Fragility and Inefficient Fire Sales in Decentralized Asset Markets*

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Abstract

I show that when market participation is endogenous, decentralized and yet competitive asset markets are prone to multiplicity of equilibria and inefficient fire sales when subjected to large enough liquidity shocks and characterize conditions for the existence of such fire sales. In the model sellers are subject to liquidity shock in the present and future dates. There can be multiple equilibria: *delayed* equilibrium where some agents wait to trade in the second period and *run* equilibrium where all agents try to trade in the first period. Fire sale equilibrium, when asset price is depressed, is a *run* (*delayed*) if the number of buyers is less (more) than sellers in the market. The two types of equilibria and hence the possibility of fire sale exist when sellers' future liquidity shock is bigger than the current shock and there is a medium degree of imbalance between the buyers and sellers in the market. Moreover, fragility exists when market liquidity is neither too high nor too low. Fire sale in the form of a *run* equilibrium is always dominated in terms of welfare by its corresponding *delayed* equilibrium with higher asset price. Fire sale in the form of *delayed* is dominated by its corresponding *run* equilibrium as long as the ratio of sellers to buyers is not too low.

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1 Introduction

Fire sale in financial markets where assets are sold at deep discounts is a prominent feature of financial crises. Two classic explanations emphasize liquidity constrained industry experts who can operate the asset productively (Shleifer and Vishny (1992)) and limited arbitrage capital by specialized investors who understand the asset (Shleifer and Vishny (1997)). Liquidity constrained experts is less applicable to financial assets and as for the limited arbitrage capital, there were non-specialized investors with abundant resources (e.g. Warren Buffet) to buy these assets and its not clear why these investors did not step in.[1] Moreover, not all buyers which may be considered experts/specialized investors, e.g., banks, were liquidity constrained during the crisis.[2] Beside lack of liquidity or expertise, what else can help explain depressed asset prices during the crisis? Can non-fundamental factors explain at least part of asset price volatility during the crisis?

Many financial assets, e.g., mortgage backed securities, as well as important real assets such as property are traded in decentralized over the counter markets. I show that when market participation is endogenous (and yet costless), decentralized markets are intrinsically fragile and prone to inefficient fire sales when subjected to large enough liquidity shocks. Fragility and fire sale require a medium degree of imbalance between buyers and sellers and a medium degree of market liquidity. A brief description of the model is as follows. Sellers and buyers can consume over three periods but sellers are assumed to have a higher propensity to consume in the first two periods because they are in need of liquidity. One can think of the sellers as banks or other financial entities that need to pay their short term debts by liquidating assets in the market. Asset is indivisible and sellers hold one unit of asset at the beginning while buyers can buy at most one unit of the asset. Limited capacity of buyers may be due to tightening borrowing constraints in the financial market during a period of financial stress. Buyers post prices in the submarkets and sellers choose the the submarket and the corresponding price to maximize their utility. Sellers can choose a higher price but at the cost of facing a lower probability of trade as higher prices attract more sellers. There are two types of equilibria which I label *delayed* and *run* equilibria. A *delayed* equilibrium is an equilibrium in which buyers or sellers are indifferent between participating or not participating in the market at the first date. In other words some of buyers and sellers may prefer to wait until the second period for trade. A *run*, on the other hand, is an equilibrium where all buyers and sellers try to buy and sell in the market.

Market is said to be fragile, where the two types of equilibria namely *run* and *delayed*

[1] For a more detailed discussion see Dow and Han (2018).

[2] He, Khang, and Krishnamurthy (2010) show that there has been a sizable redistribution of leverage across the financial markets rather than a uniform deleveraging in the banking system.

coexist. When market is fragile, fire sale equilibrium is the equilibrium with the lower asset price. A fire sale equilibrium may be of either type of equilibria: asset price falls either because sellers *run* to sell immediately as they don't expect to find many opportunities to trade later or buyers *delay* their purchase in the hope of finding even more desperate sellers in the future. If the number of buyers is less (more) than the sellers in the market, fire sale happens in the form of a *run* (*delayed*) equilibrium. There are four critical elements making an asset market fragile in the model: a decentralized (yet competitive) asset market where agents can decide when to enter the market, liquidity shocks at current and future dates which are increasing in magnitude over time and a medium degree of imbalance between potential supply and demand for the asset as well as a market liquidity that is neither too high nor too low. I show that equilibrium is unique in the absence of any of these elements.

Fire sale equilibrium is often Pareto inferior to the equilibrium with the higher asset price. When there are multiple equilibria and if buyers are less than sellers, the *run* equilibrium is always dominated by the *delayed* equilibrium in terms of welfare in that the former generates a lower aggregate trade surplus than the latter. A social planner can create a Pareto superior allocation to the *run* equilibrium by doing lump sum transfers in the *delayed* equilibrium and therefore, *run* equilibrium is always inefficient in this case. In contrast, when sellers are less than buyers and as long as the ratio of sellers to buyers is not too low, the *run* equilibrium dominates *delayed* equilibrium and the latter is inefficient.

In a centralized market, where trade takes place with certainty, the effect of each agent's decision to enter the market on others is fully priced and there are no non-priced externalities. But in a decentralized (yet competitive) market, on the other hand, agent's decision to participate affects the probability of trade at current and future dates. A competitive market can price at most one of the two margins but not both. This leaves room for the presence of non-priced externalities and coordination failure which is at the heart of market fragility.

To better understand these externalities, consider the case where there are more sellers than buyers. In this case, a seller's decision to wait to enter the market in the future period increases market tightness, i.e., ratio of buyers to sellers, in both current and future dates.[3]This implies higher probability of trade and hence higher payoff of participating at the margin for other sellers as well as lower probability of trade and lower payoff for buyers at both dates. If the increase in probability and hence welfare for other sellers and the decrease in probability and welfare for buyers is lower in the future date, seller's decision to wait will encourage other sellers and buyers to wait as well. In other words, sellers and buyers decision

[3]Market tightness increases in the second period because the volume of trade, i.e., number of matches, decreases in the initial period. The ration of remaining buyers to remaining sellers increases as the number of sellers are higher in the market.

to delay trade become complement which makes multiple equilibria a possibility. For this to happen, the relative change in market tightness in current and future dates induced by a seller's decision to wait, should not be too high or too low: if its too high (too low) buyers (other sellers) would prefer to trade on the initial date instead of waiting.

The relative marginal change in market tightness in turn depends on size of sellers relative to buyers as well as the matching efficiency which determines market liquidity. These externailities can lead to market fragility and fire sale of the asset when there is a medium degree of imbalance between buyers and sellers as well as market liquidity. In the limit when the number of sellers and buyers are very close or too far apart, or market liquidity is too high or too low, agents' actions will not be complement. These externalities are the source of inefficiency in the fire sale equilibria.

Another implication of the model is that when there are multiple equilibria, fire sale prices coincide with lower (higher) volume of trade when sellers (buyers) are the short side of the market. In other words, fire sale is accompanied by a surge in the volume of sales in the market (a *run* equilibrium) only if there are too many sellers relative to buyers. In contrast, when there are too few sellers, fire sale follows a decline in sales (a *delayed* equilibrium). This is an empirically testable prediction.

1.1 Related Literature

Guerrieri and Shimer (2014) is a model of fire sale of assets in decentralized markets with competitive search. Key features are competitive decentralized market and the presence of private information about the quality of asset. There is a unique equilibrium in which sellers signal quality of their asset by waiting longer for trade to take place. In contrast, in this paper adverse selection doesn't play any role.

Guerrieri (2010) is a model of competitive search in the labor market with private information and limited commitment on the side of workers. Unlike typical models of directed search, the equilibrium is inefficient outside the steady state. The reason is that firms offering contracts at a given point in time do not internalize their externality on workers' outside option in previous periods. Similar to Guerrieri (2010), an intertemporal externality is the source of inefficiency and multiplicity in this model. However, the externality is from the past actions on future probability of trade.

This paper is also related to the large body of research on multiple equilibria in currency markets. Obstfeld (1996) provides a review of the relevant models. Unlike this model, models of currency attacks feature centralized markets. The key ingredient which makes the centralized market vulnerable to bad equilibria with large currency devaluations is the presence of a non-negligible strategic agent namely the government.

Bernardo and Welch (2004) is an example of a model of fire sale and run in a stock market. Risk neutral investors fear that they need to liquidate shares after a run takes place and before prices recover. This fear may force investors to sell today and may cause the run itself. Similar to this model, future liquidity shocks play a key role in causing a run today.

2 Model

There are three periods $t = 0, 1, 2$ and two types of agents, measure 1 of buyers and $m > 0$ of sellers, trading an indivisible asset. The asset pays off its only dividend $d_2 > 0$ units of consumption goods at $t = 2$. Agents preferences are as follows:

$$\begin{cases} \boldsymbol{U}_S = \delta_0 C_0 + \delta_1 C_1 + C_2 \,, \\ \boldsymbol{U}_B = C_0 + C_1 + C_2 \,, \end{cases}$$

Where subscripts S and B indicate the seller and buyer. We assume that buyers have big enough endowments in $t = 0, 1$ to pay for the asset.[4] Both buyers and sellers can save at the real rate of zero in $t = 0, 1$. Timing of events are as follows. In each period, both types of agents first decide how much to save, then if they want to participate in the market and finally consume whatever has not been saved or spent on market transactions.[5]

Coefficients δ_0 and δ_1 are seller's marginal utility of consumption in periods $t = 0, 1$. They capture a liquidity shock today, e.g., need to liquidate assets to pay off the debt, or an expected liquidity shock tomorrow respectively. Therefore we make the following assumption:

Assumption 1. *$\delta_0 > 1$ and $\delta_1 > 1$.*

Sellers hold a unit of the asset in $t = 0$ and buyers may buy at most one unit of the asset in either $t = 0, 1$. Note that given Assumption 1, buyers (sellers) have no incentives to sell (buy) if they purchase the asset in the market at $t = 0$.

Trade takes place in a decentralized market in both $t = 0, 1$ with competitive search and random matching. In each period, buyers post prices and form submarkets each of which represent the subset of buyers who have posted the same price. Sellers of the asset observe these prices and choose the submarket and the corresponding price to maximize their utility. The buyers' problem at $t = 1$ if she participates is:

[4]The value of sellers' endowment is irrelevant due to risk neutrality.
[5]We can allow for saving after market transactions if we introduce risk aversion.

$$\begin{cases} V_1^B = \max_{\sigma_1, p_1} q_1^B(\sigma_1)(d_2 - p_1) \\ s.t. \quad \bar{U}_1^S \leq q_1^S(\sigma_1)\delta_1 p_1 + (1 - q_1^S(\sigma_1))d_2 \end{cases} \tag{1}$$

Buyer strictly prefers to participate in the market as long as $V_1^B > R_1^B$ where R_1^B is the reservation utility of the buyer at $t = 1$ and is equal to zero. If $V_1^B = R_1^B$ buyer is indifferent between participating and staying out of the market. σ_1 is ratio of sellers to buyers or the length of the queue for the submarket with posted price of p_1. Moreover, q_1^B and q_1^S are probabilities of being matched with a seller and buyer respectively in the submarket with the queue length of σ_1 which depends on the matching technology, i.e., market microstructure. The participation constraint by the seller in the problem above requires that the seller's utility from trade is no less than the maximum, $\bar{U}_1^S$, she can obtain outside the match where:

$$\bar{U}_1^S = \max(V_1^S, R_1^S)$$

Where V_1^S and R_1^S are sellers continuation utility from participating and reservation utility of not participating in the market at $t = 1$. We note that $R_1^S = d_2$. Using $q_1^B = \sigma_1 q_1^S$ and that the participation constraint is binding in equilibrium, one has:

$$\begin{cases} q_1^S(\sigma_1)(\delta_1 p_1 - d_2) = \bar{U}_1^S - d_2 \Rightarrow \\ \delta_1 q_1^B(\sigma_1)(d_2 - p_1) = \sigma_1(\bar{U}_1^S - d_2) - (\delta_1 - 1)q_1^B(\sigma_1)d_2 \end{cases} \tag{2}$$

Hence one can simplify the buyers problem:

$$V_1^B = \max_{\sigma_1} \big\{ \frac{1}{\delta_1}((\delta_1 - 1)q_1^B(\sigma_1)d_2 - \sigma_1(\bar{U}_1^S - d_2)) \big\} \tag{3}$$

$$\tag{4}$$

Similarly, the problem of buyers at $t = 0$ is:

$$\begin{cases} V_0^B = \max_{\sigma_0, p_0} \left\{ q_0^B(\sigma_0)(d_2 - p_0) + (1 - q_0^B(\sigma_0))\bar{U}_1^B \right\} \\ s.t. \quad \bar{U}_0^S \leq q_0^S(\sigma_0)\delta_0 p_0 + (1 - q_0^S(\sigma_0))\bar{U}_1^S \end{cases} \tag{5}$$

Where $\bar{U}_1^B = \max(V_1^B, R_1^B)$ is the maximum utility for buyers if they wait until $t = 1$. Note that given our notation this is the reservation utility at $t = 0$ for buyers as well, i.e., $R_0^B = \bar{U}_1^B$. And $\bar{U}_0^S$ is the maximum utility sellers can obtain in the market.

2.1 Competitive Search Equilibrium

For any measure s of sellers and b of buyers at any given period who are active in any of the submarkets, the number of matches is given by:

$$\mathbb{M}(s, b) = \gamma s^{1-\alpha} b^{\alpha}\,, \tag{6}$$

where $0 < \alpha < 1$ and $0 < \gamma < 1$ is the efficiency of the matching function. Note that the market liquidity of the asset is affected by both γ and m.

We restrict our parameters so that the implied probabilities will be always less than one:

Assumption 2. $\gamma^{\frac{1}{\alpha}} < \frac{m - \gamma m^{1-\alpha}}{1 - \gamma m^{1-\alpha}} < \gamma^{\frac{-1}{1-\alpha}}$.

To solve for the equilibrium we start at $t = 1$. Using 3, the first order conditions (FOC) for buyers at $t = 1$ are:

$$\begin{cases} (\delta_1 - 1)\frac{dq_1^B(\sigma_1)}{d\sigma_1} d_2 = \bar{U}_1^S - d_2 \Rightarrow \\ \\ \frac{dq_1^B(\sigma_1)}{d\sigma_1} = \frac{\bar{U}_1^S - d_2}{(\delta_1 - 1)d_2} \end{cases} \tag{7}$$

Given our matching technology 6, we have $q_1^B(\sigma_1) = \gamma\sigma_1^{1-\alpha}$, $q_1^S(\sigma_1) = \gamma\sigma_1^{-\alpha}$ and hence using 7 and 2 we can obtain the unique equilibrium price at $t = 1$:

$$\begin{cases} (\delta_1 - 1)d_2 = \delta_1 p_1 - d_2 \Rightarrow \\ \\ p_1^* = \frac{1 + (1-\alpha)(\delta_1)}{\delta_1} d_2 \end{cases} \tag{8}$$

Note that we have:

$$\frac{d_2}{\delta_1} < p_1^* < d_2$$

As $0 < \alpha < 1$. This implies that both buyers and sellers strictly prefer to participate in the market at $t = 1$. In other words $\bar{U}_1^S = V_1^S > R_1^S$ and $\bar{U}_1^B = V_1^B > R_1^B$. Therefore, there is a unique equilibrium price in the market at $t = 1$ with full participation of both buyers and sellers.

Using 2 and the equilibrium price 8, we can obtain the continuation utilities $\bar{U}_1^S$ and $\bar{U}_1^B$ (or equivalently V_1^S and V_1^B) as follows:

$$\begin{cases} \bar{U}_1^S = V_1^S = (1 + (1-\alpha)(\delta_1 - 1)\gamma\sigma_1^{*-\alpha})d_2 \\ \\ \bar{U}_1^B = V_1^B = \frac{\alpha(\delta_1-1)}{\delta_1}\gamma\sigma_1^{*1-\alpha}d_2 \end{cases} \tag{9}$$

Turning to $t = 0$, we can use the participation constraint in 5, which must hold with equality in equilibrium, to obtain:

$$\begin{cases} q_0^S(\sigma_0)\delta_0 p_0 = \sigma_0(\bar{U}_0^S - (1 - q_0^S(\sigma_0))) \Rightarrow \\ \\ q_0^B(\sigma_0)p_0 = \frac{\sigma_0}{\delta_0}(\bar{U}_0^S - (1 - q_0^S(\sigma_0))) \end{cases} \tag{10}$$

Since $q_0^B(\sigma_0) = \sigma_0 q_0^S(\sigma_0)$. This simplifies the buyer's maximization at $t = 0$:

$$V_0^B = \max_{\sigma_0} \left\{ q_0^B(\sigma_0)(d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_0^B) - \frac{\bar{U}_0^S - \bar{U}_1^S}{\delta_0}\sigma_0 + \bar{U}_1^B \right\} \tag{11}$$

The FOC for an interior solution with full participation is:

$$\begin{cases} (d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_0^B)\frac{dq_1^B(\sigma_0)}{d\sigma_0} = \frac{\bar{U}_0^S - \bar{U}_1^S}{\delta_0} \Rightarrow \\ \\ \sigma_0^{*-\alpha} = \frac{\bar{U}_0^S - \bar{U}_1^S}{\delta_0\gamma(1-\alpha)(d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_0^B)} \end{cases} \tag{12}$$

Full participation requires $d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_0^B > 0$ and $\bar{U}_0^S - \bar{U}_1^S > 0$. The price at $t = 0$ can be obtained participation constraint of the seller at $t = 0$ and 12:

$$p_0^* = (1-\alpha)(d_2 - \bar{U}_1^B) + \alpha\frac{1}{\delta_0}\bar{U}_1^S \tag{13}$$

$$\tag{14}$$

Where $\bar{U}_1^B$ and $\bar{U}_1^S$ are given by 9. Full participation equilibrium, however, is not the only possible outcome. Another type of equilibria may exist where agents are indifferent between participating or staying out of the market at $t = 0$. This happens only when:

$$\begin{cases} \bar{U}_0^S - \bar{U}_1^S = 0, \\ \\ d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_0^B = 0 \end{cases} \tag{15}$$

Equations 15 imply that buyers cannot raise their lifetime utility by changing the probability or the price they post in the market. The term $d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_0^B$ is total excess surplus (conditional on trade taking place), a fraction α of which accrues to the buyer. It is evident from the first equation in 15 that for buyers to be indifferent, sellers must be indifferent as well. The second equation above is in terms of continuation utilities of sellers and buyers at $t = 1$ which are given by 9. We can use 9 to obtain the following equation in the inverse of market tightness at $t = 1$:

$$\frac{1}{\delta_0} + \frac{(1-\alpha)(\delta_1 - 1)}{\delta_0}\gamma\sigma_1^{*-\alpha} + \frac{\alpha(\delta_1 - 1)}{\delta_1}\gamma\sigma_1^{*1-\alpha} = 1 \tag{16}$$

Any solution to 17 pins down the market tightness and continuation utilities at $t = 1$. We will examine later the conditions under which 17 admits a feasible solution. To save space, it is convenient to define the following function for future reference:

Definition 1. *Function $f(\sigma)$ is defined as:*

$$f(\sigma) \equiv \frac{1}{\delta_0} + \frac{(1-\alpha)(\delta_1 - 1)}{\delta_0}\gamma\sigma_1^{-\alpha} + \frac{\alpha(\delta_1 - 1)}{\delta_1}\gamma\sigma_1^{1-\alpha}, \tag{17}$$

The market price will be given by the sellers' participation constraint in 5 combined with

15:

$$
\begin{cases}
p_0^{**} = \frac{1}{\delta_0}\bar{U}_1^S \Rightarrow \\
\\
p_0^{**} = \{\frac{1}{\delta_0} + \frac{(1-\alpha)(\delta_1-1)}{\delta_0}\gamma\sigma_1^{*-\alpha}\}d_2
\end{cases}
\tag{18}
$$

The following defines a competitive search equilibrium in the model:

DEFINITION 2. *An equilibrium is a set of prices, probabilities of trade in the market, (inverse of) market tightness, measures of sellers and buyers participating in the markets and utilities at $t = 0, 1$, denoted by $\{p_t^*, q_t^{*j}, \sigma_t^*, \mu_t^{*j}, V_t^j\}$ for $t = 0, 1$ and $j \in \{B, S\}$, such that $\sigma_0^* = \frac{\mu_0^{*S}}{\mu_0^{*B}}$ and $\sigma_1^* = \frac{\mu_1^{*S}}{\mu_1^{*B}}$ and sellers and buyers are maximizing their welfare according to 1, 3, 7, 5 and 12 and $\mu_0^{*S} = m$ if $V_0^S > R_0^S$ and $\mu_0^{*B} = 1$ if $V_0^B > R_0^B$.*

Before moving to the next section, we can define some of the notions which will be used throughout:

DEFINITION 3. *Given a set of parameters, $\{m, \gamma, \delta_0, \delta_1\}$, market is said to be fragile or subject to run, if both full participation and limited participation equilibria exist. When market is fragile we call the equilibrium with full and limited participation a run and delayed equilibrium denoted by subscripts R and D respectively. The strictly lower price at $t = 0$ between the run and delayed equilibria (p_{0D}^* or p_{0R}^*) is called a fire-sale price.*

2.2 Fragility and Fire-Sale

We have characterized competitive equilibria in the previous section. In this section, we examine the conditions under which markets are fragile and whether and when any fire-sale takes place.

First, we want to characterize the behavior of the price at $t = 0$ for the *delayed* and *run* equilibria when we have fragility in the market. We restate the equilibrium price for the full participation, i.e., *run*, equilibrium:

$$
\begin{cases}
p_0^* = (1-\alpha)(d_2 - \bar{U}_1^B) + \alpha\frac{1}{\delta_0}\bar{U}_1^S \Rightarrow \\
\\
p_0^*(\sigma_1^*) = \{(1-\alpha)\big(1 - \frac{\alpha(\delta_1-1)}{\delta_1}\gamma\sigma_1^{*1-\alpha}\big) + \frac{\alpha}{\delta_0}\big(1 + (1-\alpha)(\delta_1-1)\gamma\sigma_1^{*-\alpha}\big)\}d_2
\end{cases}
\tag{19}
$$

The above equation for the price at $t = 0$ holds across different equilibria: for the *delayed* equilibrium the equation collapses to 18 as we have $d_2 - \bar{U}_1^B = \frac{1}{\delta_0}\bar{U}_1^S$. Taking the derivative with respect to the (inverse of) market tightness gives:

$$\frac{dp_0^*(\sigma_1^*)}{d\sigma_1^*} = \big(-\frac{\alpha}{\delta_0}\sigma_1^{*-(1+\alpha)} - \frac{1-\alpha}{\delta_1}\sigma_1^{*-\alpha}\big) d_2 < 0 \tag{20}$$

Hence the change in price at $t = 0$, when agents switch from one equilibrium to another, depends on the change in σ_1^*. As the equilibrium in the market at $t = 1$ always entails full participation of buyers and sellers we have:

$$\sigma_1^* = \frac{m - \nu_0}{1 - \nu_0} \tag{21}$$

Where ν_0 is the total number of matches formed or the volume of trade in $t = 0$. We note that $\frac{d\sigma_1^*}{d\nu} > 0$ if and only if $m > 1$. We know that full participation always has the highest volume of trade and therefore we can summarize the results as follows:

LEMMA 1. *Whenever we have market fragility, the following hold for the delayed and run equilibria:*

$$p_0^{*R} < p_0^{*D} \iff m > 1$$

Where superscripts D and R denote delayed and run equilibria. In other words, we have fire sale either in a run equilibrium where $m > 1$ or in a delayed equilibrium where $m < 1$. Moreover, ν_0 or the volume of trade at $t = 0$ is higher when the price is higher if and only if $m < 1$.

When there are more (less) sellers than buyers in the market, the *run* equilibrium has a lower (higher) price than the *delayed* as more participation in the market at $t = 0$ lowers (raises) the continuation value and therefore the current market price. The latter is true because when $m > 1$ ($m < 1$) the resulting market tightness at $t = 1$ is a decreasing (increasing) function of the volume of trade at $t = 0$.

2.2.1 Conditions for Fragility

Before any further analysis, its helpful to take a look at a benchmark where trade takes place in a centralized competitive asset market:

Lemma 2. *(Centralized market as a benchmark) In a centralized competitive asset market, and except for the knife-edge case of $m = 1$, the equilibrium prices at which trade takes place are unique. When $m > 1$ the following are true. All trades take place in $t = 0$ at $p_0 = d_2/\delta_0$ if $\delta_0 > \delta_1$ or in $t = 1$ at $p_1 = d_2/\delta_1$ if $\delta_0 < \delta_1$. And $p_0 = p_1 = d_2/\delta_0 = d_2/\delta_1$ and the volumes of trade in $t = 0, 1$ are indeterminate if $\delta_0 = \delta_1$. In this case buyers enjoy all of the trade surplus. When $m < 1$ all trades take place in $t = 0$ at $p_0 = d_2$ when $\delta_1 < \delta_0$ and sellers enjoy all of trade surplus. When $\delta_1 = \delta_0$ we have $p_0 = d_2$ but the volume of trade at $t = 0$ is indeterminate. Finally if $\delta_1 > \delta_0$ no trade takes place at $t = 0$ and asset is traded only at $t = 1$.*

The above lemma implies that market fragility, at least within this setup, is not a feature of centralized trading. This is because there are no externalities and no possibility of coordination failure among sellers or buyers. When sellers and buyers can trade with certainty if they choose to, delaying trade by one agent at $t = 0$ has no externality on others' decisions as it doesn't affect the probability of trade for other agents.

By contrast, in a decentralized market each agent's decision to participate at $t = 0$ may change the probability of trade at both dates, $t = 0, 1$, for others. The following lemma establishes the existence of multiplicity at $t = 0$:

Lemma 3. *There are potentially two types of equilibria with strictly positive volume of trade at $t = 0$ in the model. One type features a unique full participation equilibrium that is characterized by 8, 9, 12, 13 and the following additional conditions:*

$$\begin{cases} V_0^S > R_0^S, \quad V_0^B > R_0^B, \quad \mu_0^{*S} = m, \quad \mu_0^{*B} = 1, \\ \\ \mu_1^{*S} = m - \gamma m^{1-\alpha}, \quad \mu_1^{*B} = 1 - \gamma m^{1-\alpha} \\ \\ \sigma_1^* = \frac{m - \gamma m^{1-\alpha}}{1 - \gamma m^{1-\alpha}}, \quad \sigma_0^* = m \end{cases}$$

In the second type or the limited participation equilibria, sellers and buyers are indifferent between participating in the market and staying out of the market at $t = 0$. Equilibrium prices and utilities at $t = 0, 1$, (inverse of) tightness measure and hence probabilities of finding a partner at $t = 1$ are pinned down using 9, 8, 15, 17, 18 and the following additional *conditions:*

$$\begin{cases} \gamma {\mu_0^{*S}}^{1-\alpha} {\mu_0^{*B}}^{\alpha} = \frac{m-\sigma_1^*}{1-\sigma_1^*}, \quad \sigma_0^* = \frac{\mu_0^{*S}}{\mu_0^{*B}} \\ \\ \mu_1^{*S} = m - \gamma {\mu_0^{*S}}^{1-\alpha} {\mu_0^{*B}}^{\alpha}, \quad \mu_1^{*B} = 1 - \gamma {\mu_0^{*S}}^{1-\alpha} {\mu_0^{*B}}^{\alpha} \end{cases}$$

Unlike random search, competitive search allows the intratemporal effect of each agent's action to be fully priced in the market. But the intertemporal effect of agent's decision at $t = 0$ on others agents' probability of trade and continuation utility at $t = 1$ is not fully captured by price mechanism at $t = 0$. This is the source of coordination problems and multiplicity in the model.

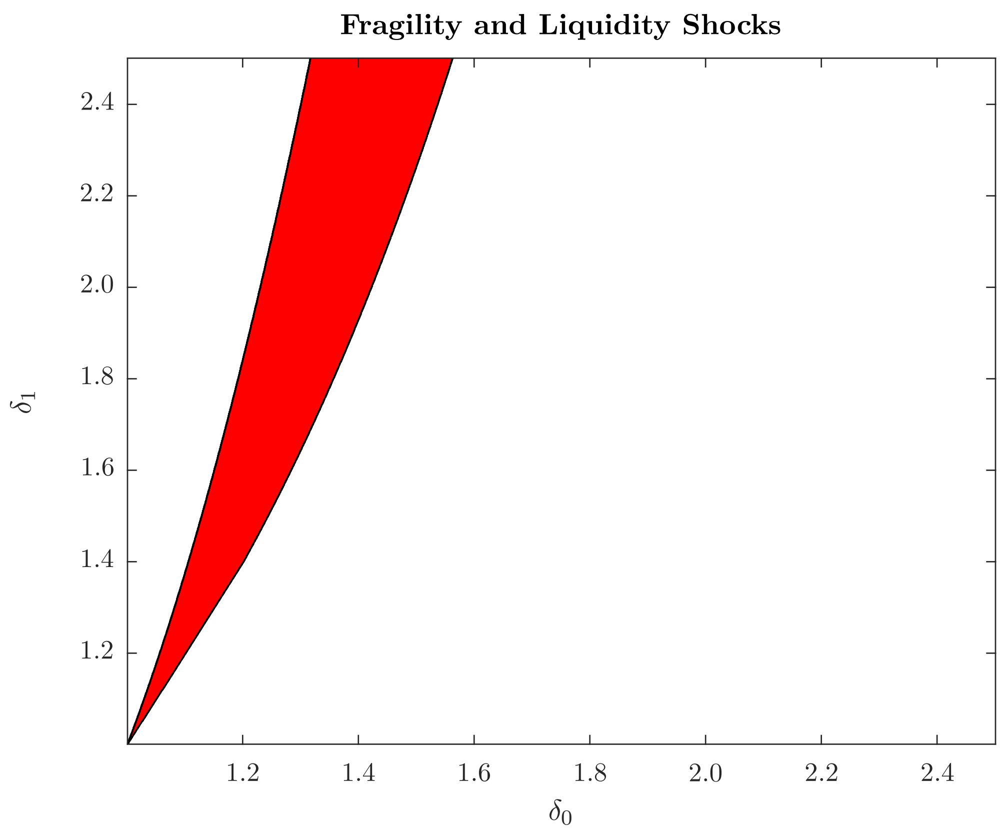

Figure 1: The region with red illustrates the values of liquidity shocks for which markets are fragile for $\alpha = 0.55$ and $\gamma = 0.3$.

We now turn to the conditions for the existence of multiple equilibria. To this end, we need to make the following assumptions:

Assumption 3. *We have* $f(\gamma^{\frac{-1}{1-\alpha}}) > 1$, $f(\gamma^{\frac{1}{\alpha}}) > 1$.

The first two inequalities in 3 ensures that limited participation equilibrium exists at least for some values of m. The following proposition characterizes the conditions on liquidity shocks for the existence of fragility in the market:

Proposition 1. *Given 3, the necessary and sufficient condition in terms of the liquidity shocks δ_0 and δ_1 that ensure the existence of fragility in the asset market at least for some values of m is:*

$$f\left(\frac{\delta_1}{\delta_0}\right) < 1 \iff \gamma\left(\frac{\delta_1}{\delta_0}\right)^{-\alpha} < \frac{\delta_0 - 1}{\delta_1 - 1},$$

The above condition implies:

$$1 < \frac{\delta_1}{\delta_0} < \gamma^{\frac{-1}{1-\alpha}},$$

Moreover there is no fragility for $m < 1$, if $f(1) > 1$.

Proposition 1 states that to have fragility in the market the future liquidity shock should not be too high or too low. For fixed levels of m and current liquidity shock, δ_0, an increasing δ_1 changes the profile of equilibria as follows. For low levels of δ_1, the only equilibrium is the full participation equilibrium. As δ_1 increases further and within the medium range of the future liquidity shock indicated by Proposition 1, both types of full and limited participation equilibria exist and market becomes fragile. Increasing δ_1 even further leads to an equilibrium in which there is no trade at $t = 0$.

This is similar to other environments such as models of currency attacks where multiple equilibria is an outcome when fundamentals are within a medium range, i.e., not too weak or too strong. It is also important to note that a higher liquidity shock today, makes it less likely to have market fragility. A non-extreme value of δ_1/δ_0 is needed for fragility because opportunities to trade in the current and future dates should be of relatively comparable values. This has to be the case for the externalities of each agent's decision to trade to have any effect on others.

The next result shows how fragility depends on m, the ratio of sellers to buyers or the size of (potential) supply relative to demand for the asset. Using Proposition 1, we restrict our attention to the case $f(1) < 1$ where markets with $m < 1$ can also be fragile[6]:

Proposition 2. *Assume that $f(1) < 1$. Given a set of all other parameters, $\{\gamma, \delta_0, \delta_1\}$, there exist a quadruple $\{\underline{m}, \overline{m}, \underline{\underline{m}}, \overline{\overline{m}}\}$ satisfying:*

$$\begin{cases} \gamma^{\frac{1}{\alpha}} < \underline{m} < \overline{m} < 1, \\ \\ 1 < \underline{\underline{m}} < \overline{\overline{m}} < \gamma^{\frac{-1}{1-\alpha}} \end{cases}$$

such that markets are fragile if and only if m satisfies one of the two following conditions:

$$\begin{cases} m < 1, \quad \underline{m} < m < \overline{m}, \\ \\ m > 1, \quad \underline{\underline{m}} < m < \overline{\overline{m}} \end{cases}$$

The above proposition suggests that there should be a minimum *imbalance* between the supply and demand for the asset for the market to be fragile. This is because when m is close to one, participation decisions at $t = 0$ by agents don't have any substantial impact on the (inverse of) market tightness, σ_1^*, at $t = 1$: in the limit when $m = 1$, there will be no impact and $\sigma_1^* = 1$ regardless of what happens in the market at $t = 0$.

Finally, we examine whether more or less liquid markets are prone to fragility. We focus on the case $m < 1$, where fire-sale can happen only by switching from a *run* to a *delayed* equilibrium. Given $m < 1$, the main parameter determining market liquidity in the model is γ. Higher values of γ means higher probability of being matched to a trading partner and hence higher market liquidity (given a certain level of participation). Fixing all other parameters one can derive the following conditions for values of γ for which markets are fragile:

Lemma 4. *For a given set of parameters $\{\alpha, \delta_0, \delta_1, m\}$, the matching efficiency, γ, has to satisfy the following necessary conditions for the existence of multiple equilibria:*

$$\frac{m^{\alpha}(\delta_0 - 1)\delta_1}{((1-\alpha)\delta_1 + \alpha m \delta_0)(\delta_1 - 1)} \leq \gamma \leq \frac{(\delta_0 - 1)\delta_1}{((1-\alpha)\delta_1 + \alpha\delta_0)(\delta_1 - 1)}$$

6

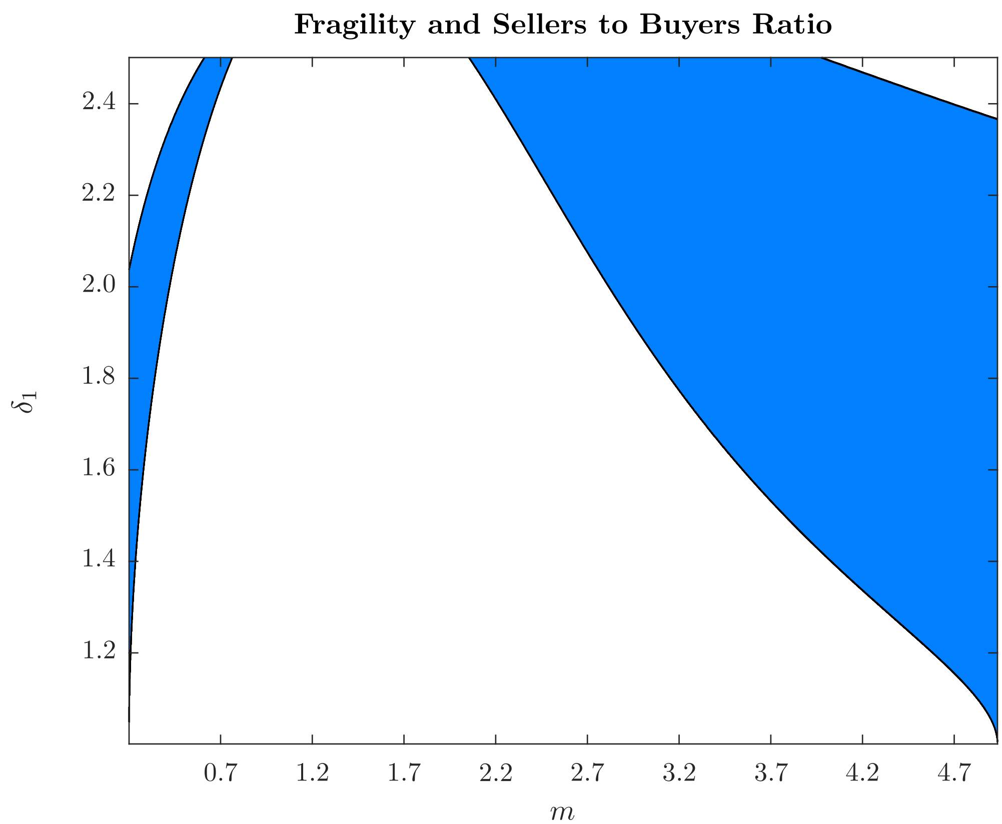

Figure 2: The blue region illustrates the values of m for any level of δ_1 for which markets are fragile. Note that $\delta_0 = 1.6$, $\alpha = 0.5$ and $\gamma = 0.45$.

Too liquid or too illiquid markets don't feature multiplicity or fire-sale. In a too illiquid market, agents' decision to participate doesn't have strong externalities on others as the probability of trade is low. This reduces the scope for coordination failure and hence multiplicity.

MORE ANALYSIS TO BE DONE

2.3 Welfare Analysis

In this section, we look at the welfare properties of different types of equilibria in the presence of fragility. To this end and as the utilities are linear and transferable, we can assume a planner who aims at maximizing the aggregate sum of the trade surplus in units of time zero consumption goods. This amounts to summing up the consumption equivalent of utilities of all agents:

$$W \equiv \bar{U}_0^B + \frac{m}{\delta_0}\bar{U}_0^S \tag{22}$$

The planner can make lump sum taxes and transfers at $t = 0$ to improve sellers and buyers welfare. The following lemma shows how we can rank equilibria according to the welfare measure defined in 22:

LEMMA 5. *If W_i and W_j are total welfare for two different equilibria i and j where $W_j > W_i$, the planner can design transfers in j to achieve an allocation j' that is Pareto superior to i.*

We now derive the total welfare in the case of full participation. Using the objective function 5 and FOC of the full participation equilibrium in 12 we have:

$$\begin{cases} \bar{U}_0^B = \gamma\sigma_0^{*1-\alpha}\big(d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_1^B\big) - (1-\alpha)\gamma\sigma_0^{*1-\alpha}\big(d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_1^B\big) + \bar{U}_1^B, \\ \\ \frac{1}{\delta_0}\bar{U}_0^S = (1-\alpha)\gamma\sigma_0^{*-\alpha}\big(d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_1^B\big) + \frac{1}{\delta_0}\bar{U}_1^S \end{cases} \tag{23}$$

Using the fact that in a full participation equilibrium $\sigma_0^* = m$, we can compute the welfare by adding the two terms in 23 and simplifying as follows:

$$W = \big(\bar{U}_1^B + \frac{\bar{U}_1^S}{\delta_0}m\big) + \gamma m^{1-\alpha}\big(d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_1^B\big) \tag{24}$$

The first term in 24 is the sum of (time zero consumption equivalent) reservation utilities of both sellers and buyers. The second term is the product of total number of transactions at $t = 0$ which is $\gamma m^{1-\alpha}$ and the total trade (extra) surplus of a match between a seller and a buyer.

Note that when there is multiplicity, 24 can be applied to both *run* and *delayed* equilibria. In the case of a *delayed* equilibrium, the extra surplus $d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_1^B$ is equal to zero and total welfare is equal to the sum of continuation utilities. Therefore, and based on Lemma 5, we can use this measure to rank different equilibria when there is fragility. The following proposition establishes the main result of this section:

PROPOSITION 3. *Given a set of parameters $\{\gamma, \delta_0, \delta_1\}$ for which market is fragile, the* *following are true for the run and delayed equilibria. When $m > 1$:*

$$m > 1 \quad \Rightarrow W^D(m, \gamma, \delta_0, \delta_1) > W^R(m, \gamma, \delta_0, \delta_1)$$

And there exists $0 < \xi < 1$ such that for $m < 1$:

$$\xi \leq m < 1 \quad \Rightarrow W^D(m, \gamma, \delta_0, \delta_1) < W^R(m, \gamma, \delta_0, \delta_1)$$

Where W^R and W^D denote the welfare for the run and delayed equilibria respectively.

Lemma 5 and Proposition 3 imply that the *run* and *delayed* equilibria can be ranked according to welfare in a meaningful sense. We note that when market is fragile, the equilibrium with the fire sale price is typically worse in terms of welfare:

Corollary 1. *When market is fragile, the equilibrium with fire sale has the lower welfare for all $m > \xi$ where $\xi < 1$ is defined above.*

Proposition 4.

Proposition 5.

3 Effects of Monetary Policy

4 Conclusion

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A Appendix: Proofs

Proof of Lemma 1. For the last part of the proposition, note that we have $\nu_0^R > \nu_0^D$, which in the case of $m < 1$ implies:

$$\sigma_1^{*R} = \frac{m - \nu_0^R}{1 - \nu_0^R} < \frac{m - \nu_0^D}{1 - \nu_0^D} = \sigma_1^{*D}$$

Given the derivations in the text, this implies that $p_0^{*R} > p_0^{*D}$. The rest of the proposition is proven in the text. ■

Proof of Lemma 2. We solve the equilibrium backward. Suppose that $1 - s$ and $m - s$ are measures of buyers and sellers at $t = 1$ who haven't already traded in the market, where s is the volume of trade at $t = 0$.

Consider the case of $m > 1$ first. Let p_1 denote the price in the market at $t = 1$. The demand curve for asset is:

$$\begin{cases} 0 \le p_1 < d_2 \Rightarrow a_1^d(p_1) = 1 - s \\ \\ p_1 = d_2 \Rightarrow a_1^d(p_1) \in [0, 1 - s] \\ \\ p_1 > d_2 \Rightarrow a_1^d(p_1) = 0 \end{cases}$$

Similarly supply curve at $t = 1$ is:

$$\begin{cases} 0 \le p_1 < \frac{d_2}{\delta_1} \Rightarrow a_1^s(p_1) = 0 \\ \\ p_1 = \frac{d_2}{\delta_1} \Rightarrow a_1^s(p_1) \in [0, m - s] \\ \\ p_1 > \frac{d_2}{\delta_1} \Rightarrow a_1^s(p_1) = m - s \end{cases}$$

Where a^d and a^s are demand and supply for the asset. It is easy to check that the only equilibrium price at $t = 1$ where supply and demand for the asset are equal is $p_1 = \frac{d_2}{\delta_1}$. And in equilibrium all $1 - s$ buyers participate in the market while only $1 - s$ measure of sellers trade and the rest $m - 1$ stay out (and they will be indifferent between the two options).

Moving to $t = 0$, the demand supply are:

$$\begin{cases} 0 \leq p_0 < \frac{d_2}{\delta_1} \Rightarrow a_0^d(p_0) = 1 \\ \\ p_0 = \frac{d_2}{\delta_1} \Rightarrow a_0^d(p_0) \in [0, 1] \\ \\ p_0 > \frac{d_2}{\delta_1} \Rightarrow a_0^d(p_0) = 0 \end{cases}$$

Similarly supply curve at $t = 0$ is:

$$\begin{cases} 0 \leq p_0 < \frac{d_2}{\delta_0} \Rightarrow a_0^s(p_0) = 0 \\ \\ p_0 = \frac{d_2}{\delta_0} \Rightarrow a_0^s(p_0) \in [0, m] \\ \\ p_0 > \frac{d_2}{\delta_0} \Rightarrow a_0^s(p_0) = m \end{cases}$$

It is easy to see that equilibrium price and quantities are as follows:

$$\begin{cases} \delta_0 > \delta_1 \Rightarrow p_0 = \frac{d_2}{\delta_0}, \quad a_0^s = a_0^d = 1 \\ \\ \delta_0 = \delta_1 \Rightarrow p_0 = \frac{d_2}{\delta_0}, \quad a_0^s = a_0^d \in [0, 1] \\ \\ \delta_0 < \delta_1 \Rightarrow p_0 \in [\frac{d_2}{\delta_1}, \frac{d_2}{\delta_0}], \quad a_0^s = a_0^d = 0 \end{cases}$$

Therefore whenever there is any trade at $t = 0$, the equilibrium price is unique and equal to $\frac{d_2}{\delta_0}$.

Now consider the case of $m < 1$. Demand and supply curves at $t = 1$ are defined as before. But because $m < 1$, the unique price will be $p_1 = d_2$. At $t = 0$ we have:

$$\begin{cases} 0 \leq p_0 < d_2 \Rightarrow a_0^d(p_0) = 1 \\ \\ p_0 = d_2 \Rightarrow a_0^d(p_0) \in [0, 1] \\ \\ p_0 > d_2 \Rightarrow a_0^d(p_0) = 0 \end{cases}$$

Similarly supply curve at $t = 0$ is:

$$\begin{cases} 0 \leq p_0 < \frac{\delta_1 d_2}{\delta_0} \Rightarrow a_0^s(p_0) = 0 \\ \\ p_0 = \frac{\delta_1 d_2}{\delta_0} \Rightarrow a_0^s(p_0) \in [0, m] \\ \\ p_0 > \frac{\delta_1 d_2}{\delta_0} \Rightarrow a_0^s(p_0) = m \end{cases}$$

And therefore the equilibrium in this case will be:

$$\begin{cases} \delta_0 > \delta_1 \Rightarrow p_0 = d_2, \quad a_0^s = a_0^d = m \\ \\ \delta_0 = \delta_1 \Rightarrow p_0 = d_2, \quad a_0^s = a_0^d \in [0, m] \\ \\ \delta_0 < \delta_1 \Rightarrow p_0 \in [d_2, \frac{\delta_1 d_2}{\delta_0}], \quad a_0^s = a_0^d = 0 \end{cases}$$

And again whenever there is trade at $t = 0$, the equilibrium price is unique which completes the proof. ■

Proof of Lemma 3. Given 11, if $d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_0^B < 0$ then $\sigma_0^* = 0$ is optimal for buyers which implies no trade at $t = 0$. Hence we focus on the case $d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_0^B \geq 0$.

Consider the case of $d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_0^B > 0$ first. In this case, if $\bar{U}_0^S - \bar{U}_1^S = 0$ buyers will set $q_0^B(\sigma_0^*) = 1$. This needs either $m \leq \gamma^{\frac{1}{\alpha}}$ or $m \geq \gamma^{\frac{-1}{1-\alpha}}$ which cannot be the case given Assumption 2. To see this note that $m > \frac{m - \gamma m^{1-\alpha}}{1 - \gamma m^{1-\alpha}}$ for $m < 1$ and $m < \frac{m - \gamma m^{1-\alpha}}{1 - \gamma m^{1-\alpha}}$ for $m > 1$. Therefore, we need to have $\bar{U}_0^S - \bar{U}_1^S > 0$. This implies the FOC in 12 are the optimality conditions for buyers. Solving for $\bar{U}_0^B$ and $\bar{U}_0^S$ gives:

$$\begin{cases} \bar{U}_0^B = \alpha\gamma\sigma_0^{*1-\alpha}\left(d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_0^B\right) + \bar{U}_1^B \\ \\ \frac{\bar{U}_0^S}{\delta_0} = (1-\alpha)\gamma\sigma_0^{*-\alpha}\left(d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_0^B\right) + \frac{\bar{U}_1^S}{\delta_0} \end{cases}$$

This shows that $V_0^B = \bar{U}_0^B > \bar{U}_1^B$ and $V_0^S = \bar{U}_0^S > \bar{U}_1^S$ and that there is full participation by buyers and sellers in equilibrium. Hence in equilibrium we have $\sigma_0^* = m$ which also implies

$\sigma_1^* = \frac{m-\gamma m^{1-\alpha}}{1-\gamma m^{1-\alpha}}$ and:

$$\begin{cases} \mu_0^{*S} = m, \quad \mu_0^{*B} = 1, \\ \\ \mu_1^{*S} = m - \gamma m^{1-\alpha}, \quad \mu_1^{*B} = 1 - \gamma m^{1-\alpha} \end{cases}$$

Now consider the case $d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_0^B = 0$. We must have $\bar{U}_0^S - \bar{U}_1^S = 0$, otherwise buyers will not enter the market and there will be no trade at $t = 0$. Therefore, in this case both buyers and sellers are indifferent between participating in or staying out of the market at time zero. If there exists a solution to $d_2 - \frac{1}{\delta_0}\bar{U}_1^S - \bar{U}_0^B = 0$ in terms of σ_1^* such that $\sigma_1^* \in [\frac{m-\gamma m^{1-\alpha}}{1-\gamma m^{1-\alpha}}, m]$ when $m < 1$ or $\sigma_1^* \in [m, \frac{m-\gamma m^{1-\alpha}}{1-\gamma m^{1-\alpha}}]$ when $m > 1$, we will have the following:

$$\sigma_1^* = \frac{m - \gamma {\mu_0^{*S}}^{1-\alpha} {\mu_0^{*B}}^{\alpha}}{1 - \gamma {\mu_0^{*S}}^{1-\alpha} {\mu_0^{*B}}^{\alpha}} \Rightarrow \gamma {\mu_0^{*S}}^{1-\alpha} {\mu_0^{*B}}^{\alpha} = \frac{m - \sigma_1^*}{1 - \sigma_1^*}$$

Which completes the proof. ■

Proof of Proposition 1. Taking the derivative of $f(\sigma_1)$ gives:

$$f'(\sigma_1) = \frac{\alpha(1-\alpha)(\delta_1 - 1)}{\delta_1} \gamma \sigma_1^{-(1+\alpha)} \{\sigma_1 - \frac{\delta_1}{\delta_0}\}$$

We note that $f(\sigma_1)$ is strictly decreasing for $\sigma_1 < \frac{\delta_1}{\delta_0}$ and strictly increasing for $\frac{\delta_1}{\delta_0} < \sigma_1$ and has a minimum at $\frac{\delta_1}{\delta_0}$. By Assumption 3, we know that f takes values higher than one at the boundaries. Hence the necessary and sufficient condition for the existence of a root is $f(\frac{\delta_1}{\delta_0}) < 1$. This implies:

$$\begin{cases} f(\frac{\delta_1}{\delta_0}) = \frac{1}{\delta_0} + \gamma \left(\frac{\delta_1}{\delta_0}\right)^{-\alpha} \frac{\delta_1 - 1}{\delta_0} < 1 \iff \\ \\ \gamma \left(\frac{\delta_1}{\delta_0}\right)^{-\alpha} < \frac{\delta_0 - 1}{\delta_1 - 1} \end{cases}$$

It is easy to verify that Assumption 3 and Assumption 2 imply $\delta_1 > \delta_0$. This and the above conditions give:

$$\gamma\left(\frac{\delta_1}{\delta_0}\right)^{-\alpha} < \frac{\delta_0 - 1}{\delta_1 - 1} < \frac{\delta_0}{\delta_1} \Rightarrow \frac{\delta_1}{\delta_0} < \gamma^{\frac{-1}{1-\alpha}}$$

Therefore $1 < \frac{\delta_1}{\delta_0} < \gamma^{\frac{-1}{1-\alpha}}$. Finally, if $m < 1$ we have $\gamma^{\frac{1}{\alpha}} < \frac{m-\gamma m^{1-\alpha}}{1-\gamma m^{1-\alpha}} < m < 1$ (Assumption 2). If $f(1) > 1$, all values of f in $[\gamma^{\frac{1}{\alpha}}, 1]$ are strictly higher than one and hence there's no $\sigma_1 \in [\frac{m-\gamma m^{1-\alpha}}{1-\gamma m^{1-\alpha}}, m]$ for which $f(\sigma_1) = 1$. This completes the proof. ■

Proof of Proposition 2. Let's define the function $\zeta(m) \equiv \frac{m-\gamma m^{1-\alpha}}{1-\gamma m^{1-\alpha}}$. Taking the derivative we have:

$$\zeta'(m) = \left\{1 - \left(\alpha\gamma m^{1-\alpha} + (1-\alpha)\gamma m^{-\alpha}\right)\right\}\frac{1}{(1-\gamma m^{1-\alpha})^2} > 0$$

The inequality above holds because $\gamma m^{1-\alpha}$ and $\gamma m^{-\alpha}$ are always less than one by Assumption 2. Moreover we have $\zeta(1) = 1$ and $\zeta(0) = 0$. Hence $\zeta : [0,1] \to [0,1]$ is an isomorphism and has a well defined inverse on $[0,1]$. Moreover, $\zeta : [1, \gamma^{\frac{-1}{1-\alpha}}) \to [1, \infty)$ is also an isomorphism strictly increasing for $1 < m < \gamma^{\frac{-1}{1-\alpha}}$.

Consider the case of $m < 1$ first. And let $\tilde{\sigma}_1$ be the solution to 15 which amounts to $f(\tilde{\sigma}_1) = 1$. We know that $\gamma^{\frac{1}{\alpha}} < \tilde{\sigma}_1 < 1$ exists because $f(1) < 1$ and $f(\gamma^{\frac{1}{\alpha}}) > 1$ by Assumption 3. We know from the proof of Proposition 1 that f is strictly decreasing for $m < 1$. Also we must have that $\tilde{\sigma}_1 \in [\zeta(m), m]$ (note that $\zeta(m) < m$ for $m < 1$) which implies:

$$f(m) < f(\tilde{\sigma}_1) < f(\zeta(m)) \Rightarrow \zeta(m) < \tilde{\sigma}_1 < m \Rightarrow \tilde{\sigma}_1 < m < \zeta^{-1}(\tilde{\sigma}_1)$$

Given that $f(\tilde{\sigma}_1) = 1$ and that ζ is an isomorphism which also imply that $\zeta^{-1}(\tilde{\sigma}_1) < 1$.

Next consider the case when $m > 1$. Let $\tilde{\tilde{\sigma}}_1 \in [\frac{\delta_1}{\delta_0}, \gamma^{\frac{-1}{1-\alpha}}]$ be the solution to $f(\tilde{\tilde{\sigma}}_1) = 1$. We know that such solution exists because of Assumption 3, that $f(\frac{\delta_1}{\delta_0}) < f(1) = 1$ since f is minimized at $\frac{\delta_1}{\delta_0}$ and that f is strictly increasing for $[\frac{\delta_1}{\delta_0}, \gamma^{\frac{-1}{1-\alpha}}]$ (Proposition 1). We must also have $\tilde{\tilde{\sigma}}_1 \in [m, zeta(m)]$ which implies $\frac{\delta_1}{\delta_0} < \zeta(m)$. This gives:

$$f(\tilde{\tilde{\sigma}}_1) < f(\zeta(m)) \Rightarrow \tilde{\tilde{\sigma}}_1 < \zeta(m) \Rightarrow \zeta^{-1}(\tilde{\tilde{\sigma}}_1) < m < \zeta^{-1}(\gamma^{\frac{-1}{1-\alpha}}) < \gamma^{\frac{-1}{1-\alpha}}$$

The last inequalities are the results of Assumption 2 and the fact that $m < \zeta(m)$ for $m > 1$. $\tilde{\tilde{\sigma}}_1 > \frac{\delta_1}{\delta_0} > 1$ and hence $1 < \zeta^{-1}(\tilde{\tilde{\sigma}}_1)$ which completes the proof.

■

Proof of Lemma 4. $m < 1$ implies that at time $t = 1$ inverse of market tightness σ_1 satisfies $m \leq \sigma_1 \leq 1$. Using the proof of Proposition 1, we know that function $f(\sigma_1)$ is strictly decreasing for $\sigma_1 \in [m, 1]$. This implies that $f(\sigma_1) = 1$ has a solution within $\sigma_1 \in [m, 1]$ if and only if $f(1) \leq 1$ and $f(m) \geq 1$. These last two conditions can be written as:

$$\begin{cases} \frac{1}{\delta_0} + \left\{\frac{(1-\alpha)(\delta_1-1)}{\delta_0} + \frac{\alpha(\delta_1-1)}{\delta_1}\right\}\gamma \leq 1 \\ \\ \frac{1}{\delta_0} + \left\{\frac{(1-\alpha)(\delta_1-1)}{\delta_0}m^{-\alpha} + \frac{\alpha(\delta_1-1)}{\delta_1}m^{1-\alpha}\right\}\gamma \geq 1 \end{cases}$$

Taking γ to one side in above and simplifying yields:

$$\frac{m^{\alpha}(\delta_0 - 1)\delta_1}{((1-\alpha)\delta_1 + \alpha m\delta_0)(\delta_1 - 1)} \leq \gamma \leq \frac{(\delta_0 - 1)\delta_1}{((1-\alpha)\delta_1 + \alpha\delta_0)(\delta_1 - 1)}$$

■

Proof of Lemma 5. To see this suppose $W_j - W_i = \Delta > 0$. Let $\bar{U}^B_{0j} - \bar{U}^B_{0i} = \epsilon$ and $\bar{U}^S_{0j} - \bar{U}^S_{0i} = \eta$. If both ϵ and η are positive, j Pareto dominates i and no transfers are needed. Suppose without loss of generality that $\epsilon > 0$ but $\eta \leq 0$. We know by assumption that $\epsilon + \frac{\eta}{\delta_0}m = \Delta > 0$. The planner can give a lump sum subsidy equal to $-\frac{\eta}{\delta_0}$ to the sellers while levy a tax equal to $-\frac{\eta}{\delta_0}m$ on buyers in equilibrium j to achieve a new allocation j'. Then we will have:

$$\begin{cases} \bar{U}^{S'}_{0j'} = \bar{U}^S_{0j} - \eta = \bar{U}^S_{0i}, \\ \\ \bar{U}^{B'}_{0j'} = \bar{U}^B_{0j} + \frac{\eta}{\delta_0}m = (\bar{U}^B_{0i} + \epsilon) + \frac{\eta}{\delta_0}m = \bar{U}^B_{0i} + \Delta > \bar{U}^B_{0i} \end{cases}$$

Hence in the new allocation j', buyers are strictly better off and sellers are at least as well off. This implies that j' is Pareto superior to i. ■

Proof of Proposition 3. Using 9 and 24 we can rewrite the the welfare as a function of the (inverse of) market tightness at $t = 1$:

$$W = \{\gamma m^{1-\alpha} + \frac{m - \gamma m^{1-\alpha}}{\delta_0} + \frac{(1-\alpha)(m - \gamma m^{1-\alpha})(\delta_1 - 1)}{\delta_0}\gamma\sigma_1^{*-\alpha} + \frac{\alpha(\delta_1 - 1)(1 - \gamma m^{1-\alpha})}{\delta_1}\gamma\sigma_1^{*1-\alpha}\}d_2$$

Taking the derivative of W with respect to σ_1^* gives:

$$\frac{dW}{d\sigma_1^*} = \frac{\alpha(1-\alpha)(\delta_1 - 1)(1 - \gamma m^{1-\alpha})}{\delta_1}\gamma\sigma_1^{*-(1+\alpha)}\left\{\sigma_1^* - \frac{\delta_1}{\delta_0}\frac{m - \gamma m^{1-\alpha}}{1 - \gamma m^{1-\alpha}}\right\}d_2$$

It is evident that W is strictly decreasing for $\sigma_1^* < \frac{\delta_1}{\delta_0}\frac{m-\gamma m^{1-\alpha}}{1-\gamma m^{1-\alpha}}$ and strictly increasing for $\sigma_1^* > \frac{\delta_1}{\delta_0}\frac{m-\gamma m^{1-\alpha}}{1-\gamma m^{1-\alpha}}$ and has a minimum at $\sigma_{1min}^* = \frac{\delta_1}{\delta_0}\frac{m-\gamma m^{1-\alpha}}{1-\gamma m^{1-\alpha}}$.

When $m > 1$, we have:

$$\sigma_1^{*D} < \sigma_1^{*R} = \frac{m - \gamma m^{1-\alpha}}{1 - \gamma m^{1-\alpha}} < \sigma_{1min}^*$$

Where σ_1^{*R} and σ_1^{*D} are the (inverse of) market tightness at $t = 1$ for the *run* and *delayed* equilibria. The last inequality follows from $\delta_0 < \delta_1$. This implies that $W^D > W^R$. For the case of $m < 1$, we know that $\sigma_1^{*R} = \frac{m-\gamma m^{1-\alpha}}{1-\gamma m^{1-\alpha}} < \sigma_1^{*D} < m$. If $m \leq \sigma_{1min}^*$ we have $W^R > W^D$ because W is strictly decreasing on the left of σ_{1min}^*. We show that there is a threshold $0 < \xi < 1$ for which $m \leq \sigma_{1min}^*$ as long as $m \geq \xi$. To this end define:

$$\xi(m) \equiv \frac{m - \gamma m^{1-\alpha}}{1 - \gamma m^{1-\alpha}}/m$$

Taking derivative with respect to m gives:

$$\xi'(m) = \left\{1 - \left(\frac{\alpha\gamma m^{1-\alpha} - 1}{\alpha\gamma m^{1-\alpha}}\right)\left(\frac{1 - \gamma m^{-\alpha}}{m^{-1} - \gamma m^{-\alpha}}\right)\right\}\frac{\alpha\gamma m^{-(1+\alpha)}}{m^{-1} - \gamma m^{-\alpha}}$$

We note that the first term above is always strictly positive for all $0 < m < 1$ and hence $\xi'(m) > 0$ and $\xi(m)$ is strictly increasing. Hence we have:

$$m \leq \frac{\delta_1}{\delta_0}\frac{m - \gamma m^{1-\alpha}}{1 - \gamma m^{1-\alpha}} \iff \xi(m) \geq \frac{\delta_0}{\delta_1} \iff m \geq \xi^{-1}(\frac{\delta_0}{\delta_1})$$

We also note that $\lim_{m\to 0}\xi(m) = 0$ which implies that $\xi(m)$ is an isomorphism on $[0, 1]$ and so it has an inverse on $[0, 1]$. Hence the threshold is $\xi = \xi^{-1}(\frac{\delta_0}{\delta_1}) < 1$. This completes the proof. ■