathbf{x})=q(\mathbf{x})\cdot exp\left\{-\left[1+\sum_{i=0}^{n}\lambda_i\mathbf{I}_i\right]\right\}.$$

Changing the notation of the Lagrange multiplier of the additivity constraint to μ, we arrive at

(35)
$$p(\mathbf{x})=q(\mathbf{x})\cdot exp\left\{-\left[1+\mu+\sum_{i=1}^{n}\lambda_i\mathbf{I}_i\right]\right\},$$

which is the general form of the solution to the CIMDO minimization problem.

4. Proof of Independence within the Default Region of the CIMDO Distribution

To prove independence, we want to show that using standard normal distribution as a prior, the following holds for the posterior CIMDO distribution and its marginals:

$$P(x_1 > \overline{\mathbf{x}}_1, x_2 > \overline{\mathbf{x}}_2, ..., x_n > \overline{\mathbf{x}}_n) = P(x_1 > \overline{\mathbf{x}}_1) \cdot P(x_2 > \overline{\mathbf{x}}_2) \cdots P(x_n > \overline{\mathbf{x}}_n) = \prod_{i=1}^{n} P(x_i > \overline{\mathbf{x}}_i), \tag{36}$$

where $P(x_1 > \overline{\mathbf{x}}_1)$, $P(x_2 > \overline{\mathbf{x}}_2)$,..., $P(x_n > \overline{\mathbf{x}}_n)$ and $P(x_1 > \overline{\mathbf{x}}_1, x_2 > \overline{\mathbf{x}}_2, ..., x_n > \overline{\mathbf{x}}_n)$ are the cumulative marginal and joint CIMDO probabilities.

Proof:

We present a *direct* proof of the statement above. We start by expressing $P(x_n > \overline{\mathbf{x}}_n)$, $P(x_1 > \overline{\mathbf{x}}_1, x_2 > \overline{\mathbf{x}}_2, ..., x_{n-1} > \overline{\mathbf{x}}_{n-1})$ and $P(x_1 > \overline{\mathbf{x}}_1, x_2 > \overline{\mathbf{x}}_2, ..., x_n > \overline{\mathbf{x}}_n)$ in terms of the prior (multivariate standard normal) distribution and the thresholds $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ to $\overline{\mathbf{x}}_n$:

$$\begin{aligned} &P(x_n > \overline{\mathbf{x}}_n) = PoD^n \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p(x_1, x_2, ..., x_n)\mathbf{I}_{[\overline{\mathbf{x}}_n,\infty)} dx_1 \cdots dx_{n-1}dx_n \\ &= \int_{\overline{\mathbf{x}}_n}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} (2\pi)^{\frac{n}{2}} e^{\left(-\frac{\sum_{i=1}^{n} x_i^2}{2}\right)} e^{\left(-(1+\mu+\sum_{i=1}^{n-1}\lambda_i \mathbf{I}_{[\overline{\mathbf{x}}_i,\infty)}+\lambda_n)\right)} dx_1 \cdots dx_{n-1}dx_n, \end{aligned} \tag{37}$$

$$P(x_1 > \overline{\mathbf{x}}_1, x_2 > \overline{\mathbf{x}}_2, ..., x_{n-1} > \overline{\mathbf{x}}_{n-1}) =$$

$$(38)\quad = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p(x_1, x_2, ..., x_n)\mathbf{I}_{[\overline{\mathbf{x}}_1,\infty)}\cdot\mathbf{I}_{[\overline{\mathbf{x}}_2,\infty)}\cdots\mathbf{I}_{[\overline{\mathbf{x}}_{n-1},\infty)}dx_1\cdots dx_{n-1}dx_n$$

$$= \int_{-\infty}^{+\infty}\int_{\overline{\mathbf{x}}_{n-1}}^{+\infty}\cdots\int_{\overline{\mathbf{x}}_1}^{+\infty}(2\pi)^{\frac{n}{2}}e^{\left(-\frac{\sum_{i=1}^{n}x_i^2}{2}\right)}e^{(-(1+\mu+\sum_{i=1}^{n-1}\lambda_i+\lambda_n\mathbf{I}_{[\overline{\mathbf{x}}_n,\infty)}))}dx_1\cdots dx_{n-1}dx_n,$$

where $\mathbf{I}_{[\overline{\mathbf{x}}_1,\infty)}$, $\mathbf{I}_{[\overline{\mathbf{x}}_2,\infty)}$ to $\mathbf{I}_{[\overline{\mathbf{x}}_n,\infty)}$ are indicator functions that take the value of one in the cases where the assets of X_1, X_2 to X_n are beyond their individual thresholds, respectively. Then, the joint probability of distress is as follows:

$$P(x_1 > \overline{\mathbf{x}}_1, x_2 > \overline{\mathbf{x}}_2, ..., x_n > \overline{\mathbf{x}}_n) =$$

$$(39)\quad = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p(x_1, x_2, ..., x_n)\mathbf{I}_{[\overline{\mathbf{x}}_1,\infty)}\cdot\mathbf{I}_{[\overline{\mathbf{x}}_2,\infty)}\cdots\mathbf{I}_{[\overline{\mathbf{x}}_n,\infty)}dx_1\cdots dx_{n-1}dx_n$$

$$= \int_{\overline{\mathbf{x}}_n}^{+\infty}\int_{\overline{\mathbf{x}}_{n-1}}^{+\infty}\cdots\int_{\overline{\mathbf{x}}_1}^{+\infty}(2\pi)^{\frac{n}{2}}e^{\left(-\frac{\sum_{i=1}^{n}x_i^2}{2}\right)}e^{(-(1+\mu+\sum_{i=1}^{n}\lambda_i))}dx_1\cdots dx_{n-1}dx_n.$$

Rearranging $P(x_n > \overline{\mathbf{x}}_n)$, we get

$$P(x_n > \overline{\mathbf{x}}_n) =$$

$$(40)\quad = \int_{\overline{\mathbf{x}}_n}^{+\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{x_n^2}{2}}e^{-\lambda_n}dx_n\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}(2\pi)^{\frac{n-1}{2}}e^{\left(-\frac{\sum_{i=1}^{n-1}x_i^2}{2}\right)}e^{(-(1+\mu+\sum_{i=1}^{n-1}\lambda_i\mathbf{I}_{[\overline{\mathbf{x}}_i,\infty)}))}dx_1\cdots dx_{n-1}.$$

Analogously, for $P(x_1 > \overline{\mathbf{x}}_1, x_2 > \overline{\mathbf{x}}_2, ..., x_{n-1} > \overline{\mathbf{x}}_{n-1})$, we come at:

(41)
$$P(x_1 > \overline{\mathbf{x}}_1, x_2 > \overline{\mathbf{x}}_2, ..., x_{n-1} > \overline{\mathbf{x}}_{n-1}) =$$
$$= \int_{\overline{\mathbf{x}}_{n-1}}^{+\infty} \cdots \int_{\overline{\mathbf{x}}_1}^{+\infty} (2\pi)^{\frac{n-1}{2}} e^{\left(-\frac{\sum_{i=1}^{n-1} x_i^2}{2}\right)} e^{(-\sum_{i=1}^{n-1} \lambda_i)} dx_1 \cdots dx_{n-1} \int_{-\infty}^{+\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{x_n^2}{2}} e^{(-(1+\mu+\lambda_n \mathbf{I}_{[\overline{\mathbf{x}}_n,\infty)}))} dx_n,$$

Hence, for the product of the latter probabilities, we have:

(42)
$$P(x_1 > \overline{\mathbf{x}}_1, x_2 > \overline{\mathbf{x}}_2, ..., x_{n-1} > \overline{\mathbf{x}}_{n-1}) \cdot P(x_n > \overline{\mathbf{x}}_n) =$$
$$= \left[\int_{\overline{\mathbf{x}}_{n-1}}^{+\infty} \cdots \int_{\overline{\mathbf{x}}_1}^{+\infty} (2\pi)^{\frac{n-1}{2}} e^{\left(-\frac{\sum_{i=1}^{n-1} x_i^2}{2}\right)} e^{(-\sum_{i=1}^{n-1} \lambda_i)} dx_1 \cdots dx_{n-1} \int_{-\infty}^{+\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{x_n^2}{2}} e^{(-(1+\mu+\lambda_n \mathbf{I}_{[\overline{\mathbf{x}}_n,\infty)}))} dx_n \right]$$
$$\cdot \left[\int_{\overline{\mathbf{x}}_n}^{+\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{x_n^2}{2}} e^{-\lambda_n} dx_n \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (2\pi)^{\frac{n-1}{2}} e^{\left(-\frac{\sum_{i=1}^{n-1} x_i^2}{2}\right)} e^{(-(1+\mu+\sum_{i=1}^{n-1} \lambda_i \mathbf{I}_{[\overline{\mathbf{x}}_i,\infty)}))} dx_1 \cdots dx_{n-1} \right]$$
$$= \left[\int_{\overline{\mathbf{x}}_n}^{+\infty} \int_{\overline{\mathbf{x}}_{n-1}}^{+\infty} \cdots \int_{\overline{\mathbf{x}}_1}^{+\infty} (2\pi)^{\frac{n}{2}} e^{\left(-\frac{\sum_{i=1}^{n} x_i^2}{2}\right)} e^{(-(1+\mu+\sum_{i=1}^{n} \lambda_i))} dx_1 \cdots dx_{n-1} dx_n \right]$$
$$\cdot \left[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (2\pi)^{\frac{n}{2}} e^{\left(-\frac{\sum_{i=1}^{n} x_i^2}{2}\right)} e^{(-(1+\mu+\sum_{i=1}^{n} \lambda_i \mathbf{I}_{[\overline{\mathbf{x}}_i,\infty)}))} dx_1 \cdots dx_{n-1} dx_n \right].$$

As the integral in the last square brackets is in fact the additivity constraint in our optimization problem, it equals 1 by definition. The remaining term equals our definition for the joint probability $P(x_1 > \overline{\mathbf{x}}_1, x_2 > \overline{\mathbf{x}}_2, ..., x_n > \overline{\mathbf{x}}_n)$. If we repeat the procedure iteratively for the joint distributions $P(x_1 > \overline{\mathbf{x}}_1, x_2 > \overline{\mathbf{x}}_2, ..., x_i > \overline{\mathbf{x}}_i)$, for $i = n-1,\ ...,\ 2$, we arrive at the following decomposition:

(43)
$$P(x_1 > \overline{\mathbf{x}}_1, x_2 > \overline{\mathbf{x}}_2, ..., x_n > \overline{\mathbf{x}}_n) = P(x_1 > \overline{\mathbf{x}}_1) \cdot P(x_2 > \overline{\mathbf{x}}_2) \cdots P(x_n > \overline{\mathbf{x}}_n)$$
$$= \prod_{i=1}^{n} P(x_i > \overline{\mathbf{x}}_i).$$

Hence, the product of the marginal probabilities of distress $P(x_1 > \overline{\mathbf{x}}_1)$, $P(x_2 > \overline{\mathbf{x}}_2)$, ..., and $P(x_n > \overline{\mathbf{x}}_n)$ equals the joint probability of distress, meaning that within the joint distress region, the entities X_1, X_2 and X_n are independent. ■