Coherence without Rationality at the ZLB*

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Abstract

Standard rational expectations (RE) models with an occasionally binding zero lower bound (ZLB) constraint either admit no solutions (incoherence) or multiple solutions (incompleteness). This paper shows that deviations from full-information RE mitigate concerns about incoherence and incompleteness. Models with no REE admit self-confirming equilibria involving the use of simple mis-specified forecasting models. Completeness and coherence is restored if expectations are adaptive or if agents are less forward-looking due to some information or behavioral friction. In the case of incompleteness, the E-stability criterion selects an equilibrium.

Keywords: incompleteness, incoherence, expectations, zero lower bound
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The irrationality of a thing is no argument against its existence, rather a condition of it. Friedrich Nietzsche

1 Introduction

In the last 15 years after the Great Financial Crisis, central banks in western economies had to face the problem of a zero (or effective) lower bound (ZLB) on the nominal interest rate. This spurred a very large and important literature on the topic. At least from the seminal contribution by Benhabib et al. (2001), it is well-known that rational expectations (RE) models with a ZLB on the nominal interest rate generally admit multiple equilibria and also multiple steady states. Most recently, however, Ascari and Mavroeidis (2022) highlight an even more serious concern regarding this type of models when stochastic shocks hit the economy, a standard assumption in macroeconomic models. They show that in models featuring a ZLB constraint, a stochastic environment and RE, equilibrium existence is not generic (incoherence), and when these model do admit an equilibrium, they generally admit more equilibria (incompleteness) than previously acknowledged.[1] Specifically, Ascari and Mavroeidis (2022) derive conditions for existence of a rational expectations equilibrium (REE), and for existence and uniqueness of a minimum state variable (MSV) equilibrium for dynamic forward-looking models with occasionally binding constraints.[2] Given that a model without an equilibrium could not be of any use, Ascari and Mavroeidis (2022) explore how modifications to the baseline New Keynesian (NK) model, such as introducing unconventional monetary policies, could avoid the incoherence problem.

In this paper, we point to another route to tackle the incoherence problem. Rather than modifying the baseline model, we abandon the full-information RE assumption. We show that the problem of incoherence and incompleteness hinges on the assumption that agents have RE. Non-existence of REE is by itself a compelling and novel reason to investigate the possibility of non-rational equilibria. Indeed, one of the main results from this paper is that a standard New Keynesian model with the ZLB constraint can fail to yield an REE and still admit other types of self-confirming equilibria. To illustrate this point, we consider two distinct equilibrium concepts which have been associated with different types of deviations from full-information RE. First, we study an economy populated by agents who have forecasting models that are mis-specified and under-parameterized relative to the forecasting models that agents would have in an REE. Under this assumption, we derive the analytical

[1] Following Ascari and Mavroeidis (2022) we will use the terms incoherence and incompleteness to mean the non-existence of equilibria and the multiplicity of equilibria, respectively. Hence, a model is coherent if it admits at least one equilibrium, and complete if the equilibrium is unique.

[2] Therein, an MSV equilibrium is defined as usually intended, that is, as a function of the state variables of the model. However, an incoherent model could in principle admit other types of equilibria, but, to the best of our knowledge, no work in the literature, including Ascari and Mavroeidis (2022), has found them. We use the terminology MSV and REE interchangeably in the case of incoherence.

conditions so that the economy settles on a *restricted perceptions equilibrium (RPE)* in which agents make optimal forecasts within their class of forecasting rules. Importantly, we prove that RPE can exist when the RE model is incoherent and hence no REE exists. Alternatively, we assume agents are less forward-looking than rational agents, for instance because they are myopic à la Gabaix (2020), have imperfect common knowledge as in Angeletos and Lian (2018), or have finite planning horizons similar to Woodford and Xie (2020). In this setting too, a unique *bounded rationality equilibrium (BRE)* may exist, even if an REE does not.

The derivation of BRE and RPE in an incoherent REE framework is a central contribution of the paper. In this respect, some remarks are noteworthy.

First, the learning literature has typically focused on the question of whether an REE can be learnable, because the underlying model admits an REE solution. Here, instead, an RPE emerges as a self-confirming equilibrium, even if the underlying model does not admit an REE, which we believe is a novel and intriguing case in the literature.

Second, and related to the previous point, whenever the NK model does not admit an REE, it is impossible for agents to form self-confirming beliefs about the state-dependent dynamics of inflation and output implied by the stochastic shocks, and the economy can easily diverge into a deflationary spiral if agents attempt to learn these dynamics using simple statistical techniques. The intuition is easy to grasp by interpreting the problem of rational incoherence in terms of income and substitution effects, following Bilbiie (forth.). On the one hand, a negative demand shock creates a deflation that raises the current real interest rate when the economy hits the ZLB, inducing higher savings (substitution effect). On the other hand, output increases more than one-to-one in response to an expected future output if the shock is very large or very persistent (income effect). The strength of this second effect depends on agents' forward-lookingness. When agents are forward-looking and rational, and the negative shock is very large or persistent, the income effect dominates and this gives rise to a situation where the policymaker cannot set an equilibrium nominal interest rate using a Taylor rule, i.e., incoherence. We show that deviations from RE weaken this income effect and restore coherence. Hence, while it is a curse to be smart, it is a blessing to be simple-minded, because the non-rationality of agents' beliefs can save the economy from spiralling and lead it to a coherent and complete (CC) self-confirming RPE. Note that a similar intuition is behind the so-called "forward guidance puzzle" and its proposed solutions that hinge on weakening agents' forward-lookingness (e.g., Del Negro et al., 2012; McKay et al., 2016b; Angeletos and Lian, 2018; Gabaix, 2020; Woodford and Xie, 2020; Eusepi et al., 2021a).

Third, a basic takeaway from the existence analysis is that the baseline NK model with RE is incoherent, but can admit RPE or BRE, *when negative shocks are sufficiently large in magnitude or sufficiently persistent*. A fundamentals-driven RE liquidity trap, thus, must be relatively short-lived compared to the duration of actual liquidity trap events experienced by Japan, the Euro Area and the U.S., because persistent shocks would make the REE incoherent. This is not true for the RPE, where a liquidity trap can be highly persistent. In this sense, one could argue that an RPE or a BRE could explain why the economy did not blow up after a large shock as the Great Financial Crisis.

The second contribution of the paper concerns the stability properties of these equilibria under learning, that is, the issue of whether RPE and REE can emerge from a process of learning. Following the adaptive learning literature, we employ the expectational stability or "E-stability" criterion to select an equilibrium that may arise through an economy-wide adaptive learning process in which agents recursively update the parameters of their subjective forecasting models using simple statistical techniques such as least squares. We find there is a unique E-stable RPE when an RPE exists. Similarly, only one MSV REE can be E-stable.

Finally, while E-stability is useful for selecting a self-confirming equilibrium in the case of incompleteness, it is worth noting that adaptive learning can ensure completeness and coherence all by itself. Specifically, we prove that a unique temporary equilibrium always exists in our model with a ZLB constraint and adaptive learning agents, provided that agents do *not* observe current endogenous variables before market clearing takes place–a very common assumption in the learning literature. If learning agents condition their forecasts on current information about endogenous variables, then a temporary equilibrium only exists under more stringent assumptions.

After a brief literature review, the paper proceeds as follows. Section 2 introduces a simple model of the ZLB that nests our different assumptions about expectations formation as special cases. Section 3 illustrates the problem of rational incoherence and the possibility of irrational coherence. Section 4 shows how adaptive learning resolves incompleteness issues, and also discusses the plausibility of the RPE concept. Section 5 suggests an additional route to irrational coherence: lagged information about economic shocks. Section 6 concludes. The proofs of all the Propositions can be found in the Appendix.

1.1 Literature Review

To the best of our knowledge, our paper is the first paper to analytically characterize the existence, uniqueness, and learnability of restricted perceptions equilibrium in a model with an occasionally binding ZLB constraint and persistent, recurring shocks to the economy. We build on earlier studies of RPE (see Branch, 2006, for a survey),[3] or related concepts such as consistent expectations equilibria (Hommes and Sorger, 1997) and behavioral learning equi-

[3]See also Evans and Honkapohja (2001). Some related issues are considered by Marcet and Sarget (1989), Evans et al. (1993), Branch and Evans (2006a), Branch and Evans (2006b), Bullard et al. (2008), Evans and McGough (2020) and Evans et al. (2021), among many others.

libria (Hommes and Zhu, 2014). A recent strand of this literature considers RPE and related self-confirming equilibria in models with regime-switching between active and passive monetary policy regimes (e.g., Airaudo and Hajdini, forth.; Ozden and Wouters, 2021), but this literature abstracts from the occasionally binding constraint, and instead, Airaudo and Hajdini (forth.) assume exogenous regime changes, and Ozden and Wouters (2021) numerically study a model with endogenous transition probabilities between regimes. More generally, the earlier literature primarily focuses on frameworks that may admit REE, whereas we prove that models with the ZLB constraint may *only* admit non-rational equilibria such as RPE.

This paper contributes to an already large literature about deviations from RE and the ZLB. Earlier work on adaptive learning at the ZLB studied monetary and fiscal policies that can prevent an economy with learning agents from getting stuck in the liquidity trap (Evans et al., 2008; Benhabib et al., 2014; Evans et al., forth.),[4] unconventional policies such as forward guidance (Cole, 2021; Eusepi et al., 2021a), make-up strategies such as price level targeting (Honkapohja and Mitra, 2020) or average inflation targeting (Honkapohja and McClung, 2021). Christiano et al. (2017) show that the E-stability criterion selects one of multiple equilibria of a model with a transitory demand shock that can drive the economy into a liquidity trap. This finding is closely related to our result about E-stability of REE in the case of incompleteness. However, their model assumes that the economy returns to a steady state after the shock dissipates, whereas our framework allows for multiple, recurring liquidity trap episodes, consistent with the recurrence of ZLB events in the U.S. and elsewhere. Thus, we extend insights from Christiano et al. (2017) to models with recurring demand shocks. More generally, the above mentioned papers do not consider existence and stability of equilibria of models with recurring, fundamentals-driven liquidity traps.

A number of earlier works, including Angeletos and Lian (2018), Gabaix (2020) and Woodford and Xie (2020), study bounded rationality equilibrium and the ZLB. Among other things, these papers show that deviations from RE that make agents less forward-looking than rational agents can resolve the so-called NK paradoxes of the ZLB, such as the prediction that forward guidance announcements can have arbitrarily large effects on the economy ("forward guidance puzzle"). Our contribution is to show that these deviations from RE also resolve the problems of incoherence and incompleteness that plague the standard NK model (Ascari and Mavroeidis, 2022).

Finally, Mertens and Ravn (2014), Nakata and Schmidt (2019, 2020), and Bilbiie (forth.), among others, study conditions for the existence of both fundamentals-driven and confidence-driven liquidity trap equilibria, which are caused by fundamental shocks to the economy and non-fundamental (sunspot) shocks, respectively.[5] One takeaway from these papers is

[4] See also Evans and McGough (2018b) for a related discussion on interest rate pegs and adaptive learning.

[5] Additionally, Bianchi et al. (2021) study implications of fundamentals-driven liquidity traps in a nonlinear New Keynesian model.

that fundamentals-driven liquidity trap equilibrium is unlikely to exist if shocks are too persistent, but sunspot equilibria can feature very persistent liquidity traps. However, to our knowledge, confidence-driven liquidity trap equilibria have only been derived in coherent models (i.e. models that admit at least one MSV solution). An incoherent model can fail to admit confidence-driven liquidity trap equilibria, and tight restrictions on the variance and persistence of *fundamental* shocks are necessary for existence of both MSV and confidence-driven liquidity trap equilibria.

2 Model and expectations formation mechanisms

We employ a model that nests the simple New Keynesian model as well as alternative bounded rationality models explored by Gabaix (2020), Angeletos and Lian (2018), Woodford and Xie (2020):

$$\begin{aligned} x_t &= M\hat{E}_t x_{t+1} - \sigma(r_t - N\hat{E}_t\pi_{t+1}) + \epsilon_t, & (1)\\ \pi_t &= \lambda y_t + M_f\beta\hat{E}_t\pi_{t+1}, & (2)\\ r_t &= \max\{\psi\pi_t, -\mu\}, & (3)\end{aligned}$$

where $0 < M, N, M_f \leq 1$, $0 < \beta < 1$, $0 < \sigma, \lambda, \mu$, and $\psi > 1$ (i.e. the "Taylor principle" holds). The model is log-linearized around the zero inflation steady state. Note that $\hat{E}$ denotes (possibly non-rational) expectations and $\hat{E} = E$ denotes model-consistent expectations. Also note that the model nests the simple New Keynesian model of Woodford (2003) if $M = M_f = N = 1$.

We follow earlier work, including Eggertsson and Woodford (2003), Nakata and Schmidt (2019), Christiano et al. (2017), and Ascari and Mavroeidis (2022), and assume that the demand shock, ϵ_t, follows a 2-state Markov process with transition matrix:

$$K = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix},$$

with $0 < p = Pr(\epsilon_t = \epsilon_1|\epsilon_{t-1} = \epsilon_1) \leq 1$, $0 < q = Pr(\epsilon_t = \epsilon_2|\epsilon_{t-1} = \epsilon_2) \leq 1$. If we assume $q = 1$ and $\epsilon_2 = 0$, similar to Eggertsson and Woodford (2003) or Christiano et al. (2017), then we have a model in which a transitory shock, $\epsilon_t = \epsilon_1 \neq 0$, displaces the economy from steady state, but the economy eventually returns to the absorbing steady state of the model when $\epsilon_t = \epsilon_2 = 0$. In the standard RE version of the model there are two non-stochastic steady states: one with zero inflation, and one with zero nominal interest rates. However, equilibrium inflation and output in the temporary state ($\epsilon_t = \epsilon_1$) depends on whether agents have full-information RE or whether they are boundedly rational in some way.

We consider three models of expectations formation. First, agents have full-information RE in the special case of the model with no discounting in the Euler equation and Phillips curve (1)-(3) and model consistent expectations.

Definition 1 *Agents have* ***full-information rational expectations (RE)*** *if and only if* $\hat{E} = E$ *and* $M = M_f = N = 1$ *in the NK model given by Equations (1)-(3).*

An REE, defined in Section 3, is a solution of the model (1)-(3) obtained under these assumptions. In keeping with the literature, we treat full-information RE as the benchmark model of expectations formation, against which we compare ZLB dynamics under alternative expectations formation mechanisms. Particular attention is paid to the possibility that agents do not have full knowledge about the structure of the economy, and consequently expectations can be model-inconsistent (i.e., $\hat{E} \neq E$). The adaptive learning literature in particular studies agents with imperfect knowledge who learn to forecast the law of motion for aggregate variables using standard statistical tools like least squares. In this setting, imperfect knowledge can imply model-inconsistent expectations, but the focus of a large swath of this literature is whether agents can form self-confirming beliefs, either by learning an REE, or some non-rational, self-confirming equilibrium if their subjective forecasting models are mis-specified with respect to the rational forecasting models. Holding fixed the structure of the model, imperfect knowledge by itself can lead us to new insights about policy and macroeconomic dynamics.

Definition 2 *Agents have* ***imperfect knowledge*** *if* $\hat{E} \neq E$; $M = M_f = N = 1$ *in the NK model given by Equations (1)-(3)*

Of course, we can deviate from RE without breaking the assumption that agents have full knowledge about the structure of their economic environment. For instance, Gabaix (2020) derives a model in which households and firms are relatively myopic due to cognitive limitations. In this setting, myopia implies a change in the model structure in the form of discounting in the aggregate demand curve (1) (i.e., $M < 1$) and additional discounting in the Phillips curve (2) (i.e. $M_f < 1$). However, nothing in Gabaix's (2020) model prevents agents from having full knowledge about the world they inhabit, and therefore nothing prevents these boundedly rational agents from having model-consistent expectations. Hence, Gabaix's (2020) behavioral model shows how we can deviate from full-information RE without sacrificing the assumption that agents have perfect knowledge. Bounded rationality models by Angeletos and Lian (2018) and Woodford and Xie (2020) may also lead to reduced-form structural models with additional discounting in the structural equations. If M, M_f or N is less than one, we say that agents are boundedly rational.

Definition 3 *Agents are said to be* ***boundedly rational*** *if and only if* $\hat{E} = E$ *and* $min\{M, M_f, N\} < 1$.

3 Coherence: Existence of an Equilibrium

This Section investigates the problem of coherence, that is, of the existence of an equilibrium, under the three three models of expectations formation just described.

3.1 Rationality without Coherence

We start by assuming full-information RE to illustrate the problem of incoherence. For simplicity, we focus on MSV REE, but some of the insights from our paper can be extended to study non-fundamental "sunspot" equilibria which feature extraneous volatility. Since our model, (1)-(3), is a purely forward looking model with a 2-state discrete-valued exogenous shock, the MSV REE law of motion for $Y_t = (x_t, \pi_t)'$ will assume the form $Y_t = \mathbf{Y}_j$ where $Y_t = \mathbf{Y}_1$ if $\epsilon_t = \epsilon_1$ and $Y_t = \mathbf{Y}_2$ otherwise.

Definition 4 ***Rational expectations equilibrium (REE).*** *$\boldsymbol{Y} = (\boldsymbol{Y}_1', \boldsymbol{Y}_2')'$ is a rational expectations equilibrium if and only if $\boldsymbol{Y}_j$ solves (1)-(3) given $\hat{E}_t(Y_{t+1}|\epsilon_t = \epsilon_j) = Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t = \epsilon_j)\,\boldsymbol{Y}_1 + Pr(\epsilon_{t+1} = \epsilon_2|\epsilon_t = \epsilon_j)\,\boldsymbol{Y}_2$ and $\epsilon_t = \epsilon_j$ for $j = 1, 2$.*

There are up to four MSV REE of (1)-(3). First, there is a possible solution in which interest rates are always positive ("PP" solution). Then, there is a potential solution with binding ZLB if and only if $\epsilon_t = \epsilon_1$, which we refer to as the "ZP" solution. Analogously, there could be a "PZ" solution with binding ZLB if and only if $\epsilon_t = \epsilon_2$. Finally, it is possible that the ZLB is always binding ("ZZ" solution). We add a superscript i to $\mathbf{Y}$ to distinguish between the REE (i.e. $\mathbf{Y}^i$ where $i = PP, ZP, PZ, ZZ$). Following Ascari and Mavroeidis (2022), if at least one of the four possible REE exist then the model is coherent.

Proposition 1 *Consider (1)-(3) and suppose $M = M_f = N = 1$, $\epsilon_2 \geq 0$. An REE exists if and only if $\epsilon_1 > \bar{\epsilon}_{REE}$, where $\bar{\epsilon}_{REE}$ is a constant that depends on the model's parameters, defined in Equation (30) in the Appendix.*

Proposition 1 generalizes Proposition 5 of Ascari and Mavroeidis (2022) to the case with $q < 1$. It establishes that under the conventional assumption that the Taylor rule (3) satisfies the Taylor Principle and recurrent demand shocks, we need to restrict the magnitude of the shocks, ϵ_t, to get an REE. For a solution to exist, ϵ_1 cannot be too negative (i.e. the shock cannot be too "big", in absolute value). The lower bound on ϵ_1, denoted as $\bar{\epsilon}_{REE}$, is increasing in p for standard parameters, which means that a model with more persistent shocks requires tighter restrictions on the magnitude of the shocks for an equilibrium to exist. This explains why fundamentals-driven liquidity trap cannot be persistent in an REE. A "big" shock is needed to take the economy into a liquidity trap, but then, for an REE to exist, it cannot be persistent. Thus, the model is not generically coherent; solutions only exist for special calibrations of the shock process and solutions do not exist if the shocks are too persistent (i.e. p is very high) or if the shock is big (ϵ_1 is very low).

Intuition from a special case. While Proposition 1 deals with the case with $q < 1$, the assumption that the high demand state is absorbing ($q = 1$) and equal to zero ($\epsilon_2 = 0$) is helpful for intuition.[6] Under this assumption, the economy under full-information RE either returns to the steady state with zero inflation (i.e. $\pi_t = x_y = i_t = 0$) or the steady state with zero interest rates (i.e. $i_t = -\mu$, $\pi_t = -\mu < 0$ $x_t = -\mu(1-\beta)/\kappa < 0$). The "temporary state" value of output when $\epsilon_t = \epsilon_1$ (assuming for brevity that we go back to the zero inflation steady state) is given by:

$$x_t = \nu(p)E_t x_{t+1} - \sigma \max\{\frac{\psi\lambda}{1-\beta p}x_t, -\mu\} + \epsilon_1, \tag{4}$$

$$\nu(p) = \left(1 + \frac{\lambda\sigma}{1-\beta p}\right) > 1, \tag{5}$$

which we obtain by substituting the Phillips curve and Taylor rule into (1). From (4), it is apparent that for any p, sufficiently low values of ϵ_1 preclude unconstrained interest rates. Thus, for a sufficiently large demand shock, output will be given by:

$$x_t = \frac{1}{1-p\nu(p)}(\sigma\mu + \epsilon_1) \tag{6}$$

if a solution of the model exists at all. However, if the negative demand shock is sufficiently persistent, so that $p\nu(p) > 1$, then x_t and therefore temporary inflation, $\pi_t = \frac{\kappa}{1-\beta p}x_t$ are decreasing in ϵ_1. This implies that sufficiently large ϵ_1 will increase x_t and π_t, precluding existence of a solution in which the ZLB binds. Therefore, for a solution to exist we need to do one of two things. First, we can restrict p to be small enough to ensure $p\nu(p) < 1$, which in turn implies a solution for any ϵ_1. Or, alternatively, we need to restrict ϵ_1 to be small (i.e. close to zero) to rule out a situation where large demand shocks preclude both unconstrained and constrained equilibrium interest rates – i.e., incoherence. Both options require restrictions on the support of the demand shock.

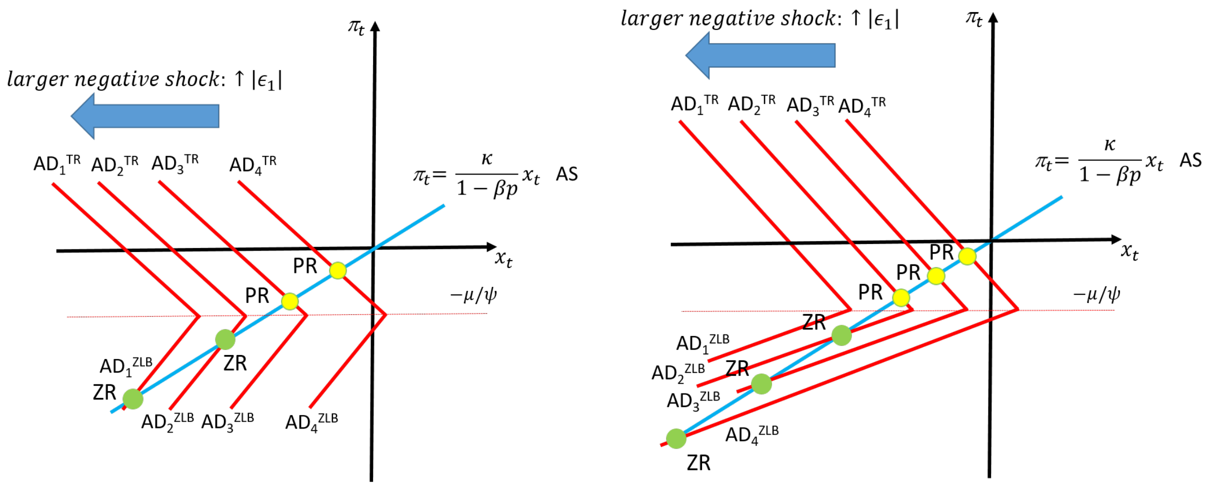
Figure 1a graphically illustrates the determination of demand for the case $p\nu(p) < 1$. It can be seen that a solution exists for any ϵ_1. Figure 1b graphically illustrates equilibrium determination when $p\nu(p) > 1$. It is apparent that two solutions exist if ϵ_1 is small, but no solution if ϵ_1 is large in magnitude.[7]

How should we interpret this restriction on p and ϵ_1? Following Bilbiie (forth.), there are two effects of the demand shock, ϵ_1, when interest rates are pegged at the zero level. First,

[6]The assumption $q = 1$ is standard in the literature (e.g., Eggertsson and Woodford, 2003; Christiano et al., 2017; Bilbiie, forth.). To explain the intuition, we borrow heavily from Ascari and Mavroeidis (2022) and Bilbiie (forth.).

[7]In fact two or four solutions exist in the two cases, respectively, depending on whether one assumes the economy returns to the zero inflation steady – as in Figures 1a and 1b – or one assumes the economy goes to the permanent liquidity trap steady state – not depicted in Figures 1a and 1b. Moreover, the Figures visualize that the condition $p\nu(p) \lesseqgtr 1$ relates to the relative slope of the AS and the AD curve under ZLB. See Ascari and Mavroeidis (2022).

Figure 1: Incoherence and Income vs. Substitution



(a) $p\nu(p) < 1$

(b) $p\nu(p) > 1$

a larger demand shock (i.e. more negative value of ϵ_1) raises real interest rates given a fixed nominal rate, and this induces households to save more. This intertemporal substitution effect should put downward pressure on inflation and output. At the same time, $\nu(p) > 1$ implies strong income effects at the ZLB; current income, x_t, responds by *more* than an increase in expected future output, $E_t x_{t+1}$. For high values of p, an exogenous increase in real interest rates (via lower ϵ_1) raises demand and inflation through this second income effect. In the case where $p\nu(p) > 1$ the income effect dominates the substitution effect, and the negative demand shock has the counter-intuitive effect of raising inflation at the ZLB, while lowering inflation away from the ZLB (see the green and yellow dots respectively in Figure 1b) . In this scenario, we need to make sure that ϵ_1 is not *too* negative. On the other hand, if $p\nu(p) < 1$ then intertemporal substitution effects dominate and more negative ϵ_1 leads to more negative inflation and output, which in turn ensures that a solution with binding ZLB always exists.

In sum, we can discuss the problem of incoherence in our model in terms of income and substitution effects. RE implies that agents are very forward-looking, which in turn can imply a scenario where income effects dominate substitution effects. Tight restrictions on persistence parameter, p, are necessary to avoid this scenario, while restrictions on ϵ_1 are essential to ensure equilibrium when income effects are strong. Much of the rest of this paper investigates whether deviations from RE ensure that these substitution effects dominate income effects when $p\nu(p) > 1$, thus opening up the possibility that non-rational solutions exist when rational solutions may not.

3.2 Coherence without Rationality

What if no REE exists? Here we investigate the possibility of the existence of non-rational equilibria. First, we look at the case of imperfect knowledge as in Definition 2. We show that the NK model with a ZLB may still admit RPE if we are willing to assume that agents omit the demand shock from their subjective forecasting model, and attempt to forecast period-ahead inflation and output using their estimates of the long-run average of both variables. Second, bounded rationality does not need to imply imperfect knowledge, and so it is important to consider what happens when agents are boundedly rational as in Definition 3. It turns out that bounded rationality in the form of discounting ($M, M_f, N < 1$) can imply an even more complete resolution of the problem of incoherence than RPE.

3.2.1 Restricted Perceptions

The model (1)-(3) has a single state variable, ϵ_t, which follows a regime-switching process. Consequently, the REE law of motion for output and inflation is a regime-switching intercept – see Definition 4. Rational agents are assumed to know the functional form of the REE solution. However, agents without RE could fail to grasp the structure of the REE – particularly in the case of incoherence when no such equilibrium exists – and consequently, they might try to forecast inflation and output using an under-parameterized forecasting model which omits the state variable, ϵ_t. Agents with these restricted perceptions instead try to forecast the *unconditional* mean of output and inflation:

$$E(Y) = \bar{q}\hat{\mathbf{Y}}_2 + (1 - \bar{q})\hat{\mathbf{Y}}_1,$$

where $Y = (x, y)'$, $\hat{\mathbf{Y}}_j$ is Y_t when $\epsilon_t = \epsilon_j$ and $\bar{q} = Pr(\epsilon_t = \epsilon_2) = (1 - p)/(2 - p - q)$. If the agents form conditional forecasts using the unconditional mean of inflation and output (i.e. if $\hat{E}_t Y_{t+j} = E(Y)$) then agents' beliefs about the long-run averages of inflation and output are true and self-confirming only if $\hat{\mathbf{Y}}_j$ solves (1)-(3) given $E_t Y_{t+j} = E(Y) = \bar{q}\hat{\mathbf{Y}}_2 + (1 - \bar{q})\hat{\mathbf{Y}}_1$ and $\epsilon_t = \epsilon_j$ for $j = 1, 2$.

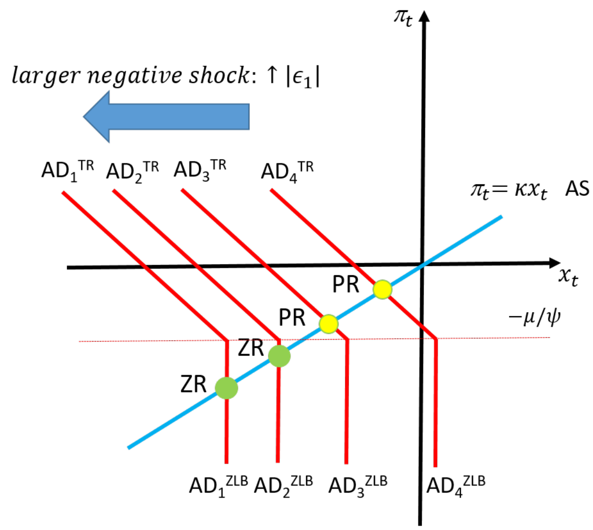
Definition 5 ***Restricted perceptions equilibrium (RPE).*** *$\hat{\mathbf{Y}} = (\hat{\mathbf{Y}}_1', \hat{\mathbf{Y}}_2')'$ is a restricted perceptions equilibrium if and only if (i) $\hat{\mathbf{Y}}_j$ solves (1)-(3) given $E_t Y_{t+1} = \bar{\mathbf{Y}} := \bar{q}\hat{\mathbf{Y}}_2 + (1 - \bar{q})\hat{\mathbf{Y}}_1$ and $\epsilon_t = \epsilon_j$ for $j = 1, 2$; and (ii) $E(Y_t) = \bar{\mathbf{Y}}$.*[8]

There are four possible RPE of (1)-(3) indexed by $i = PP, ZP, PZ, ZZ$, which are analogous to the REE discussed earlier. Notice that the *actual* law of motion for inflation and output in the RPE is still a regime-switching process. A sufficiently attentive learning agent might be expected to notice that their forecasting model is misspecified in an RPE and

[8] See Evans and Honkapohja (2001, sec. 3.6 and 13.1) and Branch (2006) for a thorough discussion of the RPE concept.

consequently, we might question whether this equilibrium concept is "reasonable." A later section discusses issues related to the plausibility of the RPE concept in the context of learning, but in any case, the simple RPE concept put forth in this section is the natural RPE concept for this model.[9] This RPE concept also makes the analysis tractable, leading to the following useful result.

Figure 2: Restricted Perceptions Equilibrium



Proposition 2 *Consider (1)-(3) and suppose $M = M_f = N = 1$, $\epsilon_2 \geq 0$. Then:*

i. An RPE exists if and only if $\epsilon_1 > \bar{\epsilon}_{RPE}$, where $\bar{\epsilon}_{RPE}$ depends on the model's parameters, see Equation (31) in the Appendix, and satisfies $\bar{\epsilon}_{RPE} = -\infty$ if $q = 1$.

ii. $\bar{\epsilon}_{REE} \geq \bar{\epsilon}_{RPE}$ if and only $p + q > 1$.

Proposition 2 is one of the main results of this paper. It tells us that models with persistent shocks (i.e. $p + q > 1$) admit non-rational equilibria but *not* rational equilibria if $\epsilon_1 \in (\bar{\epsilon}_{RPE}, \bar{\epsilon}_{REE})$. Thus we can gain traction in an otherwise incoherent model of the ZLB by assuming restricted perceptions.

As in the case of REE, it is useful to study RPE when $q = 1$ and $\epsilon_2 = 0$ to develop intuition, see Figure 2. In this case, we have $\bar{q} = 1$ and so the RPE forecast is simply equal to one of the two non-stochastic steady states of the model. Substituting the forecast consistent with the economy reverting to the zero inflation steady state into the model – so $\hat{E}_t x_{t+1} = \hat{E}_t \pi_{t+1} = 0$ in (1)-(3) – and solving for equilibrium output in the temporary state

[9] In an RPE, agents have "restricted perceptions" in the sense that they omit key fundamental state variables from their forecasting models. In our simple model, ϵ_t is the only state variable. Consequently, the natural RPE for this model involves the PLM that omits ϵ_t.

with $\epsilon_t = \epsilon_1$ gives: $x_t = \sigma\mu + \epsilon_1$, assuming the ZLB binds. Thus, effectively the perceived p is equal zero and the slope of the aggregate demand curve becomes vertical in the temporary state under a ZLB. It follows that an RPE exists for *any* p and ϵ_1. No support restrictions for the shock distribution are needed. Restricted perceptions ensures that income effects of raising real rates do not dominate substitution effects, and thus equilibrium is ensured for any assumptions about p and ϵ_1, in accordance with Proposition 2.

3.2.2 Bounded Rationality

Assuming bounded rationality in the form of discounting ($M, M_f, N < 1$) yields the following proposition that illustrates how deviations from RE ameliorate incoherence concerns, as in Proposition 2.

Proposition 3 *Consider (1)-(3) and suppose min*$\{M, M_f, N\} < 1$ *and* $\epsilon_2 \geq 0$. *Then:*

i. A BRE exists if and only if $\epsilon_1 > \bar{\epsilon}_{BR}$, *for some constant* $\bar{\epsilon}_{BR}$ *that depends on the model's parameters, see Equation (34) in the Appendix.*

ii. If $(M-1)(1-M_f\beta) + \lambda\sigma N < 0$ *then* $\bar{\epsilon}_{BR} = -\infty$.

However, bounded rationality provides a larger resolution of the problem with respect to imperfect knowledge, as coherence can be ensured for any assumption about p, q and ϵ_t if M, M_f, N are sufficiently small. Thus, if we are willing to explore deviations from RE that alter the decision problem of the agents (e.g. the Euler equation), and not just the expectations formation of agents (e.g. the RPE approach), then we can robustly resolve coherence problems.

Again, we can understand the coherence result in terms of the income and substitution effect of shocks that raises real interest rates at the ZLB. Assume $q = 1$ and $\epsilon_2 = 0$. The BRE value of output in the temporary state binding ZLB is given by:

$$\begin{aligned} x_t &= \nu^{BR}(p)E_t x_{t+1} - \sigma \max\{\frac{\psi\lambda}{1-M_f\beta p}x_t, -\mu\} + \epsilon_1, \\ \nu^{BR}(p) &= \left(M + N\frac{\lambda\sigma}{1-\beta M_f p}\right). \end{aligned} \tag{7}$$

In this bounded rationality model therefore, output at the ZLB is given by

$$x_t = \frac{1}{1-p\nu^{BR}(p)}(\sigma\mu + \epsilon_1). \tag{8}$$

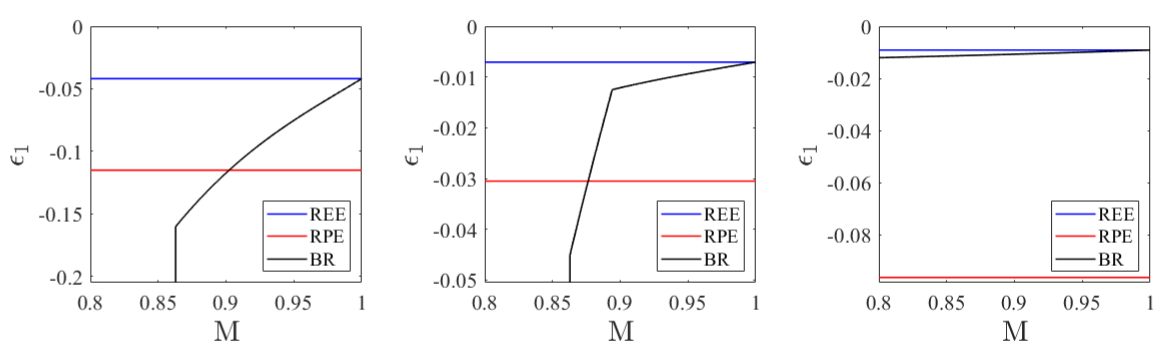
Clearly, substitution effects dominate income effects if and only if $p\nu^{BR}(p) < 1$, similar to the RE case. However, unlike the RE case, we have $\nu^{BR}(p) < 1$ if and only if

$$N\lambda\sigma + (M-1)(1-\beta M_f) < 0,$$

which is the condition in Proposition 3. Therefore, myopia can ensure that substitution effects dominate income effects for any p (i.e. existence of a MSV solution for any p and ϵ_1).

Though bounded rationality can provide a larger resolution of the problem if $(M-1)(1-M_f\beta)+\lambda\sigma N<0$, the RPE can provide a larger resolution of the problem with respect to bounded rationality if $(M-1)(1-M_f\beta)+\lambda\sigma N>0$. Figure 3 depicts different combinations of values of the negative shock, ϵ_1, and of the bounded rationality discount factor, M, that yield coherence in the REE, RPE and BRE cases. The blue line and red line depict $\bar{\epsilon}_{REE}$ and $\bar{\epsilon}_{RPE}$, respectively, and the black line depicts $\bar{\epsilon}_{BRE}$ for different values of ϵ_1 and $M=M_f$. Panels (a), (b) and (c) shows that the difference between $\bar{\epsilon}_{REE}$, $\bar{\epsilon}_{RPE}$, and $\bar{\epsilon}_{BR}$ can be substantial. Panel (a) shows that larger values of M can rule out existence of BRE in cases where an RPE exists. Panel (b) shows that the same result holds even if the expected low demand state duration is calibrated to match the duration of the 2008-2015 U.S. ZLB episode (i.e. $p=0.965$ implies an expected low state duration of 28 quarters). However, if $M<0.86$ in the calibrated model then $(M-1)(1-M_f\beta)+\lambda\sigma N<0$ and $\bar{\epsilon}_{BRE}=-\infty$. Panel (c) reveals that in addition to small M, a high degree of price stickiness (small λ) is necessary for the BRE approach to provide a fuller solution of the incoherence problem than the RPE concept. For high values of λ even heavy cognitive discounting in the Euler equation and Phillips curve will not resolve the problem of incoherence.[10] The so-called "curse of flexibility" is therefore a much more pronounced problem for both REE and BRE than for RPE.

Figure 3: Region of Coherence of the REE, RPE, and of the BRE



(a) (b) $p=0.965$ (c) $\lambda=0.2$

Note: The area above the blue (red) curve depicts values of ϵ_1 for which at least one REE (RPE) exists. The area above the black curve depicts values of ϵ_1 and $M=M_f$ for which at least one BRE exists. Other parameter values: $\beta=0.99$, $\sigma=1$, $\lambda=0.02$, $q=0.98$, $p=0.85$, $N=1$, $\epsilon_2=0.01$.

Not only does $(M-1)(1-M_f\beta)+\lambda\sigma N<0$ ensure coherence in the case of bounded

[10]For any M, Mf, N, there is always a large enough value of the product $\lambda\sigma$ to ensure that $(M-1)(1-M_f\beta)+\lambda\sigma N>0$. Thus, price rigidity and the intertemporal elasticity of substitution play a key role in the existence of BRE.

rationality, it also ensures existence of a unique BRE ("completeness").

Proposition 4 *Consider the model given by (1)-(3) and assume $\psi > 1$. A unique bounded rationality equilibrium (BRE) exists for any $p, q, \epsilon_1, \epsilon_2$ if and only if $\lambda\sigma N + (M-1)(1-M_f\beta) < 0$. Further, there exist $\epsilon^{PP,BR}$ and $\epsilon^{ZP,BR}$ such that $\epsilon^{PP,BR} > \epsilon^{ZP,BR}$ and*

i. The PP solution is the unique BRE if and only $\epsilon_1 > \epsilon^{PP,BR}$.

ii. The ZP solution is the unique BRE if and only if $\epsilon^{PP,BR} > \epsilon_1 > \epsilon^{ZP,BR}$.

iii. The ZZ solution is the unique BRE if and only if $\epsilon_1 < \epsilon^{ZP,BR}$.

The condition $(M-1)(1-M_f\beta) + \lambda\sigma N < 0$ completely mitigates concerns about incoherence and incompleteness, but the condition requires a rather high degree of discounting in the Euler and Phillips curve equations. As it turns out, the condition is satisfied by Gabaix's preferred calibration: $M = 0.85$, $M_f = 0.8$, $N = 1$, $\beta = 0.99$, $\lambda = 0.11$, $\sigma = 0.2$. For that calibration, we have:

$$\lambda\sigma N + (M-1)(1-M_f\beta) = -0.0092 < 0.$$

On the other hand, it is not satisfied for the calibration in McKay et al. (2016a): $M = 0.97$, $M_f = N = 1$, $\beta = 0.99$, $\lambda = 0.02$, $\sigma = 0.375$. That calibration yields:

$$\lambda\sigma N + (M-1)(1-M_f\beta) = 0.0072 > 0.$$

Thus bounded rationality offers a full solution of the problems of incoherence and incompleteness for some, but not all, calibrations featured in the literature.

4 Learning to solve the incompleteness problem

We just saw that a BRE can ensure coherence and completeness with sufficient discounting, without any restrictions on the support of the shock. What about completeness in the REE and RPE cases? The coherence condition guarantees existence, but generally that implies a multiplicity of admissible MSV solutions in the case of RE (e.g., Ascari and Mavroeidis, 2022). Incompleteness is by itself a problem which can only be solved using some criterion for selecting an equilibrium. Here we investigate whether learning can provide any guidance, that is, whether the "E-stability" criterion can select an equilibrium of the model as the outcome of an adaptive learning process.

4.1 Learning the REE

In order to define when an REE is E-stable, we first need to be precise about what it means for agents to be learning. Under adaptive learning, agents are assumed to have imperfect knowledge, as defined in Section 2. Since they lack RE, they also lack sufficient knowledge to compute the REE analytically. However, adaptive learning agents still need to forecast inflation and output in order to make consumption, labor, savings and pricing decisions consistent with (1)-(2).[11] Consequently, adaptive learning agents are assumed to have a subjective forecasting model, or "perceived law of motion" (PLM) for output and inflation. If the learning agents choose a PLM that is also consistent with how expectations are formed in an REE, then it is possible for learning agents to "learn" an REE if their beliefs about the PLM converge to RE, as beliefs are updated recursively using some statistical scheme for estimating the coefficients of the PLM and observable macro data.

Recall from Section 3.1 that our model admits four possible REE in which output and inflation follow a 2-state process, which are indexed by superscript i to $\mathbf{Y}$, i.e. $\mathbf{Y}^i$ where $i = PP, ZP, PZ, ZZ$. Agents could conceivably learn one of these REE if their PLM for output and inflation is a 2-state process which is estimated recursively using least squares. Consider the following model of learning, in which agents' PLM is a 2-state process for inflation and output, like the REE, and beliefs about the state-contingent means are updated recursively using least squares:

$$Y^e_{j,t} = Y^e_{j,t-1} + t^{-1}\mathcal{I}_{j,t-1}\nu^{-1}_{j,t-1}\left(Y_{t-1} - Y^e_{j,t-1}\right), \tag{9}$$

$$\nu_{j,t} = \nu_{j,t-1} + t^{-1}\left(\mathcal{I}_{j,t-1} - \nu_{j,t-1}\right), \tag{10}$$

$$\hat{E}_t Y_{t+1} = Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t)Y^e_{1,t} + (1 - Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t))Y^e_{2,t}, \tag{11}$$

where $j = 1, 2$, $k\nu_{j,k}$ is the number of periods that $\epsilon_t = \epsilon_j$ up until time k, and $\mathcal{I}_{j,t} = 1$ if $\epsilon_t = \epsilon_j$ and $\mathcal{I}_{j,t} = 0$ otherwise (i.e. $\mathcal{I}_{j,t} = 1$ is the indicator function for state j). $Y^e_{j,t}$ is the agents' most recent estimate of the state-contingent average of Y_t when $\epsilon_t = \epsilon_j$. According to equation (9), agents revise their beliefs about the state-contingent average of Y in state j (i.e. $Y^e_{j,t}$) in the direction of their time-$t-1$ forecast error only if $\epsilon_{t-1} = \epsilon_j$ (otherwise, $Y^e_{j,t} = Y^e_{j,t-1}$). Equation (11) then gives agents' time-t forecast of period-ahead inflation and forecast. It is assumed that agents observe ϵ_t when forecasting at time-t and also that $Pr(\epsilon_{t+1}|\epsilon_t)$ coincides

[11] Throughout this paper we restrict our attention to the "Euler equation" approach in which adaptive learning agents are assumed to treat the RE decision rules, (1)-(2), as the decision rules given subjective forecasts. However, under this assumption, agents are not making optimal decision *given* non-rational expectations, as demonstrated by Preston (2005). An alternative approach which accounts for the true optimal consumption and pricing decisions under non-rational expectations is the "infinite horizon learning" approach advanced by Preston (2005) and others. Preliminary results included in the Appendix show that identical RPE existence results can obtain under Euler equation and infinite horizon learning. We leave the full topic of RPE existence and E-stability under infinite horizon learning for future research.

with the actual transition probabilities – e.g. agents know $Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t = \epsilon_1) = p$ and $Pr(\epsilon_{t+1} = \epsilon_2|\epsilon_t = \epsilon_2) = q$. After agents form time-t expectations, we obtain the time-t market-clearing equilibrium, Y_t by substituting equation (11) into the model (1)-(3). The process repeats itself at time $t+1$ and so on.[12]

We are interested in knowing if $(Y^e_{1,t}, Y^e_{2,t}) \to (\mathbf{Y}^i_1, \mathbf{Y}^i_2)$ for some REE i as time goes on ($t \to \infty$) and agents' expectations evolve according to (9)-(11). We say that REE i is "stable under learning" if $(Y^e_{1,t}, Y^e_{2,t}) \to (\mathbf{Y}^i_1, \mathbf{Y}^i_2)$ almost surely. When might this convergence of subjective beliefs to RE occur? To make this question tractable, assume that $Y^e_t = (Y^{e'}_{1,t}, Y^{e'}_{2,t})'$ is sufficiently near REE i, such that the ZLB binds under adaptive learning if and only if the ZLB would bind in REE i. This implies the following actual law of motion for Y:

$$Y_t \;=\; A^i_t \left(Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t)Y^e_{1,t} + (1 - Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t))Y^e_{2,t}\right) + B^i_t, \tag{12}$$

where $A^{PP}_t = A_P$ and $B^{PP}_t = B_{P,t}$ for all t; $A^{ZZ}_t = A_Z$ and $B^{ZZ}_t = B_{Z,t}$ for all t; $A^{ZP}_t = A_P$ and $B^{ZP}_t = B_{P,t}$ if $\epsilon_t = \epsilon_2$ and $A^{ZP}_t = A_Z$ and $B^{ZP}_t = B_{Z,t}$ otherwise; $A^{PZ}_t = A_P$ and $B^{PZ}_t = B_{P,t}$ if $\epsilon_t = \epsilon_1$ and $A^{PZ}_t = A_Z$ and $B^{PZ}_t = B_{Z,t}$ otherwise, and

$$\begin{aligned} A_P &= \begin{pmatrix} \frac{1}{\lambda\sigma\psi+1} & \frac{\sigma-\beta\sigma\psi}{\lambda\sigma\psi+1} \\ \frac{\lambda}{\lambda\sigma\psi+1} & \frac{\beta+\lambda\sigma}{\lambda\sigma\psi+1} \end{pmatrix} \qquad & A_Z = \begin{pmatrix} 1 & \sigma \\ \lambda & \beta + \lambda\sigma \end{pmatrix} \\ B_{P,t} &= \begin{pmatrix} \frac{\epsilon_t}{1+\lambda\psi\sigma} \\ \frac{\lambda\epsilon_t}{1+\lambda\psi\sigma} \end{pmatrix} & B_{Z,t} = \begin{pmatrix} \epsilon_t + \sigma\mu \\ \lambda\epsilon_t + \lambda\sigma\mu \end{pmatrix}. \end{aligned}$$

Given beliefs that are local to RE beliefs, we assess the learnability of equilibrium using the E-stability principle. An REE i is said to be E-stable if it is a locally fixed point of the ordinary differential equation (ODE):

$$\frac{\partial \tilde{Y}^e}{\partial \tau} = H^i(\tilde{Y}^e), \qquad \text{where} \qquad H^i(\tilde{Y}^e) = \begin{pmatrix} Y^i_1(Y^e_1, Y^e_2) \\ Y^i_2(Y^e_1, Y^e_2) \end{pmatrix} - \begin{pmatrix} Y^e_1 \\ Y^e_2 \end{pmatrix} \tag{13}$$

and $Y^i_j(Y^e_1, Y^e_2)$ is the value of Y when $\epsilon_t = \epsilon_j$ as a function of expectations, $\tilde{Y}^e = (Y^{e'}_1, Y^{e'}_2)'$. The relevant Jacobian for assessing the E-stability of REE i is: $DT_{Y^i} = \frac{\partial H^i(\tilde{Y}^e)}{\partial \tilde{Y}^e}|_{\tilde{Y}^e = \mathbf{Y}^i}$. An REE i is E-stable if the eigenvalues of DT_{Y^i} have negative real parts, see Evans and Honkapohja (2001).

There is an intuition for the link between the E-stability condition and stability of beliefs. The ODE (13) is an approximation of the dynamics of Y^e_t near the REE for large t, and it tells us that agents' expectations are revised in the direction of the forecast error, $\bar{Y}^i(Y^e) - Y^e$. If

[12] Closely related learning algorithms are used by Woodford (1990), Evans and Honkapohja (1994) and (Evans and Honkapohja, 2001, p.305-308) to study the E-stability of sunspot equilibria involving discrete-valued shocks, and by Evans and Honkapohja (1998) to study learnability of fundamental equilibria with exogenous shocks following a finite state Markov chain. We arrive at identical E-stability results if we alternatively assume least squares estimation of a PLM of the form: $Y^e_t = \hat{a} + \hat{b}\mathcal{I}_t$ where $\mathcal{I}_t = 1$ if $\epsilon_t = \epsilon_2$ and 0 otherwise.

the roots of $DT_{\bar{Y}^i}$ have negative real parts, then agents' expectation about the unconditional mean of inflation and output are also revised in the direction of their REE values.

We note the E-stability conditions applied to the REE of the occasionally binding constraint model are identical to the E-stability conditions applied to a model that features exogenous Markov-switching in the monetary policy stance driven entirely by ϵ_t (e.g., see Branch et al., 2013; McClung, 2020).[13] For example, the E-stability condition associated to the ZP equilibrium of (1)-(3) is the same condition associated to the MSV solution of a model that assumes $r_t = \psi \pi_t$ if $\epsilon_t = \epsilon_2$ and $r_t = -\mu$ if $\epsilon_t = \epsilon_1$ regardless of whether the ZLB binds.

Applying the E-stability to the model at hand leads us to the conclusion that only one REE has the property of being E-stable.

Proposition 5 *Consider (1)-(3) and suppose $M = M_f = N = 1$, $\epsilon_2 \geq 0$. Then:*

i. At most one E-stable REE exists.

ii. The E-stable REE is either the PP REE or the ZP REE.

Proposition 5 somewhat extends insights from Christiano et al. (2017) to models with recurring low demand states (i.e. $q < 1$). Thus Proposition 5 can be applied to study an economy such as the U.S. economy, which has visited the ZLB twice since 2007, following two distinct negative shocks to the economy. The result in Proposition 5 makes it clear that while multiple solutions exist, only one of them can be understood as the outcome of an adaptive learning process. Hence, incompleteness is resolved by E-stability.

Pinning down the forecasting model Proposition 5 assumes that agents believe that output and inflation follow a 2-state process, consistent with REE. However, the REE law of motion can be represented in a variety of different ways. For instance, consider the following perceived laws of motion for inflation and output:

$$Y_t^e = a_{\epsilon_{t-k}}, \tag{14}$$
$$Y_t^e = a_{\epsilon_{t-k}} + b\epsilon_{t-k}, \tag{15}$$
$$Y_t^e = a + b_{\epsilon_{t-k}}\epsilon_{t-k}, \tag{16}$$
$$Y_t^e = a_{\epsilon_{t-k}} + b_{\epsilon_{t-k}}\epsilon_{t-k}, \tag{17}$$
$$Y_t^e = a + b\epsilon_{t-k}, \tag{18}$$
$$z_t^e = a_z + b_z z_{t-1} \tag{19}$$

[13] Mertens and Ravn (2014) also derive E-stability conditions for an equilibrium of a simple New Keynesian model with ZLB constraint, assuming a 2-state discrete sunspot shock with an absorbing regime.

where $z \in \{\pi, x\}$, $k = 0, 1$ and $a_{\epsilon_{t-k}}$, $b_{\epsilon_{t-k}}$ may assume different values depending on ϵ_{t-k}. Again, Y_t^e denotes the subjective forecast of Y_t implied by the forecasting model.

If learning agents instead had one of the PLMs (14)-(19) and estimated the parameters of those models recursively, e.g. using least squares, would they eventually have self-confirming views about inflation and output? In other words, would the data confirm their belief that Y_t follows one of the processes (14)-(19)? If agents observe ϵ_t and Y_t when forecasting at time t, then beliefs formed under PLMs of the form (14)-(19) can only become self-confirming if an REE exists. Hence, we refer to (14)-(19) as "REE-consistent beliefs."

Proposition 6 *Suppose agents condition time-t forecasts on current (time-t) variables. Then REE-consistent beliefs (14)-(19) can only be self-confirming if an REE exists.*

Proposition 6 makes it apparent that agents including the demand shock, ϵ_t, in their (piecewise) linear forecasting model (or Y_t in the case of (19)) cannot develop self-confirming views about the economy if an REE does not exist (incoherence). This result has implications for how we should think about learning and equilibrium in the case of incoherence.

4.2 Learning the RPE

What about the learnability of RPE? Proposition 5 tells us that agents with imperfect knowledge using any of the subjective forecasting models discussed in the previous section will not learn any self-confirming equilibrium. However, Proposition 2 shows that an RPE can exist even if an REE does not. It turns out multiple RPE may exist when the restrictions in Proposition 2 hold. Can one or more of these RPE emerge as the outcome an econometric learning process, similar to what we considered in the case of REE? The answer is yes. Here we show that the model may still admit one unique learnable, self-confirming RPE.

First, we must assume agents have a subjective PLM for output and inflation that is consistent with how expectations are formed in an RPE:

$$\hat{E}_t Y_{t+j} = Y_t^e = Y_{t-1}^e + t^{-1}\left(Y_{t-k} - Y_{t-1}^e\right), \tag{20}$$

where Y_t^e is the agents' most recent least squares estimate of the unconditional mean of $Y = (x, \pi)'$ using all data available from $t = 0, \ldots, t-k$ where $k = 0$ if agents have current information and $k = 1$ if agents have lagged information and only observe endogenous variables after markets clear. If we substitute (20) into the model and assume Y_t^e is sufficiently near RPE i then we have the following actual law of motion for Y:

$$Y_t \;\; = \;\; A_t^i Y_t^e + B_t^i, \tag{21}$$

where $A_t^{PP} = A_P$ and $B_t^{PP} = B_{P,t}$ for all t; $A_t^{ZZ} = A_Z$ and $B_t^{ZZ} = B_{Z,t}$ for all t; $A_t^{ZP} = A_P$ and $B_t^{ZP} = B_{P,t}$ if $\epsilon_t = \epsilon_2$ and $A_t^{ZP} = A_Z$ and $B_t^{ZP} = B_{Z,t}$ otherwise; $A_t^{PZ} = A_P$ and $B_t^{PZ} = B_{P,t}$ if $\epsilon_t = \epsilon_1$ and $A_t^{PZ} = A_Z$ and $B_t^{PZ} = B_{Z,t}$ otherwise.

We say that RPE i is stable under learning if $Y_t^e \to \bar{\mathbf{Y}}^i$ almost surely, where $\bar{\mathbf{Y}}^i$ denotes the unconditional mean of Y_t^i. Analogous to the discussion of E-stability of REE above, we say that RPE i is said to be E-stable if it is a locally stable fixed point of the ODE, $\partial Y^e/\partial\tau = h^i(Y^e)$, where $h^i(Y^e) = \bar{Y}^i(Y^e) - Y^e$, where $\bar{Y}^i(Y^e)$ is the unconditional mean of Y as a function of expectations, Y^e. Formally, E-stability obtains if the eigenvalues of the Jacobian, $DT_{\bar{Y}^i} = \frac{\partial h^i(Y_e)}{\partial Y^e}|_{Y^e=\bar{\mathbf{Y}}^i}$ have negative real parts. An E-stable RPE is stable under learning if agents estimate Y_t^e using least squares, as in (20), or related estimation routines.

Proposition 7 *Consider (1)-(3) and suppose $M = M_f = N = 1$, $\epsilon_2 \geq 0$. If an RPE exists, then:*

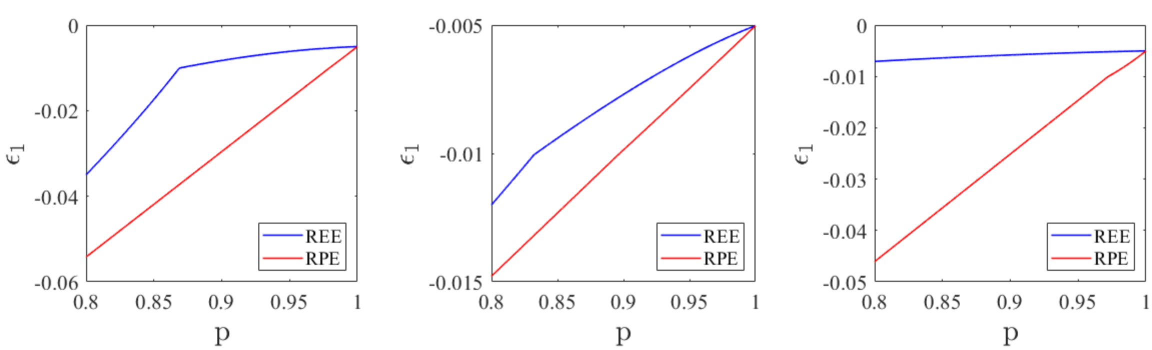
i. There is a unique E-stable RPE.

ii. The E-stable RPE is either the PP RPE or the ZP RPE.

4.3 Is the RPE reasonable?

In an RPE, agents have badly misspecified beliefs. Agents forecast the mean of inflation and output as if they believe those variables are constant or mean-plus-noise, despite the fact that these variables would obviously follow a persistent 2-state Markov chain in an RPE. Why would we consider RPE reasonable? Should agents be expected to detect their mis-specification over time simply by looking at time series data? Few comments are in order.

Figure 4: Region of Coherence of the REE and of the RPE



(a) (b) $q = 0.9$ (c) $\lambda = 0.2$

Note: The area above the blue (red) curve depicts values of ϵ_1 and p for which at least one REE (RPE) exists. Other parameter values: $\beta = 0.99$, $\sigma = 1$, $\lambda = 0.02$, $q = 0.98$, $\epsilon_2 = 0$.

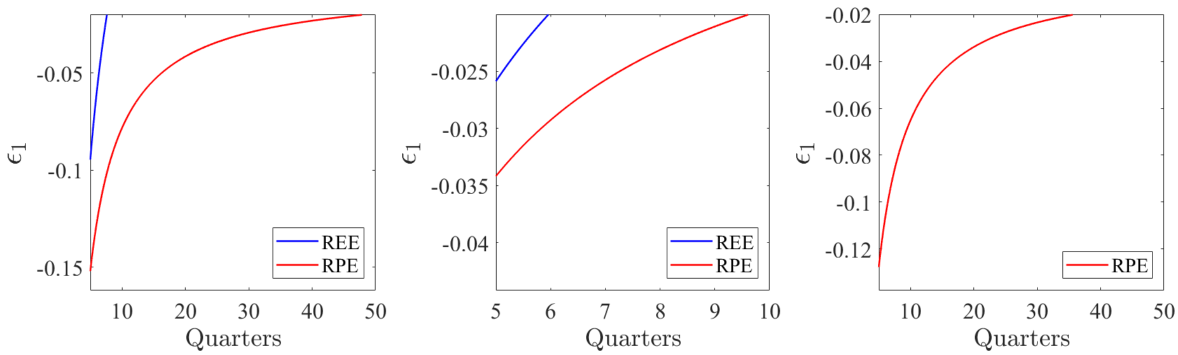
First, if an REE exists, then we could argue these RPE are implausible. In this case, agents could learn to do better, because there would likely be a learnable REE. But incoherence precludes REE, and as shown in Proposition 6, it implies that agents fail to form self-confirming expectations using a variety of different forecasting models that condition on the demand shock or even a lag of the endogenous variables. In the case of incoherence of REE, the RPE is thus a potentially reasonable alternative, because it relaxes the condition for the existence a self-confirming equilibria. Figure 4 visualizes the difference between the combination of values of the negative shock, ϵ_1, and of its persistence, i.e., p, that yields coherence in the REE and in the RPE cases. The area above the blue line and the red line defines the set of pairs (ϵ_1, p) so that at least one REE and RPE exist, respectively. Panel (a) shows that the difference between the region of the parameter space for which there is coherence in the two cases is substantial. In particular, unless the persistence, p, of the negative demand shock falls below 0.87, RE admits an equilibrium only for very small negative shocks. Panel (b) shows that both regions are quite sensitive – they shrink by around a quarter – to the value of the persistence of the other state where $\epsilon_2 = 0$. Finally, panel (c) shows that the region of coherence of REE shrinks quite substantially as prices becomes more flexible, while this is not the case for the RPE. The curse of flexibility is therefore a much more pronounced problem for REE than for RPE, just as Figure 3 (c) shows, which is very intuitive because the curse hinges on the rationality and forward-lookingness of the agents.

The Figure 4 results suggest that a fundamentals-driven RE liquidity trap must be relatively short-lived in the case of an REE compared to the duration of actual liquidity trap events experienced by Japan, the Euro Area and the U.S. In contrast, a fundamentals-driven RPE liquidity trap can be more persistent. Figure 5 depicts the maximum expected duration of the liquidity trap (equal to $(1-p)^{-1}$) that we can generate in a ZP REE or ZP RPE for different combinations of demand shock, ϵ_1. It can be seen that liquidity traps cannot be very persistent in an REE, whereas the RPE liquidity traps can be highly persistent, particularly if q is relatively large as in panel (a).[14] Panel (c) again shows that the curse of flexibility is a more pronounced problem for the REE. The BRE results are not depicted in Figure 5, but Proposition 4 implies that we can generate permanent ZLB events in a BRE for very negative shocks.

Second, suppose the model is incoherent under RE, but an E-stable RPE exists and the economy is in it. One could argue that agents inhabiting the RPE would notice that RPE inflation and output follow a 2-state process. Hence, agents would then stop setting one-period ahead inflation and output expectations equal to the long run average of those variables, and start to estimate a 2-state forecasting model in their attempt to learn these dynamics. Our previous propositions already suggest this might be a bad idea. Indeed, Proposition 5 establishes that such beliefs cannot be self-confirming. Can they reach another – not self-confirming – equilibrium? Figure 6 (a) depicts the results from simulating the

[14]Note that $p = 0.965$ produces an expected liquidity trap duration of around 28 quarters, which is the length of the 2008-2015 ZLB episode in the U.S.

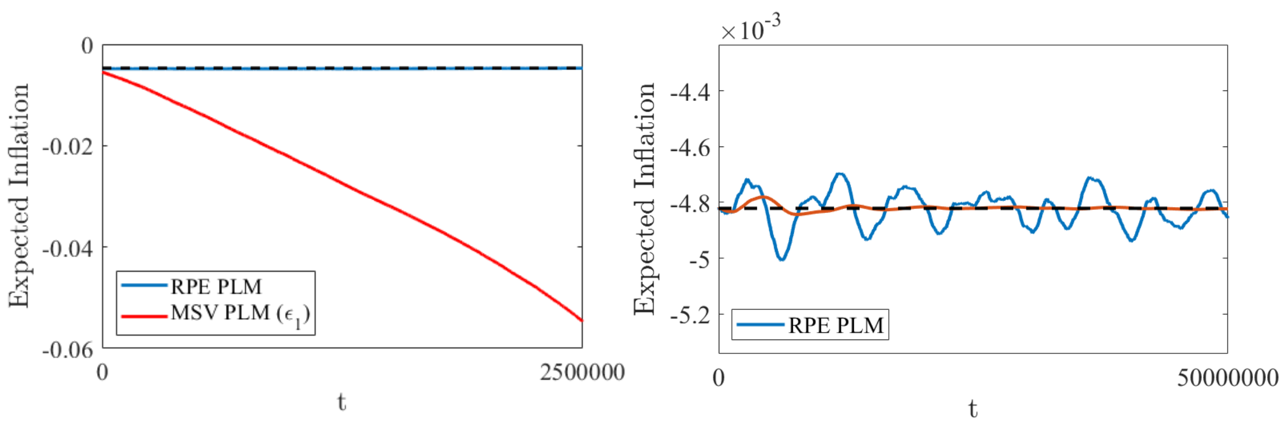
Figure 5: Maximum Expected ZLB Duration in a ZP Solution



(a) (b) $q = 0.9$ (c) $\lambda = 0.2$

Note: The blue (red) curve depicts the maximum expected duration ZLB ($(1-p)^{-1}$) we can generate for given ϵ_1 in an REE (RPE) ZP solution. The figure only depicts values of ϵ_1 for which an REE ZP or RPE ZP solution exists. Other parameter values: $\beta = 0.99$, $\sigma = 1$, $\lambda = 0.02$, $q = 0.98$, $\epsilon_2 = 0.01$.

Figure 6: Simulations when REE doe not exist and an RPE exists



(a) RPE vs MSV beliefs (b) RPE beliefs

Note: The model is calibrated so that an E-stable RPE ZP solution exists, but no MSV REE exists. The constant gain is small and set to $g_t = 0.00001$ for all t. $\beta = 0.99$, $\sigma = 1$, $\lambda = 0.02$, $p = 0.85$, $q = 0.98$, $\epsilon_1 = -0.04$, $\epsilon_2 = 0$.

learning dynamics for the case of MSV-consistent beliefs and also for the case of RPE-consistent beliefs, assuming a small constant gain.[15] It clearly shows that MSV-consistent beliefs are explosive even with very small gain parameter, while, on the contrary, the RPE-consistent beliefs are not. Panel (b) in Figure 6 displays the dynamics of expected inflation (and its cumulative average in red) from which it is evident that RPE expectations remain in some neighborhood around their RPE values.[16] Numerical simulation therefore suggests another reason why the RPE might be a good alternative. If an RPE exists – and an REE does not – and if agents try to learn using the REE PLM, then the economy will derail into deflationary spirals. On the contrary, if agents try to learn the RPE, then expectations remain stable and "centered" on the correct RPE values – provided that the gain parameter is small and initial inflation and output expectations are not too far away from the average inflation and output rate in the RPE.

Third, it is important to note that the assumption of learning by itself ensures coherence and completeness, provided that agents have lagged information, which is the most common assumption in the learning literature. The following proposition highlights this important implication of learning.

Proposition 8 *The model (20)-(21) is coherent and complete if Y_t is not observed contemporaneously (i.e. $k = 1$).*

The preceding proposition makes it clear that learning ensures the existence of a *temporary* equilibrium given any $p, q, \epsilon_1, \epsilon_2$, provided $k = 1$.[17] Intuitively, learning implies that expectations are predetermined, and this simplifies the task of computing the market clearing equilibrium allocation relative to the nontrivial fixed point problem needed to solve for the REE. While expectations are not self-confirming in a temporary equilibrium – unless inflation and output forecasts are identically equal to RPE values – a temporary equilibrium for the economy always exists when agents are in the process of learning the RPE.

[15] For MSV learning simulation, we initialize the forecast, $Y^e_{j,1}$ to match the state-contingent mean of inflation/output in the RPE when $\epsilon_t = \epsilon_j$. In other words, we assume that agents observe actual endogenous variables in the RPE switching with ϵ_t during periods $t < 1$ and then they decide to make their forecasts consistent with the switching at $t = 1$. We use the same initialization for RPE beliefs.

[16] Moreover, simulations – not reported – also show that RPE-consistent beliefs tend to revert to RPE values even with decreasing gain and when initial beliefs are a small distance from RPE values. Intuitively, the RPE-consistent beliefs could also be explosive (into deflationary spirals) whenever the gain parameter is too large or initial beliefs are very far from the RPE value.

[17] If $k = 0$ then a temporary equilibrium can fail to exist for small values of t with decreasing gain. Therefore, under contemporaneous information – which is the less common assumption in the adaptive learning literature – we need to restrict the magnitude of the gain parameter to get a solution. This same result applies to a model with "constant gain" parameter, where t^{-1} is replaced by some scalar, γ. If $k = 1$, the model is coherent for any $\gamma \in [0, 1]$, but is incoherent for high values of γ if $k = 0$. Evans and McGough (2018b) documents that constant gain learning models with contemporaneous information can lead to unreasonable predictions when interest rates are pegged. Proposition 8 is a complementary result that favors the lagged information assumption.

Of course there could be other non-rational equilibria such as the consistent expectations equilibrium considered by Jorgensen and Lansing (2021), the stochastic consistent expectations equilibria (SCEE) of Hommes and Zhu (2014) or Airaudo and Hajdini (forth.). In particular, an SCEE arises if agents use a forecasting model akin to (19) with lagged information about endogenous variables and the agents' beliefs about the mean *and* autocorrelation of inflation and output implied by the forecasting model is confirmed by the observable economic data. In a SCEE, agents' forecasts introduce a lag of inflation and output into the model, which prevents us from analyzing the existence of SCEE in our model with an occasionally binding constraint.[18] Our numerical analysis indicates that these more sophisticated non-rational equilibria may not exist for some plausible calibrations of the model.[19] Thus, the RPE may even be the best alternative among non-rational equilibria of our model with $M = M_f = N = 1$, but CEE or SCEE existence remains an open question. However, whether or not these alternative non-rational equilibria exist is not relevant for the main result of this paper: rationally incoherent models are non-rationally coherent, i.e., admit non-rational equilibria.

4.4 RPE and Continuous Shocks

To get closed-form solutions for both REE and RPE, we must assume that ϵ_t follows a discrete-valued Markov chain. To the best of our knowledge, no paper provides conditions for existence and uniqueness of RE equilibrium which can be applied to a model similar to our model under the assumption that ϵ_t is both persistent and continuously distributed.[20] However, while it is hard to characterize REE in a model with continuous shocks and an occasionally binding constraint, it is relatively easy to derive RPE.

To illustrate, consider the model (1)-(3) and suppose instead that $\epsilon_t = \rho\epsilon_{t-1} + v_t$ where $\rho \in [0, 1)$ and $v_t \sim \mathcal{N}(0, \sigma_v^2)$. In an RPE of this economy, agents' forecasts are given by $\hat{E}_t\pi_{t+1} = a_\pi$, $\hat{E}_t x_{t+1} = \frac{1-\beta}{\lambda} a_\pi$ consistent with the RPE studied in the previous sections. Substituting these expectations into the model gives the following RPE law of motion for inflation:

$$\pi_t = \begin{cases} (1+\lambda\sigma)a_\pi + \lambda\sigma\mu + \lambda\epsilon_t & \text{if } s_t = 0, \\ \frac{1+\lambda\sigma}{1+\lambda\sigma\psi} a_\pi + \frac{\lambda}{1+\lambda\sigma\psi}\epsilon_t & \text{if } s_t = 1. \end{cases} \tag{22}$$

Let $h(a_\pi)$ denote $E(\pi_t)$ as a function of a_π. Then:

$$h(a_\pi) = Pr\,(s_t = 0)\,E(\pi_t|s_t = 0) + (1 - Pr\,(s_t = 0))\,E\,(\pi_t|s_t = 1) \tag{23}$$

[18] See, e.g., Ascari and Mavroeidis (2022) for a discussion on the difficulty of studying models with occasionally binding constraints and lagged endogenous variables

[19] Results are available on request.

[20] See Mendes (2011) for analytical existence results under the assumption that ϵ_t is a mean-zero, i.i.d process.

To compute RPE, we need to compute $Pr(s_t = 0)$, $E(\pi_t|s_t = 0)$ and $E(\pi_t|s_t = 1)$ as functions of a_π. Let Φ and ϕ denote the standard normal probability distribution function and standard normal probability density function, respectively. Further, define:

$$L(a_\pi) := (\sigma_v \lambda)^{-1} \left(-\mu/\psi - (1 + \lambda\sigma)a_\pi - \lambda\sigma\mu\right) \tag{24}$$

It follows that:

$$\begin{aligned}
Pr(s_t = 0) &= \Phi(L(a_\pi)),\\
E(\pi_t|s_t = 0) &= (1 + \lambda\sigma)a_\pi + \lambda\sigma\mu - \frac{\lambda\sigma_v \phi(L(a_\pi))}{\Phi(L(a_\pi))},\\
E(\pi_t|s_t = 1) &= \frac{1 + \lambda\sigma}{1 + \lambda\sigma\psi} a_\pi + \frac{\lambda\sigma_v \phi(L(a_\pi))}{(1 + \lambda\sigma\psi)(1 - \Phi(L(a_\pi)))}.
\end{aligned}$$

Therefore, we have :

$$h(a_\pi) = \frac{1 + \lambda\sigma}{1 + \lambda\sigma\psi} a_\pi + \Phi(L(a_\pi)) \left(\frac{(1 + \lambda\sigma)\lambda\sigma\psi}{1 + \lambda\sigma\psi} a_\pi + \lambda\sigma\mu\right) - \frac{\phi(L(a_\pi))\lambda^2\sigma_v\sigma\psi}{1 + \lambda\sigma\psi}. \tag{25}$$

There is an RPE if and only if there exists $\bar{a}_\pi \in \mathbb{R}$ such that $h(\bar{a}_\pi) = \bar{a}_\pi$. One can show there exists a unique maximum of $h(\bar{a}_\pi)$, denoted a^*_π, and consequently there is either no RPE solution or there are exactly two RPE solutions.[21] A necessary and sufficient condition for existence of the RPE is $h(a^*_\pi) - a^*_\pi > 0$. We summarize the result as a proposition.

Proposition 9 *Consider (1)-(3) and suppose that* $\epsilon_t = \rho\epsilon_{t-1} + v_t$ *where* $v_t \sim \mathcal{N}(0, \sigma_v^2)$. *Then:*

i. Two RPE exist if and only if $h(a^*_\pi) > a^*_\pi$ *where* a^*_π *is given by*

$$a^*_\pi = L^{-1}\left(\Phi^{-1}\left(\frac{\psi - 1}{(1 + \lambda\sigma)\psi}\right)\right).$$

ii. An RPE does not exist if and only if $h(a^*_\pi) < a^*_\pi$.

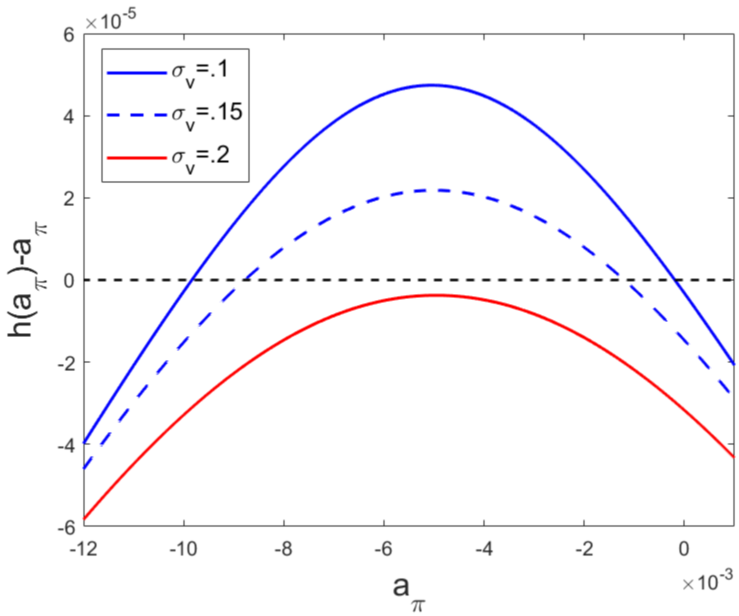
By inspecting (25), one can see that increasing the variance and persistence of the shocks (i.e. increasing σ_v and ρ) or decreasing price rigidity (i.e. increasing λ) reduces $h(a)$, that needs to be positive for an (actually two) RPE to exist. Consequently, sufficiently high values of σ_v, ρ or λ preclude existence of RPE in the model with continuous, persistent shocks. Figure 7 plots $h(a) - a$ for three different values of σ_v, assuming $\rho = 0.8$. It is evident that larger values of σ_v shifts $h(a)$ down.[22] Notice that the RPE levels of inflation are always less

[21]To see this, note that Φ is strictly decreasing in a_π and Φ and L are injective functions and that $h'(a_\pi) = \frac{\lambda\sigma(1-\psi)}{1+\lambda\sigma\psi} + \frac{\Phi(L(a_\pi))\lambda\sigma\psi(1+\lambda\sigma)}{1+\lambda\sigma\psi}$. Then under the Taylor Principle ($\psi > 1$), there exists a unique maximum, a^*_π, such that $h'(a^*_\pi) = 0$ and $h'(a_\pi) > 0$ ($h'(a_\pi) < 0$) for all $a_\pi < a^*_\pi$ ($a_\pi > a^*_\pi$).

[22]Figures 7 and 8 plot $h(a^*_\pi) - a^*_\pi$ for different calibrations of key parameters. In both figures we use the following benchmark calibration unless otherwise noted: $\beta = 0.99$, $\sigma = 1$, $\psi = 2$, $\lambda = 0.02$, $\rho = 0.8$, $\sigma = 0.1$.

than the zero inflation steady state level, and hence the RPE display a deflationary bias akin to the deflationary bias studied under RE in Nakata and Schmidt (2019) or Bianchi et al. (2021). Figure 8 plots $h(a_\pi^*) - a_\pi^*$ for different values of other key parameters in calibrated models. To interpret the panels in the Figure recall that $h(a_\pi^*) - a_\pi^* > 0$ for the RPE to exist. The figure shows that the RPE is less likely to exist if the shock variance or persistence is high, or if prices are more flexible. Hence, the same insights from the simple 2-state process example carry over to the case of continuous shocks (see Figure 4).

Figure 7: Existence and Multiplicity of RPE with Continuous Shocks

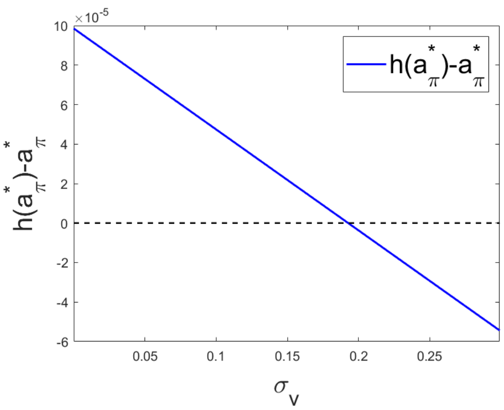


5 Variation on a Theme: Lagged Expectations

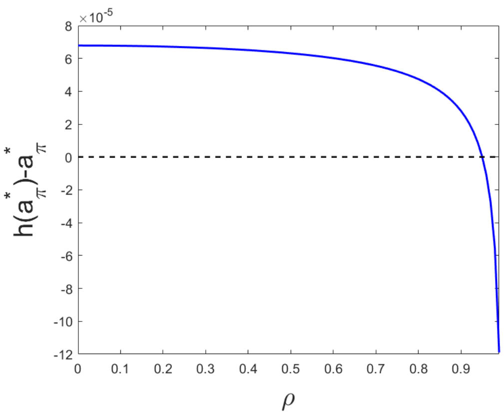
Throughout this paper we stuck to the standard assumption that agents observe the demand shock contemporaneously (i.e. ϵ_t is included in agents' time-t information set). This would be a natural assumption if for example ϵ_t is a shock to the households' preferences as in Eggertsson and Woodford (2003). However, the assumption that agents observe ϵ_t with a lag (so that ϵ_{t-1}, but not ϵ_t, is included in agents' time-t information set) permits the study of some additional non-rational equilibria which may exist in rationally incoherent models.

To illustrate existence of these additional "lagged expectations equilibria" (LEE), consider the model (1)-(3) and suppose $q = 1$, $\epsilon_2 = 0$. Further suppose that agents believe inflation and output follows a persistent 2-state Markov chain (just like rational agents) but instead agents do not know ϵ_t and hence agents attach p^2 probability to the prospect that

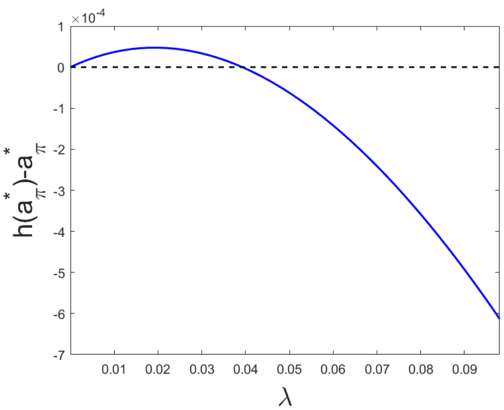
Figure 8: RPE Existence



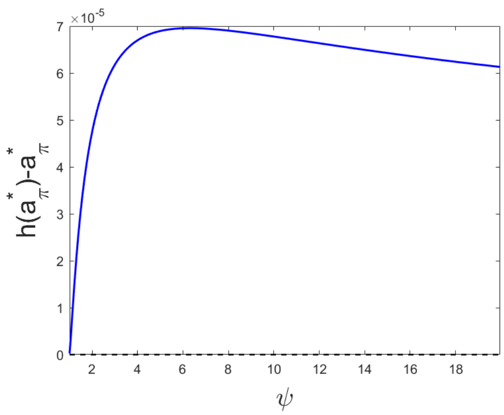
(a) Shock Variance



(b) Shock Persistence



(c) Price Flexibility



(d) Activeness of Policy

$\epsilon_{t+1} = \epsilon_1$ when forecasting at time t in the temporary state, instead of attaching p probability to this event as agents with full-information RE would do. Under this assumption about agents' time-t information set, the economy either returns to the steady state with zero inflation or the steady state with zero interest rates after $\epsilon_t = \epsilon_2$. The "temporary state" value of output when $\epsilon_t = \epsilon_1$ (assuming for simplicity that we go back to the zero inflation steady state) is given by:

$$\begin{aligned} x_t &= \nu(p^2)E_t x_{t+1} - \sigma \max\{\frac{\psi\lambda}{1-\beta p^2}x_t, -\mu\} + \epsilon_1 \qquad (26) \\ \text{where} \qquad \nu(p^2) &= \left(1 + \frac{\lambda\sigma}{1-\beta p^2}\right) > 1, \end{aligned}$$

which we obtain by substituting the Phillips curve and Taylor rule into (1). From this equation, it is apparent that for any p, sufficiently low values of ϵ_1 preclude unconstrained interest rates, just as in the case of full information RE. Thus, for a sufficiently large demand shock, output will be given by;

$$x_t = \frac{1}{1 - p^2\nu(p^2)}(\sigma\mu + \epsilon_1), \qquad (27)$$

if a solution of the model exists at all. We call this solution a lagged expectation equilibrium (LEE). It is a self-confirming equilibrium because agents correctly forecast the conditional mean of output and inflation (e.g. $E(x_t|\epsilon_t = \epsilon_1) = \frac{1}{1-p^2\nu(p^2)}(\sigma\mu+\epsilon_1)$ and $E(x_t|\epsilon_t = \epsilon_2) = 0$).[23] Note that $p^2\nu(p^2) < p\nu(p)$, and therefore if $p^2\nu(p^2) < 1 < p\nu(p)$ we will have a LEE given any ϵ_1, but only an REE if ϵ_1 is sufficiently close to zero. REE existence always implies existence of LEE, so the opposite is not true. This simple exercise reveals that there can be additional deviations from RE, beyond the scope of this paper, which are useful for understanding an incoherent model.

6 Concluding Remarks

Standard RE models with an occasionally binding zero lower bound (ZLB) constraint either admit no solutions (incoherence) or multiple solutions (incompleteness). This paper shows that the problem of incompleteness and incoherence hinges on the assumption of RE.

Models with no rational equilibria may admit self-confirming equilibria involving the use of simple mis-specified forecasting models. The main message of the paper from the existence analysis is that when negative shocks are sufficiently large in magnitude or sufficiently persistent, the baseline NK model is incoherent, but can admit RPE or BRE. Completeness and

[23]In the first period such that $\epsilon_t = \epsilon_2$, we have $x_t \neq 0$. However, $E(x_t|\epsilon_t = \epsilon_2) = E(\pi_t|\epsilon_t = \epsilon_2) = 0$ because state 2 is an absorbing state. Thus, the LEE is a non-rational equilibrium in which agents have self-confirming beliefs about the state-contingent conditional means of endogenous variables.

coherence can be restored if expectations are adaptive or if agents are less forward-looking due to some information or behavioral friction.

In the case of multiple solutions, the E-stability criterion selects an equilibrium. An RPE can exist as a self-confirming equilibrium, even if the underlying model does not admit an REE. Thus, non-rationality of agents' beliefs can save the economy from blowing up into infinite deflationary spirals, while it yields persistent liquidity traps. These results highlight how deviations from RE help us understand persistent liquidity traps in theoretical models and interpret the recent episodes of liquidity traps in Japan, the Euro Area, and the U.S.

We leave room for future work. In particular, we used the RPE and BRE concepts to make our point simple and clear, and consequently we abstracted from other self-confirming equilibria that could emerge under adaptive learning, such as CEE or SCEE. Similarly, we excluded other popular forms of non-rationality from our analysis, such as level-k reasoning. Finally, we put a premium on analytical results and therefore we focused on a simple theoretical model. Future work could examine related issues in larger, empirically-rich DSGE models.

Appendix

Proof of Proposition 1

Consider Proposition 1 and define $a = \lambda\sigma$, $Q = I_2 - (1+\beta+\lambda\sigma)K + \beta K^2$ and $\hat{\pi}^i = (\pi_1^i, \pi_2^i)'$, and let e_j denote the j-th column of the 2 by 2 identity matrix, I_2. The PP solution is given by:

$$\hat{\pi}^{PP} = (Q + \lambda\sigma\psi I_2)^{-1} \begin{pmatrix} \lambda\epsilon_1 \\ \lambda\epsilon_2 \end{pmatrix}.$$

The PP solution exists if and only if $\psi\pi_j^{PP} > -\mu$ for $j = 1, 2$. From $\hat{\pi}^{PP}$ we see that π_1^{PP} and π_2^{PP} are linear in ϵ_1 and

$$\begin{aligned} \frac{\partial \pi_1^{PP}}{\partial \epsilon_1} &= \frac{\lambda((1-q)(1+a-(p+q-1)\beta) + a(\psi-1))}{a(\psi-1)(a(\psi+1-p-q) + (2-p-q)(1-\beta(p+q-1)))} > 0 \\ \frac{\partial \pi_2^{PP}}{\partial \epsilon_1} &= \frac{\lambda(1-q)(a-\beta(p+q-1)+1)}{a(\psi-1)(a(\psi+1-p-q) + (2-p-q)(1-\beta(p+q-1)))} > 0. \end{aligned}$$

Thus, PP exists if and only if $\epsilon_1 > \epsilon^{PP} = min(\epsilon_1^{PP}, \epsilon_2^{PP})$ where ϵ_1^{PP} and ϵ_2^{PP} solve $\psi\pi_1^{PP} = -\mu$ and $\psi\pi_2^{PP} = -\mu$, respectively. We have

$$\epsilon_1^{PP} - \epsilon_2^{PP} = \frac{a(\psi-1)(a\mu(\psi-1) + \lambda\epsilon_2\psi)(a(-p-q+\psi+1) + (-p-q+2)(1-\beta(p+q-1)))}{\lambda(1-q)\psi(a-\beta(p+q-1)+1)(a(\psi-q) + (1-q)(1-\beta(p+q-1)))} > 0$$

Therefore, the PP solution exists if and only if $\epsilon_1 > \epsilon^{PP} = \epsilon_1^{PP}$ where

$$\epsilon^{PP} = \frac{a^2\mu(\psi-1)(p+q-\psi-1)}{\lambda\psi(-(a+1)q+a\psi+\beta-\beta p+\beta q(p+q-2)+1)} + \frac{a(\lambda\epsilon_2(p-1)\psi+\mu(\psi-1)(-p-q+2)(\beta(p+q-1)-1))-\lambda\epsilon_2(p-1)\psi(\beta(p+q-1)-1)}{\lambda\psi(-(a+1)q+a\psi+\beta-\beta p+\beta q(p+q-2)+1)}. \tag{28}$$

The ZP solution is given by

$$\hat{\pi}^{ZP} = (Q+\lambda\sigma\psi e_2 e_2')^{-1}\begin{pmatrix}\lambda\epsilon_1+\lambda\sigma\mu\\ \lambda\epsilon_2\end{pmatrix}.$$

The ZP solution exists if and only if $\psi\pi_2^{ZP} > -\mu > \psi\pi_1^{ZP}$. From $\hat{\pi}^{ZP}$ we see that π_1^{ZP} and π_2^{ZP} are linear in ϵ_1 and

$$\begin{aligned}\frac{\partial\pi_1^{ZP}}{\partial\epsilon_1} &= \frac{-\lambda((1-q)(a-\beta(p+q-1)+1)+a(\psi-1))}{a(-a(p+q-1)+ap\psi-(\beta(p+q-1)-1)(p(\psi-1)-q-\psi+2))}\\ \frac{\partial\pi_2^{ZP}}{\partial\epsilon_1} &= \frac{\lambda(q-1)(a-\beta(p+q-1)+1)}{a(-a(p+q-1)+ap\psi-(\beta(p+q-1)-1)(p(\psi-1)-q-\psi+2))}\end{aligned}$$

From the last equations it is clear that $\frac{\partial\pi_1^{ZP}}{\partial\epsilon_1} > 0$ and $\frac{\partial\pi_2^{ZP}}{\partial\epsilon_1} > 0$ if and only if $den^{ZP} > 0$ where $den^{ZP} = -(-a(p+q-1)+ap\psi-(\beta(p+q-1)-1)(p(\psi-1)-q-\psi+2))$. Solving for ϵ_1^{ZP} and ϵ_2^{ZP} such that $\psi\pi_1^{ZP} = -\mu$ and $\psi\pi_2^{ZP} = -\mu$, respectively, we have

$$\begin{aligned}\epsilon_1^{ZP}-\epsilon_2^{ZP} &= \frac{a(a\mu(\psi-1)+\lambda\epsilon_2\psi)den^{ZP}}{\epsilon_{\Delta ZP,den}}\\ \epsilon_{\Delta ZP,den} &= (1-q)\lambda\psi(a+\beta(1-p-q)+1)((1-q)(a-\beta(p+q-1)+1)+a(\psi-1)) > 0\end{aligned}$$

Therefore, if $den^{ZP} > 0$ then $\epsilon_2^{ZP} < \epsilon_1 < \epsilon_1^{ZP}$ is necessary and sufficient for existence of ZP; otherwise, $\epsilon_1^{ZP} < \epsilon_1 < \epsilon_2^{ZP}$ is necessary and sufficient for existence of ZP. Further, we can show:

$$\begin{aligned}\epsilon_1^{ZP} &= \frac{a^2\mu(\psi-1)(p+q-\psi-1)}{\lambda\psi(-(a+1)q+a\psi+\beta-\beta p+\beta q(p+q-2)+1)}\\ &+ \frac{a(\lambda\epsilon_2(p-1)\psi+\mu(\psi-1)(-p-q+2)(\beta(p+q-1)-1))-\lambda\epsilon_2(p-1)\psi(\beta(p+q-1)-1)}{\lambda\psi(-(a+1)q+a\psi+\beta-\beta p+\beta q(p+q-2)+1)}\\ &= \epsilon^{PP}\end{aligned}$$

and

$$\begin{aligned}\epsilon_2^{ZP} &= \frac{a^2\mu(\psi-1)(p+q-1)-\lambda\epsilon_2(p-1)\psi(\beta(p+q-1)-1)}{\lambda(q-1)\psi(-a+\beta(p+q-1)-1)}\\ &+ \frac{a(\lambda\epsilon_2 p\psi+\mu(\psi-1)(2-p-q)(\beta(p+q-1)-1))}{\lambda(q-1)\psi(-a+\beta(p+q-1)-1)}\end{aligned} \tag{29}$$

Therefore, $\epsilon_1 > \epsilon^{ZP} = \min\{\epsilon^{PP}, \epsilon_2^{ZP}\}$ is necessary and sufficient for existence of ZP or PP solution.

The PZ solution is given by

$$\hat{\pi}^{PZ} = (Q + \lambda\sigma\psi e_1 e_1')^{-1} \begin{pmatrix} \lambda\epsilon_1 \\ \lambda\epsilon_2 + \lambda\sigma\mu \end{pmatrix}.$$

The PZ solution exists if and only if $\psi\pi_1^{PZ} > -\mu > \psi\pi_2^{PZ}$. One can show

$$\begin{aligned}
\frac{\partial\pi_1^{PZ}}{\partial\epsilon_1} &= \frac{\lambda(1-(a+1)q+\beta(q-1)(p+q-1))}{a(a(p-q\psi+q-1)-(\beta(p+q-1)-1)(p-q\psi+q+\psi-2))} \\
&= \frac{\lambda num_1^{PZ}}{den^{PZ}} \\
\frac{\partial\pi_2^{PZ}}{\partial\epsilon_1} &= \frac{\lambda(1-q)(a-\beta(p+q-1)+1)}{a(a(p-q\psi+q-1)-(\beta(p+q-1)-1)(p-q\psi+q+\psi-2))} \\
&= \frac{\lambda num_2^{PZ}}{den^{PZ}}.
\end{aligned}$$

Clearly $num_2^{PZ} \geq 0$. Now $\frac{\partial den^{PZ}}{\partial\psi} = a num_1^{PZ}$, so den^{PZ} is increasing in ψ if and only if $num_1^{PZ} > 0$. Since den^{PZ} is linear in ψ there exists a unique ψ^{PZ} such that $den^{PZ} = 0$:

$$\psi^{PZ} = 1 + \frac{(1+a+(1-p-q)\beta)(1-p)}{num_1^{PZ}}$$

Therefore, if $num_1^{PZ} < 0$, $\psi^{PZ} < 1$ and den^{PZ} is decreasing in ψ, which implies $den^{PZ} < 0$ for $\psi > 1$. Otherwise, $den^{PZ} < 0$ if $num_1^{PZ} > 0$ and $1 < \psi < \psi^{PZ}$ and $den^{PZ} > 0$ if $num_1^{PZ} > 0$ and $\psi > \psi^{PZ}$.

Solving for ϵ_1^{PZ} and ϵ_2^{PZ} such that $\psi\pi_1^{PZ} = -\mu$ and $\psi\pi_2^{PZ} = -\mu$, respectively, we have

$$\epsilon_1^{PZ} - \epsilon_2^{PZ} = \frac{(a\mu(\psi-1)+\lambda\epsilon_2\psi)den^{PZ}}{\lambda((1-q)(a-\beta(p+q-1)+1))\psi num_1^{PZ}}$$

There are three cases to consider. First, if $den^{PZ} > 0$ (which implies $num_1^{PZ} > 0$), then $\partial\pi_1^{PZ}/\partial\epsilon_1 > 0$, $\partial\pi_2^{PZ}/\partial\epsilon_1 > 0$ and $\epsilon_1^{PZ} > \epsilon_2^{PZ}$ so $\epsilon_1 > \epsilon_1^{PZ} > \epsilon_2^{PZ} > \epsilon_1$ is necessary for PZ existence, but not possible. Second, if $den^{PZ} < 0$ and $num_1^{PZ} > 0$, then $\partial\pi_1^{PZ}/\partial\epsilon_1 < 0$, $\partial\pi_2^{PZ}/\partial\epsilon_1 < 0$ and $\epsilon_1^{PZ} < \epsilon_2^{PZ}$, so $\epsilon_1 < \epsilon_1^{PZ} < \epsilon_2^{PZ} < \epsilon_1$ is necessary for PZ existence, but not possible. In the third case, $den^{PZ} < 0$ and $num_1^{PZ} < 0$, which implies $\partial\pi_1^{PZ}/\partial\epsilon_1 > 0$, $\partial\pi_2^{PZ}/\partial\epsilon_1 < 0$, and $\epsilon_2^{PZ} < \epsilon_1^{PZ}$ so that $\epsilon_2^{PZ} < \epsilon_1^{PZ} < \epsilon_1$ is necessary and sufficient for PZ existence in this case.

One can show:

$$\epsilon_1^{PZ} - \epsilon^{PP} = \frac{a(p-1)(a\mu(\psi-1)+\lambda\epsilon_2\psi)(a-\beta(p+q-1)+1)}{(num_1^{PZ})\lambda((1-q)(1+a-(-1+p+q)\beta)+a(\psi-1))} > 0$$

if PZ exists (since this requires $num_1^{PZ} < 0$). Therefore, if PZ exists then $\epsilon_1 > \epsilon^{PP}$ and hence the PP or ZP solution also exists.

The ZZ solution is given by

$$\hat{\pi}^{ZZ} = (Q)^{-1} \begin{pmatrix} \lambda\epsilon_1 + \lambda\sigma\mu \\ \lambda\epsilon_2 + \lambda\sigma\mu \end{pmatrix}.$$

The ZZ solution exists if and only if $\psi\pi_j^{ZZ} < -\mu$ for $j = 1, 2$. One can show

$$\begin{aligned}
\frac{\partial\pi_1^{ZZ}}{\partial\epsilon_1} &= \frac{\lambda(1-(a+1)q+\beta(q-1)(p+q-1))}{a(a(p+q-1)-(p+q-2)(\beta(p+q-1)-1))} \\
&= \frac{\lambda num_1^{ZZ}}{aden^{ZZ}} \\
\frac{\partial\pi_2^{ZZ}}{\partial\epsilon_1} &= \frac{\lambda(1-q)(a-\beta(p+q-1)+1)}{a(a(p+q-1)-(p+q-2)(\beta(p+q-1)-1))} \\
&= \frac{\lambda num_2^{ZZ}}{aden^{ZZ}}.
\end{aligned}$$

Clearly $num_2^{ZZ} \geq 0$. We can further show that $-num_1^{ZZ} = den^{ZZ} + (1-p)(1+a-(p+q-1)\beta) \geq den^{ZZ}$. Hence $den^{ZZ} > 0$ implies $num_1^{ZZ} < 0$. Solving for ϵ_1^{ZZ} and ϵ_2^{ZZ} such that $\psi\pi_1^{ZZ} = -\mu$ and $\psi\pi_2^{ZZ} = -\mu$, respectively, we have

$$\epsilon_1^{ZZ} - \epsilon_2^{ZZ} = \frac{aden^{ZZ}(a\mu(\psi-1)+\lambda\epsilon_2\psi)}{\lambda num_1^{ZZ}\psi(1-q)(a-\beta(p+q-1)+1)}$$

There are three cases to consider. First, if $den^{ZZ} > 0$ (which implies $num_1^{ZZ} < 0$) then $\partial\pi_1^{ZZ}/\partial\epsilon_1 < 0$, $\partial\pi_2^{ZZ}/\partial\epsilon_1 > 0$, $\epsilon_2^{ZZ} > \epsilon_1^{ZZ}$, so that ZZ existence requires $\epsilon_2^{ZZ} > \epsilon_1 > \epsilon_1^{ZZ}$. Second, if $den^{ZZ} < 0$ and $num_1^{ZZ} > 0$ then $\partial\pi_1^{ZZ}/\partial\epsilon_1 < 0$, $\partial\pi_2^{ZZ}/\partial\epsilon_1 < 0$, $\epsilon_2^{ZZ} > \epsilon_1^{ZZ}$, so that ZZ existence requires $\epsilon_1 > \epsilon_2^{ZZ} > \epsilon_1^{ZZ}$. In the third case, $den^{ZZ} < 0$ and $num_1^{ZZ} < 0$ which implies $\partial\pi_1^{ZZ}/\partial\epsilon_1 > 0$, $\partial\pi_2^{ZZ}/\partial\epsilon_1 < 0$, $\epsilon_1^{ZZ} > \epsilon_2^{ZZ}$, so that ZZ existence requires $\epsilon_1^{ZZ} > \epsilon_1 > \epsilon_2^{ZZ}$.

Now it can be shown that $\epsilon_2^{ZZ} = \epsilon_2^{ZP}$ and

$$\epsilon_1^{ZZ} - \epsilon^{PP} = \frac{a(p-1)(a\mu(\psi-1)+\lambda\epsilon_2\psi)(a-\beta(p+q-1)+1)}{\lambda num_1^{ZZ}((1-q)(1+a-(-1+p+q)\beta)+a(\psi-1))} > 0$$

if $num_1^{ZZ} < 0$. Since $\epsilon_2^{ZZ} = \epsilon_2^{ZP}$ and existence of ZZ only hinges on $\epsilon_1 > \epsilon_1^{ZZ}$ if $num_1^{ZZ} < 0$ it follows that the ZP or PP solution will exist if the ZZ solution exists.

We conclude that an REE exists if and only if

$$\epsilon_1 > \bar{\epsilon}_{REE} = \min\{\epsilon^{PP}, \epsilon_2^{ZP}\} \tag{30}$$

where ϵ^{PP} and ϵ_2^{ZP} are defined in (28) and (29), respectively. ■

Case. $q = 1$

Here we show that Proposition 1 nests Proposition 5 of Ascari and Mavroeidis (2022) as a special case. Specifically, we compute the condition from $\lim_{q\to 1}\bar{\epsilon}_{REE}$ and show that this recovers the result in Proposition 5 of Ascari and Mavroeidis (2022). Alternatively, we could repeat the preceding analysis in the model with $q = 1$, but this gives the same result.[24]

Define $\theta = \frac{(1-p)(1-p\beta)}{\lambda\sigma p} = \frac{(1-p)(1-p\beta)}{ap}$. From the preceding analysis, an REE exists if and only if $\epsilon_1 > \bar{\epsilon}_{REE} = \min\{\epsilon^{PP}, \epsilon_2^{ZP}\}$ where ϵ_2^{ZP} can be expressed as $\epsilon_2^{ZP} = \chi(1-q)^{-1}$. In the limit $q \to 1$ we have:

$$\begin{aligned}\epsilon^{PP} &= \mu\left(\frac{a(p-\psi)}{\lambda\psi} - \frac{pa\theta}{\lambda\psi}\right) + \frac{\lambda\epsilon_2(p-1)(a-\beta p+1)}{a\lambda(\psi-1)} \\ \chi &= \frac{(p(1+a+\beta)-p^2\beta-1)(a\mu(\psi-1)+\lambda\epsilon_2\psi)}{(1+a-p\beta)\psi\lambda}.\end{aligned}$$

Now, $p(1+a+\beta) - 1 - p^2\beta < 0$ if and only if $\theta > 1$. Therefore, $\bar{\epsilon}_{REE} = \epsilon_2^{ZP} \to -\infty$ as $q \to 1$ if $\theta > 1$. We conclude that any value of ϵ_1 ensures existence of a solution when $\theta > 1$ and $q = 1$. If $\theta < 1$, then $\chi \to +\infty$ and $\bar{\epsilon}_{REE} = \epsilon^{PP}$.

Now we show that our conditions recover Proposition 5 in Ascari and Mavroeidis (2022). First, we have $\mu = log(r\pi_*) > 0$ which implies $r^{-1} \leq \pi_*$ where r and π_* are the steady state gross real interest rate and inflation rate, respectively. Further, we set $\epsilon_2 = 0$ and $\epsilon_1 = -\sigma\hat{M}_{t+1|t} = \sigma p r_L$. The critical threshold, ϵ^{PP} becomes

$$-r_L \leq \mu\left(\frac{\theta}{\psi} + \frac{(\psi-p)}{p\psi}\right)$$

Thus, a solution exists if and only if either $\theta > 1$ or $\theta < 1$ and $-r_L \leq \mu\left(\frac{\theta}{\psi} + \frac{(\psi-p)}{p\psi}\right)$ as in Ascari and Mavroeidis (2022).

Proof of Proposition 2

The proof of Proposition 2 is a straightforward extension of the proof of Proposition 1. Define $\bar{q} = Pr(s_t = 2) = (1-p)/(2-p-q)$. The regime-specific levels of inflation in RPE i, $\hat{\pi}^i = (\pi_1^i, \pi_2^i)'$, are given by fixed point restrictions that have the same basic form as the REE fixed point restrictions except we replace q with $\bar{q}$ and p with $1-\bar{q}$. Therefore, RPE will exist if

$$\epsilon_1 > \bar{\epsilon}_{RPE} = \min(\epsilon^{PP,RPE}, \epsilon_2^{ZP,RPE}), \tag{31}$$

where $\epsilon^{PP,RPE}, \epsilon_2^{ZP,RPE}$ have the same form as $\epsilon^{PP}, \epsilon_2^{ZP}$ given in (28),(29) except we replace q and p with $\bar{q}$ and $1-\bar{q}$, respectively. One can show:

$$\begin{aligned}\epsilon^{PP} - \epsilon^{PP,RPE} &= \Xi_{PP}(1-p-q) \\ \epsilon^{ZP} - \epsilon^{ZP,RPE} &= \Xi_{ZP}(1-p-q)\end{aligned}$$

[24] Mathematica routine available on request.

$\Xi_{PP} = \frac{a(1+a-\beta(p+q-2))(1-p)(\psi-1)(a\mu(\psi-1)+\lambda\epsilon_2\psi)}{\lambda\psi(a(1-\psi)(2-p-q)+(a+1)(q-1))((1-q)(1+a-\beta(p+q-1))+a(\psi-1))} \leq 0$
and $\Xi_{ZP} = \frac{a(a\mu(\psi-1)+\lambda\epsilon_2\psi)(1+a-\beta(p+q-2))}{\lambda(a+1)(q-1)\psi(1+a-\beta(p+q-1))} < 0$. Hence, $\bar{\epsilon}_{REE} \geq \bar{\epsilon}_{RPE}$ if $p+q>1$. ■

Proof of Proposition 3

Consider Proposition 3 and define $a = \lambda\sigma$, $Q = I_2 - (M + M_f\beta + \lambda\sigma N)K + \beta M M_f K^2$ and $\hat{\pi}^i = (\pi_1^i, \pi_2^i)'$, and let e_j denote the j-th column of the 2 by 2 identity matrix, I_2. The PP solution is given by:

$$\hat{\pi}^{PP,BR} = (Q + \lambda\sigma\psi I_2)^{-1} \begin{pmatrix} \lambda\epsilon_1 \\ \lambda\epsilon_2 \end{pmatrix}.$$

The PP solution exists if and only if $\psi\pi_j^{PP,BR} > -\mu$ for $j = 1, 2$. From $\hat{\pi}^{PP,BR}$ we see that $\pi_1^{PP,BR}$ and $\pi_2^{PP,BR}$ are linear in ϵ_1 and

$$\begin{aligned} \frac{\partial\pi_1^{PP,BR}}{\partial\epsilon_1} &= \frac{num_1^{PP,BR}}{den^{PP,BR}} \geq 0 \\ \frac{\partial\pi_2^{PP,BR}}{\partial\epsilon_1} &= \frac{num_2^{PP,BR}}{den^{PP,BR}} \geq 0. \end{aligned}$$

where

$$\begin{aligned} num_1^{PP,BR} &= \lambda\psi(a\psi + \beta M M_f(p(q-1) + q(q-1) + 1) - Mq - q(\beta M_f + aN) + 1) \geq 0 \\ num_2^{PP,BR} &= \lambda(q-1)\psi(\beta M_f(M(p+q) - 1) - M - aN) \geq 0 \\ den^{PP,BR} &= (a(\psi - N) + (1-M)(1-\beta M_f))den_1^{PP,BR} \geq 0 \\ den_1^{PP,BR} &= a\psi + M(p+q-1)(\beta M_f(p+q-1) - 1) \\ &+ \beta M_f - (p+q)(\beta M_f + aN) + aN + 1 \geq 0 \end{aligned}$$

Thus, PP exists if and only if $\epsilon_1 > \epsilon^{PP,BR} = min(\epsilon_1^{PP,BR}, \epsilon_2^{PP,BR})$ where $\epsilon_1^{PP,BR}$ and $\epsilon_2^{PP,BR}$ solve $\psi\pi_1^{PP,BR} = -\mu$ and $\psi\pi_2^{PP,BR} = -\mu$, respectively. We have

$$\begin{aligned} &\epsilon_1^{PP,BR} - \epsilon_2^{PP,BR} = \\ &\frac{\psi den^{PP,BR}(\mu(a(\psi - N) + (1-M)(1-\beta M_f)) + \lambda\epsilon_2\psi)}{\lambda num_1^{PP,BR} num_2^{PP,BR}} \geq 0 \end{aligned}$$

Therefore, the PP solution exists if and only if $\epsilon_1 > \epsilon^{PP,BR} = \epsilon_1^{PP,BR}$ where

$$\begin{aligned} \epsilon^{PP,BR} &= \frac{\eta_1\eta_2\eta_3}{\lambda\psi(-q(aN + \beta M_f) + a\psi + \beta M M_f(p(q-1) + q(q-1) + 1) - Mq + 1)} \qquad (32) \\ \eta_1 &= (a(\psi - N) + (1-M)(1 - M_f\beta)) \geq 0 \\ \eta_2 &= (-(p+q)(aN + \beta M_f) + a(N + \psi) + M(p+q-1)(\beta M_f(p+q-1) - 1) + \beta M_f + 1) \\ \eta_3 &= \frac{\lambda\epsilon_2(1-p)\psi(-aN + \beta M_f(M(p+q) - 1) - M)}{den^{PP,BR}} - \mu \end{aligned}$$

The ZP solution is given by

$$\hat{\pi}^{ZP,BR} = (Q + \lambda\sigma\psi e_2 e_2')^{-1} \begin{pmatrix} \lambda\epsilon_1 + \lambda\sigma\mu \\ \lambda\epsilon_2 \end{pmatrix}.$$

The ZP solution exists if and only if $\psi\pi_2^{ZP,BR} > -\mu > \psi\pi_1^{ZP,BR}$. From $\hat{\pi}^{ZP,BR}$ we see that $\pi_1^{ZP,BR}$ and $\pi_2^{ZP,BR}$ are linear in ϵ_1 and

$$\begin{aligned} \frac{\partial\pi_1^{ZP,BR}}{\partial\epsilon_1} &= \frac{\lambda\psi(a(\psi - Nq) + \beta M_f(M(p(q-1) + q(q-1) + 1) - q) - Mq + 1)}{den^{ZP,BR}} \\ \frac{\partial\pi_2^{ZP,BR}}{\partial\epsilon_1} &= \frac{\lambda(1-q)\psi(aN + \beta M_f(1 - M(p+q)) + M)}{den^{ZP,BR}} \end{aligned}$$

From the last equations it is clear that $\frac{\partial\pi_1^{ZP}}{\partial\epsilon_1} > 0$ and $\frac{\partial\pi_2^{ZP}}{\partial\epsilon_1} > 0$ if and only if $den^{ZP,BR} > 0$ where $den^{ZP,BR} = M(aN(p+q-1)(\beta M_f(p+q) - 2) + a\psi(\beta M_f(-p(p+q-1) + q - 1) + p) + (\beta M_f - 1)(p+q)(\beta M_f(p+q-1) - 1)) + a\psi(aNp + \beta M_f p - 1) - (aN + \beta M_f - 1)(aN(p+q-1) + \beta M_f(p+q-1) - 1) + M^2(\beta M_f - 1)(-(p+q-1))(\beta M_f(p+q-1) - 1)$. Solving for $\epsilon_1^{ZP,BR}$ and $\epsilon_2^{ZP,BR}$ such that $\psi\pi_1^{ZP,BR} = -\mu$ and $\psi\pi_2^{ZP,BR} = -\mu$, respectively, we have

$$\begin{aligned} \epsilon_1^{ZP,BR} - \epsilon_2^{ZP,BR} &= \frac{(\mu((1-M)(1-M_f\beta) + a(\psi - N)) + \lambda\epsilon_2\psi)den^{ZP,BR}}{\lambda\epsilon_{\Delta ZP,BR}} \\ \epsilon_{\Delta ZP,BR} &= (1-q)num_1^{ZP,BR}(M + aN + M_f\beta(1 - M(p+q))) > 0 \\ num_1^{ZP,BR} &= \psi(a(\psi - Nq) + \beta M_f(M(p(q-1) + q(q-1) + 1) - q) - Mq + 1) \geq 0 \end{aligned}$$

Therefore, if $den^{ZP,BR} > 0$ then $\epsilon_2^{ZP,BR} < \epsilon_1 < \epsilon_1^{ZP,BR}$ is necessary and sufficient for existence of ZP; otherwise, $\epsilon_1^{ZP,BR} < \epsilon_1 < \epsilon_2^{ZP,BR}$ is necessary and sufficient for existence of ZP. Further, we can show:

$$\epsilon_1^{ZP,BR} = \epsilon^{PP,BR}$$

and

$$\begin{aligned} \epsilon_2^{ZP,BR} &= \frac{\mu\eta_1(-(p+q)(aN + \beta M_f) + aN + M(p+q-1)(\beta M_f(p+q-1) - 1) + \beta M_f + 1)}{\lambda(q-1)\psi(aN - \beta M M_f(p+q) + M + \beta M_f)} \\ &- \frac{\psi\epsilon_2\lambda(aNp + \beta M_f(M(-p(p+q-1) + q - 1) + p) + Mp - 1)}{\lambda(q-1)(aN - \beta M M_f(p+q) + M + \beta M_f)} \end{aligned} \quad (33)$$

Therefore, $\epsilon_1 > \min\{\epsilon^{PP,BR}, \epsilon_2^{ZP,BR}\}$ is necessary and sufficient for existence of ZP or PP solution.

The PZ solution is given by

$$\hat{\pi}^{PZ,BR} = (Q + \lambda\sigma\psi e_1 e_1')^{-1} \begin{pmatrix} \lambda\epsilon_1 \\ \lambda\epsilon_2 + \lambda\sigma\mu \end{pmatrix}.$$

The PZ solution exists if and only if $\psi\pi_1^{PZ,BR} > -\mu > \psi\pi_2^{PZ,BR}$. One can show

$$
\begin{aligned}
\frac{\partial \pi_1^{PZ,BR}}{\partial \epsilon_1} &= \frac{\lambda(1-(M+aN)q + M_f(M+Mp(-1+q)-q+M(-1+q)q)\beta)\psi)}{den^{PZ,BR}} \\
&= \frac{\lambda num_1^{PZ,BR}}{den^{PZ,BR}} \\
\frac{\partial \pi_2^{PZ,BR}}{\partial \epsilon_1} &= \frac{\lambda(1-q)(M+aN+M_f(1-M(p+q))\beta)\psi}{den^{PZ,BR}} \\
&= \frac{\lambda num_2^{PZ,BR}}{den^{PZ,BR}}.
\end{aligned}
$$

where $den^{PZ,BR} = -M(aN(p+q-1)(\beta M_f(p+q)-2) + a\psi(\beta M_f(p-1) - \beta M_f q(p+q-1)+q) + (\beta M_f - 1)(p+q)(\beta M_f(p+q-1)-1)) + (aN+\beta M_f - 1)(aN(p+q-1)+\beta M_f(p+q-1)-1) - a\psi(aNq+\beta M_f q - 1) + M^2(\beta M_f - 1)(p+q-1)(\beta M_f(p+q-1)-1)$. Clearly $num_2^{PZ,BR} \geq 0$.

Solving for $\epsilon_1^{PZ,BR}$ and $\epsilon_2^{PZ,BR}$ such that $\psi\pi_1^{PZ,BR} = -\mu$ and $\psi\pi_2^{PZ,BR} = -\mu$, respectively, we have

$$
\epsilon_1^{PZ,BR} - \epsilon_2^{PZ,BR} = \frac{\psi(\eta_1\mu + \psi\lambda\epsilon_2)den^{PZ,BR}}{\lambda(1-q)\psi(M+aN+M_f\beta(1-M(p+q)))num_1^{PZ,BR}}
$$

There are four cases to consider. First, if $den^{PZ,BR} > 0$ and $num_1^{PZ,BR} > 0$, then $\partial\pi_1^{PZ,BR}/\partial\epsilon_1 > 0$, $\partial\pi_2^{PZ,BR}/\partial\epsilon_1 > 0$ and $\epsilon_1^{PZ,BR} > \epsilon_2^{PZ,BR}$ so $\epsilon_1 > \epsilon_1^{PZ,BR} > \epsilon_2^{PZ,BR} > \epsilon_1$ is necessary for PZ existence, but not possible. Second, if $den^{PZ,BR} < 0$ and $num_1^{PZ,BR} > 0$, then $\partial\pi_1^{PZ,BR}/\partial\epsilon_1 < 0$, $\partial\pi_2^{PZ,BR}/\partial\epsilon_1 < 0$ and $\epsilon_1^{PZ,BR} < \epsilon_2^{PZ,BR}$, so $\epsilon_1 < \epsilon_1^{PZ,BR} < \epsilon_2^{PZ,BR} < \epsilon_1$ is necessary for PZ existence, but not possible. In the third case, $den^{PZ,BR} < 0$ and $num_1^{PZ,BR} < 0$, which implies $\partial\pi_1^{PZ,BR}/\partial\epsilon_1 > 0$, $\partial\pi_2^{PZ,BR}/\partial\epsilon_1 < 0$, and $\epsilon_2^{PZ,BR} < \epsilon_1^{PZ,BR}$ so that $\epsilon_2^{PZ,BR} < \epsilon_1^{PZ,BR} < \epsilon_1$ is necessary and sufficient for PZ existence in this case. In the fourth case, $den^{PZ,BR} > 0$ and $num_1^{PZ,BR} < 0$. One can show:

$$
\begin{aligned}
den^{PZ,BR} &= \delta(p-1)(M+aN+M_f\beta(1-M(p+q))) \\
&+ num_1^{PZ,BR}\eta_1\psi^{-1} \\
\delta &= (M-1)(1-M_f\beta) + aN
\end{aligned}
$$

Hence, $den^{PZ,BR} > 0$ and $num_1^{PZ,BR} < 0$ implies $\delta < 0$, since $M+aN+M_f\beta(1-M(p+q)) > 0$. Now, for PZ to exist it must be the case that

$$
\begin{aligned}
\pi_1^{PZ,BR} - \pi_2^{PZ,BR} &= \frac{-\psi\delta\lambda\epsilon_1 - \psi\eta_1(\lambda\epsilon_2 + a\mu)}{den^{PZ,BR}} > 0 \\
&\rightarrow\ \epsilon_1 > \frac{\eta_1(\lambda\epsilon_2 + a\mu)}{-\delta\lambda} > 0
\end{aligned}
$$

where the last three inequalities hold if $\delta < 0$ and $den^{PZ,BR} > 0$. Also, if $den^{PZ,BR} > 0$ then $\epsilon_1 < \epsilon_2^{PZ,BR}$ is necessary for existence of PZ solution. If $\delta < 0$, we have:

$$\epsilon_2^{PZ,BR} = \frac{\lambda\epsilon_2\psi(aNp - a\psi + M(\beta M_f(-p(p+q-1)+q-1)+p)+\beta M_f p-1)}{\lambda(1-q)\psi(aN+\beta M_f(1-M(p+q))+M)}$$
$$- \quad \frac{\mu\eta_1(-(p+q)(aN+\beta M_f)+a(N+\psi)+M(p+q-1)(\beta M_f(p+q-1)-1)+\beta M_f+1)}{\lambda(1-q)\psi(aN+\beta M_f(1-M(p+q))+M)} < 0$$

Since $\epsilon_1 > 0 > \epsilon_2^{PZ,BR} > \epsilon_1$ does not hold, the PZ solution does not exist if $den^{PZ,BR} > 0$ and $num_1^{PZ,BR} < 0$.

Hence, a PZ solution can only exist if $den^{PZ,BR} < 0$ and $num_1^{PZ,BR} < 0$ and $\epsilon_1 > \epsilon_1^{PZ,BR}$. One can show:

$$\epsilon_1^{PZ,BR} - \epsilon^{PP,BR} = \frac{\psi^2 a(1-p)(aN+M+M_f\beta(1-M(p+q)))(\lambda\psi\epsilon_2+\mu\eta_1)}{-\lambda num_1^{ZP,BR} num_1^{PZ,BR}} > 0$$

if PZ exists (since this requires $num_1^{PZ,BR} < 0$). Therefore, if the PZ exists then $\epsilon_1 > \epsilon^{PP,BR}$ and hence the PP or ZP solution also exists.

The ZZ solution is given by

$$\hat{\pi}^{ZZ,BR} = (Q)^{-1}\begin{pmatrix}\lambda\epsilon_1+\lambda\sigma\mu \\ \lambda\epsilon_2+\lambda\sigma\mu\end{pmatrix}.$$

The ZZ solution exists if and only if $\psi\pi_j^{ZZ,BR} < -\mu$ for $j = 1, 2$. One can show

$$\begin{aligned}
\frac{\partial\pi_1^{ZZ,BR}}{\partial\epsilon_1} &= \frac{\lambda((1-(M+aN)q+M_f(M+Mp(-1+q)-q+M(-1+q)q)\beta)\psi)}{den^{ZZ,BR}} \\
&= \frac{\lambda num_1^{ZZ,BR}}{den^{ZZ,BR}} \\
\frac{\partial\pi_2^{ZZ,BR}}{\partial\epsilon_1} &= \frac{\lambda(1-q)(M+aN+M_f\beta(1-M(p+q)))\psi}{den^{ZZ,BR}} \\
&= \frac{\lambda num_2^{ZZ,BR}}{den^{ZZ,BR}}.
\end{aligned}$$

where $den^{ZZ,BR} = -\delta(1+aN+M_f\beta-(p+q)(aN+M_f\beta)+M(p+q-1)(M_f\beta(p+q-1)-1)$ and clearly $num_2^{ZZ,BR} \geq 0$. Solving for $\epsilon_1^{ZZ,BR}$ and $\epsilon_2^{ZZ,BR}$ such that $\psi\pi_1^{ZZ,BR} = -\mu$ and $\psi\pi_2^{ZZ,BR} = -\mu$, respectively, we have

$$\epsilon_1^{ZZ,BR} - \epsilon_2^{ZZ,BR} \quad = \quad \frac{den^{ZZ,BR}(\eta_1\mu+\lambda\psi\epsilon_2)}{\lambda(1-q)(M+aN+M_f\beta(1-M(p+q)))num_1^{ZZ,BR}}$$

There are four cases to consider. First, if $den^{ZZ,BR} > 0$ and $num_1^{ZZ,BR} < 0$ then $\partial\pi_1^{ZZ,BR}/\partial\epsilon_1 < 0$, $\partial\pi_2^{ZZ,BR}/\partial\epsilon_1 > 0$, $\epsilon_2^{ZZ,BR} > \epsilon_1^{ZZ,BR}$, so that ZZ existence requires $\epsilon_2^{ZZ,BR} > \epsilon_1 > \epsilon_1^{ZZ,BR}$. Second, if $den^{ZZ,BR} < 0$ and $num_1^{ZZ,BR} > 0$ then $\partial\pi_1^{ZZ,BR}/\partial\epsilon_1 < 0$, $\partial\pi_2^{ZZ,BR}/\partial\epsilon_1 <$

0, $\epsilon_2^{ZZ,BR} > \epsilon_1^{ZZ,BR}$, so that ZZ existence requires $\epsilon_1 > \epsilon_2^{ZZ,BR} > \epsilon_1^{ZZ,BR}$. In the third case, $den^{ZZ,BR} < 0$ and $num_1^{ZZ,BR} < 0$ which implies $\partial\pi_1^{ZZ,BR}/\partial\epsilon_1 > 0$, $\partial\pi_2^{ZZ,BR}/\partial\epsilon_1 < 0$, $\epsilon_1^{ZZ,BR} > \epsilon_2^{ZZ,BR}$, so that ZZ existence requires $\epsilon_1^{ZZ,BR} > \epsilon_1 > \epsilon_2^{ZZ,BR}$.

Now it can be shown that $\epsilon_2^{ZZ,BR} = \epsilon_2^{ZP}$ and

$$\epsilon_1^{ZZ,BR} - \epsilon^{PP,BR} = \frac{-\psi a(1-p)(M+aN+M_f\beta(1-M(p+q)))(\psi\lambda\epsilon_2+\eta_1\mu)}{\lambda num_1^{ZZ,BR}\eta_4} > 0$$

if $num_1^{ZZ,BR} < 0$, where

$$\begin{aligned}\eta_4 &= (1-q)(aN+\beta M_f(1-M(p+q))+M)+\\ &\quad a(\psi-N)+(1-M)(1-\beta M_f) \geq 0\end{aligned}$$

Since existence of ZZ in the first three cases only hinges on $\epsilon_1 > \epsilon_1^{ZZ,BR}$ if $num_1^{ZZ,BR} < 0$ and $\epsilon_2^{ZZ,BR} = \epsilon_2^{ZP,BR}$ it follows that the ZP or PP solution will exist if the ZZ solution exists in the first three cases.

In the fourth case, $den^{ZZ,BR} > 0$ and $num_1^{ZZ,BR} > 0$. One can show that:

$$\begin{aligned}den^{ZZ,BR} &= -\delta(-\delta+(2-p-q)(M+aN+M_f\beta(1-(p+q)M)))\\ num_1^{ZZ,BR} &= \psi(-\delta^{-1}den^{ZZ,BR}+\eta_5)\\ &= \psi(-\delta+(1-q)(M+aN+M_f\beta(1-(p+q)M)))\\ \eta_5 &= (p-1)(M+aN+M_f\beta(1-(p+q)M)) \leq 0\end{aligned}$$

Therefore, $\delta < 0$ if and only if the fourth case ($num_1^{ZZ,BR} > 0$ and $den^{ZZ,BR} > 0$) applies. In the fourth case, $\partial\pi_1^{ZZ,BR}/\partial\epsilon_1 > 0$, $\partial\pi_2^{ZZ,BR}/\partial\epsilon_1 > 0$, $\epsilon_1^{ZZ,BR} > \epsilon_2^{ZZ,BR} = \epsilon_2^{ZP,BR}$, so that ZZ existence requires $\epsilon_2^{ZP,BRE} > \epsilon_1$.

We conclude that a BRE exists if and only if

$$\epsilon_1 > \bar{\epsilon}_{BR} := \begin{cases} \min\left\{\epsilon^{PP,BR}, \epsilon_2^{ZP,BR}\right\}, & \text{if } \delta > 0 \\ -\infty, & \text{if } \delta < 0, \end{cases} \tag{34}$$

where $\epsilon^{PP,BR}$ and $\epsilon_2^{ZP,BR}$ are defined in (32) and (33), respectively.

■

Proof of Proposition 4

Suppose $\delta = (M-1)(1-M_f\beta)+aN < 0$. First, $\delta < 0$ implies $num_1^{PZ,BR} = \psi((1-q)(M+aN+M_f\beta(1-M(p+q)))-\delta) > 0$ and hence no PZ solution exists since $num_1^{PZ,BR} < 0$ is necessary for existence of PZ solution, as demonstrated in the proof of Proposition 3.

Also from the proof of Proposition 3, we know that the ZZ solution exists if and only if $\epsilon_1 < \epsilon_2^{ZP,BR} = \epsilon_2^{ZZ,BR}$. From the proof of Proposition 3, we have

$$\begin{aligned}
\epsilon_1^{ZZ,BR} - \epsilon_2^{ZZ,BR} &= \\
\epsilon_1^{ZZ,BR} - \epsilon_2^{ZP,BR} &= \frac{den^{ZZ,BR}(\eta_1\mu + \lambda\psi\epsilon_2)}{\lambda(1-q)(M+aN+M_f\beta(1-M(p+q)))num_1^{ZZ,BR}} > 0
\end{aligned}$$

where the last inequality follows from the fact that $den^{ZZ,BR} > 0$ and $num_1^{ZZ,BR} > 0$ if and only if $\delta < 0$. Further, we have the following from the proof of Proposition 3:

$$\epsilon^{PP,BR} - \epsilon_1^{ZZ,BR} = \frac{\psi a(1-p)(M+aN+M_f\beta(1-M(p+q)))(\psi\lambda\epsilon_2+\eta_1\mu)}{\lambda num_1^{ZZ,BR}\eta_4} > 0$$

where the last inequality follows from the fact that $num_1^{ZZ,BR} > 0$ if and only if $\delta < 0$. Therefore, $\epsilon^{PP,BR} > \epsilon_2^{ZP,BR}$ and the ZP solution exists if and only if $\epsilon^{PP,BR} > \epsilon_1 > \epsilon_2^{ZP,BR}$. Define $\epsilon^{ZP,BR} \equiv \epsilon_2^{ZP,BR}$. We conclude that the PP solution is the unique BRE when $\epsilon_1 > \epsilon^{PP,BR}$, the ZP solution is the unique BRE when $\epsilon^{PP,BR} > \epsilon_1 > \epsilon^{ZP,BR}$, and otherwise the ZZ solution is the unique solution. ■

Alternatively, one can show that $(M+1)(1-M_f\beta)+\lambda\sigma N < 0$ ensures completeness and coherence using techniques developed by Ascari and Mavroeidis (2022).[25]

Proof of Proposition 5

Consider Proposition 5. To assess E-stability of an REE, we express $Y^i = (Y_1^{i\prime}, Y_2^{i\prime})'$ as a function of agents' expectations, $\tilde{Y}^e = (Y_1^{e\prime}, Y_2^{e\prime})'$:

$$\begin{aligned}
Y^{PP}(\tilde{Y}^e) &= \begin{pmatrix} pA_P & (1-p)A_P \\ (1-q)A_P & qA_P \end{pmatrix} \tilde{Y}^e + \Gamma^{PP}, \\
Y^{ZP}(\tilde{Y}^e) &= \begin{pmatrix} pA_Z & (1-p)A_Z \\ (1-q)A_P & qA_P \end{pmatrix} \tilde{Y}^e + \Gamma^{ZP}, \\
Y^{PZ}(\tilde{Y}^e) &= \begin{pmatrix} pA_P & (1-p)A_P \\ (1-q)A_Z & qA_Z \end{pmatrix} \tilde{Y}^e + \Gamma^{PZ}, \\
Y^{PP}(\tilde{Y}^e) &= \begin{pmatrix} pA_Z & (1-p)A_Z \\ (1-q)A_Z & qA_Z \end{pmatrix} \tilde{Y}^e + \Gamma^{ZZ},
\end{aligned}$$

where Γ^i collect terms that do not depend on beliefs, $\tilde{Y}^e$. It immediately follows that

$$\begin{aligned}
DT_{Y^{PP}} &= K \otimes A_P - I, \\
DT_{Y^{ZP}} &= \begin{pmatrix} pA_Z & (1-p)A_Z \\ (1-q)A_P & qA_P \end{pmatrix} - I, \\
DT_{Y^{PZ}} &= \begin{pmatrix} pA_P & (1-p)A_P \\ (1-q)A_Z & qA_Z \end{pmatrix} - I, \\
DT_{Y^{ZZ}} &= K \otimes A_Z - I.
\end{aligned}$$

[25] Results available on request.

REE i is E-stable if the real parts of the eigenvalues of DT_{Y^i} are negative for $i = PP, ZP, PZ, ZZ$. It is straightforward to show that the real parts of the eigenvalues of $DT_{Y^{PP}}$ are negative and the real parts of the eigenvalues of $DT_{Y^{ZZ}}$ are positive. Therefore, the PP solution is always E-stable and the ZZ solution is always E-unstable. We now proceed to show that only one REE can be E-stable in two steps.

First, we show that the PP solution does not exist if the ZP solution is E-stable. Because $DT_{Y^{ZP}}$ is a 4 by 4 matrix the following condition is necessary for E-stability of the ZP solution:

$$Det(DT_{Y^{ZP}}) = \frac{a}{1+a\psi} den^{ZP} > 0$$

where den^{ZP} is defined in the proof of Proposition 1. Therefore, E-stability of the ZP solution implies $den^{ZP} > 0$. Furthermore, since $den^{ZP} > 0$ implies $\epsilon^{PP} > \epsilon_2^{ZP}$ (see proof of Proposition 1), E-stability of the ZP solution implies $\epsilon^{PP} > \epsilon_2^{ZP}$, where $\epsilon^{PP}, \epsilon_2^{ZP}$ are defined in the proof of Proposition 1. Also from the Proposition 1 proof, if $\epsilon^{PP} > \epsilon_2^{ZP}$ then $\epsilon_1 > \epsilon^{PP}$ is necessary for existence of PP and $\epsilon_1 < \epsilon^{PP}$ is necessary for existence of ZP. It follows that the E-stability and existence of the ZP solution precludes existence of the PP solution.

Second, we show that the PZ solution is never E-stable. The following condition is necessary for E-stability of the PZ solution:

$$Det(DT_{Y^{PZ}}) = \frac{1}{1+a\psi} den^{PZ} > 0$$

where den^{PZ} is defined in the proof of Proposition 1. Therefore, $den^{PZ} > 0$ is necessary for the PZ solution to be E-stable. From the proof of Proposition 1, $den^{PZ} < 0$ is necessary for existence of the PZ solution. We conclude that the PZ solution can never be E-stable.

In sum, if the PP solution exists it is E-stable. If the ZP solution exists and is E-stable then the PP solution does not exist. The ZZ and PZ solutions are never E-stable. ■

Proof of Proposition 6

Consider (14)-(19), let Y_t^e denote the subjective forecast of Y_t implied by a given forecasting model, and assume that agents observe ϵ_t and Y_t when forecasting at time t. Furthermore, to deal with possible multiplicity of time-t temporary equilibria, i.e. a time-t solution of (1)-(3) given forecasts and ϵ_t with binding ZLB ($s_t = 0$) and a solution with slack ZLB constraint ($s_t = 1$), we simply assume that ϵ_t determines s_t. E.g. if $\epsilon_t = \epsilon_j$, $s_t = 0$ and $s_t = 1$ are both possible in temporary equilibrium, and $s_k = 0$ for some $k < t$ such that $\epsilon_k = \epsilon_j$, then we select the temporary equilibrium characterized by $s_t = 0$.

(i) First consider (14)-(18).

Case. $k = 0$.

If $k=0$ and expectations are formed under PLMs (14)-(18) then Y_t^e follows a 2-state process: $Y_t^e = Y_j^e$ if $\epsilon_t = \epsilon_j$. Further, $\hat{E}_t Y_{t+1} = Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t)Y_1^e + (1 - Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t))Y_2^e$ is a 2-state process. Therefore, if $k=0$ then $Y_j^e = Y_j$ is necessary and sufficient for the agents to have self-confirming beliefs under the PLMs (14)-(18). These self-confirming beliefs imply: $\hat{E}_t Y_{t+1} = Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t)Y_1 + (1 - Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t))Y_2$. Substituting $\hat{E}_t Y_{t+1}$ into the model and solving for Y_1 and Y_2 straightforwardly implies that Y_1, Y_2 is an REE. Hence, beliefs formed under (14)-(18) with $k=0$ are only self-confirming if an REE exists.

Case. $k = 1$**.**

Beliefs are only self-confirming under the PLMs (14)-(18) with $k=1$ if $Y_j^e = E(Y_t|\epsilon_{t-1} = \epsilon_j)$ for $j = 1, 2$ where E denotes the true mathematical expectation operator. Further, $\hat{E}_t Y_{t+1}$ formed under (14)-(18) follows a 2-state process and therefore temporary equilibrium Y_t follows a 2-state process: Y_j, where Y_j is the actual equilibrium value of Y given Y_j^e and $\epsilon_t = \epsilon_j$ for $j = 1, 2$. It follows that beliefs are self-confirming if and only if $E(Y_t|\epsilon_{t-1} = \epsilon_1) = pY_1 + (1-p)Y_2$ and $E(Y_t|\epsilon_{t-1} = \epsilon_2) = (1-q)Y_1 + qY_2$. Therefore, if agents have self-confirming beliefs under PLMs (14)-(18) with $k=1$ then $\hat{E}_t Y_{t+1} = Y_{t+1}^e = pY_1 + (1-p)Y_2$ if $\epsilon_t = \epsilon_1$ and $\hat{E}_t Y_{t+1} = Y_{t+1}^e = (1-q)Y_1 + qY_2$ otherwise. Substituting $\hat{E}_t Y_{t+1}$ into the model reveals that Y_1, Y_2 is an REE.

(ii) Now consider (19). If agents observe time$-t$ information when forming time$-t$ expectations then

$$\hat{E}_t z_{t+1} = a_z + b_z z_t \tag{35}$$

where $z \in \{\pi, x\}$. We say that (19) yields self-confirming beliefs if agents correctly understand the mean and serial correlation of x and π, i.e., $a_z = (1-b_z)E(z_t)$, $b_z = (E(z_t z_{t-1}) - a_z E(z_t))/E(z_{t-1}^2)$. Given fixed a_z, b_z and expectations (35), Y_t is a 2-state process: Y_j, where Y_j is the actual value of Y_t given expectations and $\epsilon_t = \epsilon_j$. This implies $E(z_t z_{t-1}) = q\bar{q}z_2^2 + ((1-q)\bar{q} + (1-p)(1-\bar{q}))z_1 z_2 + p(1-\bar{q})z_1^2$, $E(z_t^2) = \bar{q}z_2^2 + (1-\bar{q})z_1^2$, $E(z_t) = \bar{q}z_2 + (1-\bar{q})z_1$. Solving for a_z and b_z and substituting these values into (35) yields:

$$\begin{aligned}
\hat{E}_t(z_{t+1}|\epsilon_t = \epsilon_1) &= pz_1 + (1-p)z_2 \\
\hat{E}_t(z_{t+1}|\epsilon_t = \epsilon_2) &= qz_2 + (1-q)z_1
\end{aligned}$$

Substituting expectations into the model and solving for z_1, z_2 straightforwardly reveals that z_1 and z_2 must be an REE. Therefore, (19) is not consistent with a non-rational equilibrium of an incoherent model if agents have current information.[26]

We conclude that if beliefs formed under PLMs (14)-(18) are self-confirming then an REE exists. Consequently, (14)-(18) are not consistent with any non-rational equilibrium of an incoherent model.

■

[26] Note that our result is related to Evans and McGough (2018a), who study E-stability of REE in linear models when agents cannot observe exogenous shocks.

Proof of Proposition 7

To assess E-stability of each RPE, we express the RPE unconditional mean of inflation and output as a function of agents' expectations, Y^e:

$$\begin{aligned}
\bar{Y}^{PP}(Y^e) &= A_P Y^e + \bar{\Gamma}^{PP}, \\
\bar{Y}^{ZP}(Y^e) &= (\bar{q}A_P + (1-\bar{q})A_Z)\, Y^e + \bar{\Gamma}^{ZP}, \\
\bar{Y}^{PZ}(Y^e) &= ((1-\bar{q})A_P + \bar{q}A_Z)\, Y^e + \bar{\Gamma}^{PZ}, \\
\bar{Y}^{ZZ}(Y^e) &= A_Z Y^e + \bar{\Gamma}^{ZZ},
\end{aligned}$$

where $\bar{\Gamma}^i$ collect terms that do not depend on beliefs, Y^e. It immediately follows that

$$\begin{aligned}
DT_{\bar{Y}^{PP}} &= A_P - I, \\
DT_{\bar{Y}^{ZP}} &= \bar{q}A_P + (1-\bar{q})A_Z - I, \\
DT_{\bar{Y}^{PZ}} &= (1-\bar{q})A_P + \bar{q}A_Z - I, \\
DT_{\bar{Y}^{ZZ}} &= A_Z - I.
\end{aligned}$$

It is straightforward to show that the real parts of the eigenvalues of $DT_{\bar{Y}^{PP}}$ are negative and the real parts of the eigenvalues of $DT_{\bar{Y}^{ZZ}}$ are always positive. Therefore, the PP RPE is always E-stable and the ZZ RPE is never E-stable.

The ZP RPE is E-stable if and only if

$$\begin{aligned}
tr(DT_{\bar{Y}^{ZP}}) &= \beta + a - \frac{a\bar{q}\psi(\beta + a + 1)}{a\psi + 1} - 1 < 0 \\
Det(DT_{\bar{Y}^{ZP}})) &= \frac{\bar{q}a(a\psi + \psi)}{a\psi + 1} - a > 0
\end{aligned}$$

where $tr(B)$ denotes the trace of matrix B. We have $tr(DT_{\bar{Y}^{ZP}}) < 0 < Det(DT_{\bar{Y}^{ZP}})$ if and only if $\bar{q}(1+a)\psi - 1 - a\psi > 0$. From the proofs of Propositions 1 and 2:

$$\begin{aligned}
\epsilon^{PP,RPE} - \epsilon_2^{ZP,RPE} &= v(\bar{q}(1+a)\psi - 1 - a\psi)) \\
v &= \frac{a(\lambda\epsilon_2\psi + a\mu(\psi - 1))}{(1-\bar{q})\lambda\psi(a+1)(a(\psi - \bar{q}) + 1 - \bar{q})} > 0
\end{aligned}$$

Therefore, if the ZP RPE is E-stable then $\epsilon^{PP,RPE} > \epsilon_2^{ZP,RPE}$ and the condition for PP existence becomes $\epsilon_1 > \epsilon^{PP,RPE}$ and the condition for ZP existence becomes $\epsilon^{PP,RPE} > \epsilon_1 > \epsilon_2^{ZP,RPE}$ as demonstrated in the proofs of Propositions 1 and 2. Hence, if the ZP RPE exists and is E-stable then the PP solution does not exist.

Next consider the PZ solution. The PZ solution is E-stable if and only if

$$\begin{aligned}
tr(DT_{\bar{Y}^{PZ}}) &= \frac{\beta - 2a\psi + a - 1}{a\psi + 1} + \frac{\bar{q}\,(\beta a\psi + a^2\psi + a\psi)}{a\psi + 1} < 0 \\
Det(DT_{\bar{Y}^{PZ}})) &= -\frac{a(1-\psi)}{a\psi + 1} - \frac{a\bar{q}(a\psi + \psi)}{a\psi + 1} > 0
\end{aligned}$$

which holds if and only if $0 < \psi - 1 - \bar{q}\psi(1+a) = den^{PZ,RPE}a^{-1}$ where $den^{PZ,RPE}$ is equal to den^{PZ} defined in the Proposition 1 proof when $q = \bar{q}$ and $p = 1 - \bar{q}$. From the proof of Proposition 2, the PZ RPE only exists if $den^{PZ,RPE} < 0$. Hence the PZ RPE is never E-stable.

Therefore, the PP RPE is the only E-stable RPE solution when $\epsilon_1 > \epsilon^{PP,RPE}$, and the ZP RPE is the only E-stable RPE solution when $\epsilon^{PP,RPE} > \epsilon_1 > \epsilon_2^{ZP,RPE}$. It follows that a unique E-stable RPE solution exists when $\epsilon_1 > \bar{\epsilon}_{RPE}$. ■

Proof of Proposition 8

Consider (1)-(3) and suppose that expectations evolve according to

$$\hat{E}_t\left(y_{t+1}\right) = \hat{E}_{t-1}\left(y_t\right) + g_{y,t}\left(y_{t-k} - \hat{E}_{t-1}\left(y_t\right)\right) \tag{36}$$

with $k = 0, 1$, $g_{y,t} \in [0,1]$, given some initial condition $\hat{E}_0\left(y_1\right) = a_{y0}$. To demonstrate coherence and completeness, we begin by showing that there is a unique solution for π_0, x_0, R_0 given initial expectations and exogenous shocks. Substituting (1), (3), $E_0 y_1 = a_{y0}$ into (2) we have

$$\pi_0 = -\lambda\sigma \max\{\psi\pi_t, -\mu\} + \lambda\left(a_{x0} + \sigma a_{\pi 0} + \epsilon_0\right) + \beta a_{\pi 0} + u_0 \tag{37}$$

Given $a_{x0}, a_{\pi 0}, u_0, \epsilon_0$ there is a unique solution for π_0 obtained from (37) if $\psi > 0$. The unique solution for R_0, x_0 is obtained from the Phillips curve (2) and interest rate rule (3). Therefore, for $t \geq 0$, we have:

$$\hat{E}_{t-1}\pi_t = \beta^{-1}\left(\pi_{t-1} - \lambda x_{t-1} - u_{t-1}\right) \tag{38}$$

$$\hat{E}_{t-1}x_t = (1 + \sigma\lambda\beta^{-1})x_{t-1} - \sigma\beta^{-1}\pi_{t-1} + \sigma R_{t-1} - \epsilon_{t-1} + \sigma\beta^{-1}u_{t-1} \tag{39}$$

Substituting (38)-(39) into (36), and then (36) into (1)-(3), the model can be written in the form

$$A_{11}Y_{1t} + A_{12}Y_{2t} + A_{12}^*Y_{2t}^* = B_{10}X_{0t} + B_{11}Y_{t-1} + B_{11}^*Y_{2t-1}^* + \epsilon_{1t} \tag{40}$$

$$A_{21}Y_{1t} + A_{22}Y_{2t} + A_{22}^*Y_{2t}^* = B_{20}X_{0t} + B_{21}Y_{t-1} + B_{21}^*Y_{2t-1}^* + \epsilon_{2t} \tag{41}$$

$$Y_{2t} = \max\{Y_{2t}^*, -\mu\} \tag{42}$$

for $t > 0$, where $Y_{1t} = (x_t, \pi_t)'$, $Y_{2t}^* = R_t^* = \psi\pi_t$, $Y_t = (Y_{1t}', Y_{2t})'$, and X_{0t} are exogenous shocks.

Case. $k = 0$.

Under contemporaneous information, we have

$$A_{11} = \begin{pmatrix} 1 - g_x & -\sigma g_x \\ -\lambda & 1 - \beta g_\pi \end{pmatrix} \qquad A_{12} = \begin{pmatrix}\sigma & 0\end{pmatrix}' \qquad A_{12}^* = \begin{pmatrix}0 & 0\end{pmatrix}'$$

$$A_{21} = \begin{pmatrix}0 & -\psi\end{pmatrix} \qquad A_{22} = 0 \qquad A_{22}^* = 1$$

Following Proposition 1 of Mavroeidis (2021), we have coherence and completeness if and only if

$$1 + \frac{\sigma\lambda\psi}{(1 - g_{x,t})(1 - \beta g_{\pi,t}) - \lambda\sigma g_{\pi,t}} > 0$$

It is easy to see that we only have coherence and completeness under the Taylor principle ($\psi > 1$) if $g_{x,t}$, $g_{\pi,t}$, or λ are not too large. For instance, as $\lambda \to \infty$ we have the coherence and completeness condition: $g_{\pi,t} > \psi$, which only holds if we violate the Taylor Principle. Further, if agents have "naive" expectations (i.e. if $g_{x,t} = g_{\pi,t} = 1$) then we have the coherence condition: $\psi < 1$. We need to restrict gain parameters and the slope of the Phillips curve to have coherence under the Taylor principle, adaptive expectations and contemporanous information ($k = 0$).

Case. $k = 1$

Under lagged information, we have

$$A_{11} = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} \sigma & 0 \end{pmatrix}' \qquad A_{12}^* = \begin{pmatrix} 0 & 0 \end{pmatrix}'$$
$$A_{21} = \begin{pmatrix} 0 & -\psi \end{pmatrix} \qquad A_{22} = 0 \qquad A_{22}^* = 1$$

Following Proposition 1 of Mavroeidis (2021), we have coherence and completeness if and only if

$$1 + \sigma\lambda\psi > 0$$

which holds for all $\psi > 0$. ■

RPE under Infinite Horizon Learning

Consider the following infinite horizon New Keynesian model:[27]

$$x_t = -\sigma r_t + \hat{E}_t \sum_{T \geq t} \beta^{T-t} \left((1-\beta)x_{T+1} + \sigma\pi_{T+1} - \sigma\beta r_{T+1} + \epsilon_T\right) \tag{43}$$

$$\pi_t = \lambda x_t + \hat{E}_t \sum_{T \geq t} (\xi\beta)^{T-t} \left(\xi\beta\lambda x_{T+1} + (1-\xi)\beta\pi_{T+1}\right) \tag{44}$$

$$i_t = \max\{\psi\pi_t, -\mu\} \tag{45}$$

where $\lambda = (1-\xi\beta)(1-\xi)/\xi$. Under infinite horizon learning, agents need to forecast the paths of the nominal interest rate and the shock, in addition to the paths of inflation and output.

[27]See Eusepi et al. (2021b) for a recent derivation of the model (43)-(45). Note that this model collapses to the standard 3-equation model in our paper if we impose rational expectations. Consequently, a stochastic process for inflation, output and the interest rate is an REE of (43)-(45) if and only if said stochastic process is an REE of (1)-(3).

Consistent with the RPE studied in section 3.2, we assume that agents set endogenous and exogenous variable forecasts at all horizons equal to the unconditional means of each variable (i.e. $\hat{E}_t z_T = E(z_T)$ for all $T > t$ and $z = \pi, x, i, \epsilon$). We have:

$$\begin{aligned} E(\pi) &= E\left(\lambda x_t + \hat{E}_t \sum_{T\geq t} (\xi\beta)^{T-t} \left(\xi\beta\lambda x_{T+1} + (1-\xi)\beta\pi_{T+1}\right)\right) \\ \implies E(x) &= \frac{1-\beta}{\lambda} E(\pi) \end{aligned}$$

and

$$\begin{aligned} \pi_t &= \lambda x_t + \sum_{T\geq t} (\xi\beta)^{T-t} \left(\xi\beta\lambda E(x) + (1-\xi)\beta E(\pi)\right) \\ &= \lambda x_t + \beta E(\pi) \\ \implies x_t &= \lambda^{-1}(\pi_t - \beta E(\pi)) \end{aligned}$$

Substituting for x_t and also for expectations in (43) gives an expression for RPE inflation:

$$\begin{aligned} x_t &= \lambda^{-1}(\pi_t - \beta E(\pi)) \\ &= -\sigma r_t + \epsilon_t + \sum_{T\geq t} \beta^{T-t} \left((1-\beta)E(x) + \sigma E(\pi) - \sigma\beta E(r) + \beta E(\epsilon_T)\right) \\ \implies \pi_t &= -\lambda\sigma r_t + \lambda\epsilon_t + (1 + \frac{\lambda\sigma}{1-\beta})E(\pi) - \frac{\beta\lambda\sigma}{1-\beta}E(r) + \frac{\lambda\beta}{1-\beta}E(\epsilon) \end{aligned}$$

Let $\hat{z} = (z_1, z_2)'$ denote the vector of state-contingent RPE values of z for any variable, z. Note that $E(z) = \bar{q}z_2 + (1-\bar{q})z_1$. Then the infinite horizon RPE solution for inflation satisfies:

$$\hat{\pi} = \left(1 + \frac{\lambda\sigma}{1-\beta}\right)\tilde{K}\hat{\pi} - \lambda\sigma\left(I - \beta\tilde{K}\right)^{-1}\hat{r} + \lambda\left(I - \beta\tilde{K}\right)^{-1}\hat{\epsilon}$$

where I is the identity matrix and

$$\tilde{K} = \begin{pmatrix} 1-\bar{q} & \bar{q} \\ 1-\bar{q} & \bar{q} \end{pmatrix}$$

Premultiplying both sides of the last equation by $\left(I - \beta\tilde{K}\right)$ and rearranging yields

$$\left(I - (1+\lambda\sigma)\tilde{K}\right)\hat{\pi} = -\lambda\sigma\hat{r} + \lambda\hat{\epsilon} \tag{46}$$

From the proof of Proposition 1 and 2, it can be seen that any solution of (46) is also a RPE of (1)-(3). Hence, the infinite horizon model (43)-(45) admits the same RPE as (1)-(3), and therefore an incoherent model can admit RPE under infinite horizon learning under some conditions. The result is summarized in the following proposition.

Proposition 10 *Consider (43)-(45) and suppose $\epsilon_2 \geq 0$. Then:*

i. An infinite-horizon RPE exists if and only if $\epsilon_1 > \bar{\epsilon}_{RPE}$.

ii. $\bar{\epsilon}_{REE} > \bar{\epsilon}_{RPE}$ if and only if $p + q - 1 > 0$.

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