Monetary Policy, Expectations and Commitment*

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Abstract

Commitment in monetary policy leads to equilibria that are superior to those from optimal discretionary policies. A number of interest rate reaction functions and instrument rules have been proposed to implement or approximate commitment policy. We assess these optimal reaction functions and instrument rules in terms of whether they lead to an RE equilibrium that is both locally determinate and stable under adaptive learning by private agents. A reaction function that appropriately depends explicitly on private expectations performs particularly well on both counts.

Key words: Commitment, interest-rate setting, adaptive learning, stability, determinacy.

JEL classification: E52, E31, D84.

1 Introduction

Many recent models of monetary policy emphasize the importance of forward looking aspects of the economy, in which expectations of private agents significantly influence the economic outcome. If expectations about the future are part of the equilibrating mechanisms in the economy it is well known that standard intertemporal optimization of economic policy by the government is in general subject to the problem of time inconsistency, as was first pointed out by (Kydland and Prescott 1977). Lack of time consistency means that a policymaker has incentives to deviate, in later periods, from the optimal plan obtained in the first period. In contrast, discretionary policies are

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obtained through policy optimization separately in each period and are time consistent, but typically the resulting sequence of discretionary policy decisions will not lead to the overall intertemporal optimum of the economy. The losses from discretionary policies can be quantitatively significant, and this has provided the impetus for finding ways to achieve the optimum or at least to improve the outcome. The discussion is often framed as the question of rules vs. discretion in policy making.

The tensions between the non-optimality arising from discretionary policy relative to full optimality and the time inconsistency of fully optimal policies have also been considered in the context of monetary policy. The first papers focused attention on the so-called inflation bias that arises from overambitious government objectives with respect to aggregate output, see the contributions by (Barro and Gordon 1983a), (Barro and Gordon 1983b) and the subsequent literature. In most recent work, the problem of inflation bias has received less attention, as it is often assumed that the policymaker does not have overambitious goals. Nevertheless, the issue of commitment vs. discretion still prevails, since discretion leads to what is called a "stabilization bias" and there are gains to commitment, see (Woodford 1999a), (Woodford 1999b), (Svensson and Woodford 2003) and (McCallum and Nelson 2000) among others.[1]

It has been suggested by (Woodford 1999a) and (Woodford 1999b) that, to implement the commitment solution, monetary policy making ought be based on the *timeless perspective*. This concept is a rule-based policy that is obtained by respecting the optimality conditions from the full intertemporal optimization under commitment, except for the current decision-making period. In other words, such a rule stipulates that the policymaker follows "the pattern of behavior to which it would have wished to commit itself at a date far in the past" (p.293 in (Woodford 1999a)). Recent work has shown that the gains from committing to this policy, relative to the discretionary policies, can be significant, see (McCallum and Nelson 2000). In this paper we will adopt the timeless perspective formulation and refer to the corresponding optimal monetary policy with commitment as the "commitment solution."

Almost all of the recent literature on monetary policy, including all of the references above, has been conducted under the hypothesis of rational expectations (RE). However, this may not be an innocuous assumption as was explicitly shown by (Bullard and Mitra 2002) and (Evans and Honkapohja 2003). In the analysis of economic policy the assumption of RE should not be taken for granted, since expectations can be out of equilibrium, at least for a period of time, as a result of exogenous events such as structural shifts in the economy. Economic policies should be designed to avoid instabilities that can arise from expectational errors and the corrective behavior of economic agents in the face of such errors.

The possibility of temporary errors in forecasting, and the consequent correction mechanisms, have been widely studied in recent research using the adaptive learning approach.[2] For monetary policy (Evans and Honkapohja 2003) show that certain standard

[1]See (Clarida, Gali, and Gertler 1999) for a recent survey on this literature on monetary policy.

[2](Evans and Honkapohja 2001) provides an extensive treatise on the analysis of adaptive learning and its implications in macroeconomics. (Evans and Honkapohja 1999), (Evans and Honkapohja 1995),

forms of optimal discretionary interest-rate setting by the central bank can lead to instability as economic agents unsuccessfully try to correct their forecast functions over time, with the result that the economy may not converge to the desired rational expectations equilibrium (REE). They also propose a new way for implementing optimal discretionary policy that always leads to stability under learning. (Bullard and Mitra 2002) consider the stability of equilibria when monetary policy is conducted using some variant of the Taylor interest-rate rule. Bullard and Mitra argue that monetary policy making should take into account the learnability constraints, which imply constraints on the parameters of policy behavior.[3] A related concern addressed by (Bernanke and Woodford 1997), (Woodford 1999b), (Svensson and Woodford 2003) and others is that it is desirable for policy rules to yield determinacy, i.e. locally unique REE, to ensure that there are no nearby suboptimal REE.

The research on adaptive learning and monetary policy has so far considered the performance of discretionary optimal policies or ad hoc interest-rate rules. A partial exception is (Evans and Honkapohja 2003). However, they restrict attention to limited forms of commitment for which the REE takes the same form (but with different parameter values) as the optimal discretionary equilibrium. Learnability of optimal policy with commitment has not been analyzed thus far. In this paper our goal is to study whether optimal monetary policy by the central bank is conducive to long-run convergence of private expectations to the optimal REE.

On intuitive grounds one might think that commitment favors stability under learning by leading to more forecastable dynamics of the economy than when policy is reoptimized every period. We will argue that while this can indeed be the case, stability depends critically on the way the monetary policy with commitment is implemented. Certain standard forms of central bank reaction functions or instrument rules that approximate the policy target do not or do not always provide stability under learning. However, there is another implementation, depending explicitly on private expectations, that performs well in this respect. We propose a specific implementation of optimal policy that always leads to both determinacy and stability under learning.

2 The Model

We use a linearized model that is very commonly employed in the literature, see (Clarida, Gali, and Gertler 1999) for this particular formulation and references to the literature. The original nonlinear framework is based on a representative consumer, a continuum of firms producing differentiated goods under monopolistic competition and subject to constraints on the frequency of price changes, as originally suggested by (Calvo 1983).

(Marimon 1997), (Sargent 1993) and (Sargent 1999) provide surveys of the field.

[3]Other papers on monetary policy using the learning approach include (Bullard and Mitra 2001), (Mitra 2003), (Honkapohja and Mitra 2001), (Honkapohja and Mitra 2003) and (Carlstrom and Fuerst 2001). A predecessor to this work is (Howitt 1992), though he did not use the New Keynesian framework.

The behavior of the private sector is described by two equations

$$x_t = -\varphi(i_t - E_t^* \pi_{t+1}) + E_t^* x_{t+1} + g_t, \tag{1}$$

which is the "IS" curve derived from the Euler equation for consumer optimization, and

$$\pi_t = \lambda x_t + \beta E_t^* \pi_{t+1} + u_t, \tag{2}$$

which is the price-setting rule for the monopolistically competitive firms. Appendix A.1.1 discusses further the interpretation of (1) and (2).

Here x_t and π_t denote the output gap and inflation for period t, respectively. i_t is the nominal interest rate, expressed as the deviation from the steady state real interest rate. The determination of i_t will be discussed below. $E_t^* x_{t+1}$ and $E_t^* \pi_{t+1}$ denote the private sector expectations of the output gap and inflation next period. Since our focus is on learning behavior, these expectations need not be rational (E_t without $*$ denotes RE). The parameters φ and λ are positive and β is the discount factor so that $0 < \beta < 1$.

The shocks g_t and u_t are assumed to be observable and follow

$$\begin{pmatrix} g_t \\ u_t \end{pmatrix} = F \begin{pmatrix} g_{t-1} \\ u_{t-1} \end{pmatrix} + \begin{pmatrix} \tilde{g}_t \\ \tilde{u}_t \end{pmatrix}, \tag{3}$$

where

$$F = \begin{pmatrix} \mu & 0 \\ 0 & \rho \end{pmatrix},$$

$0 < |\mu| < 1$, $0 < |\rho| < 1$ and $\tilde{g}_t \sim iid(0, \sigma_g^2)$, $\tilde{u}_t \sim iid(0, \sigma_u^2)$ are independent white noise. g_t represents shocks to government purchases and/or potential output. u_t represents any cost-push shocks to marginal costs other than those entering through x_t.[4] The u_t shock is important for the policy issues since the g_t shock can be fully offset by appropriate interest-rate setting. For simplicity, we assume throughout the paper that μ and ρ are known (if not, they could be estimated).

Assume RE for the moment. Monetary policy is derived from minimization of a quadratic loss function

$$E_t \sum_{s=0}^{\infty} \beta^s (\pi_{t+s}^2 + \alpha x_{t+s}^2). \tag{4}$$

This type of optimal policy is often called "flexible inflation targeting" in the current literature, see e.g. (Svensson 1999) and (Svensson 2003). α is the relative weight on the output target and pure inflation targeting would be the case $\alpha = 0$. Note that, first, the policymaker is assumed to have the same discount factor as the private sector and, second, the target value of the output gap is set at zero implying that the classical problem of inflation bias does not arise.[5] Thus the target for output is set at its efficient

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[4] For possible interpretations of the u_t shock, see (Clarida, Gali, and Gertler 1999), (Erceg, Henderson, and Levin 2000) and (Woodford 2003), Chapter 6. The latter interpretation is suitable for rationalizing the policy-maker's loss function given below.

[5] See e.g. (Clarida, Gali, and Gertler 1999) for a discussion of the inflationary bias in the context of this kind of model.

level. For brevity, the inflation target is also set at zero (introducing non-zero targets would not change the conclusions of our analysis regarding determinacy and stability under learning). We treat the policymaker's preferences as exogenously given. It is also well known, see (Rotemberg and Woodford 1999), that the quadratic loss function (4) can be viewed as an approximation of the utility function of the representative consumer.[6]

The full intertemporal optimum under RE, usually called the commitment solution, is obtained by maximizing (4) subject to (2) for all periods $t, t+1, t+2, \ldots$ The first order conditions are given by[7]

$$2\alpha x_{t+s} + \lambda\omega_{t+s} = 0 \text{ for } s = 0, 1, 2, \ldots \tag{5}$$

$$2\pi_{t+s} + \omega_{t+s-1} - \omega_{t+s} = 0 \text{ for } s = 1, 2, \ldots \tag{6}$$

and

$$2\pi_t - \omega_t = 0. \tag{7}$$

Here ω_{t+s}, $s = 0, 1, 2, ...$, denote Lagrange multipliers associated with the constraints (2) for each time period.

The time inconsistency of the commitment solution is evident from (7), since this places a requirement that is specific to the current period and is different from the corresponding requirement (6) for later periods. The current decision places a constraint for the future periods that is non-optimal when the future periods actually arrive. A planned re-optimization for such a period would lead to violation of (6), so that the optimal plan would not be continued.

As noted in the Introduction, the timeless perspective resolution to the problem of the time inconsistency of optimal policy is that the policymaker should respect the optimality conditions above, except for the current period when the optimization is done. In our context this amounts to using (5) and (6) also for the current period (and neglecting (7)). This yields the *commitment optimality condition*[8]

$$\lambda\pi_t = -\alpha(x_t - x_{t-1}). \tag{8}$$

We remark that (8) is sometimes called a "specific targeting rule" in the literature.

We next compute the REE of interest. It can be shown that the dynamic system in x_t and π_t defined by (2) and (8) has a unique nonexplosive solution. This solution can be expressed as a linear function of the state variables x_{t-1} and u_t and is known as the "minimal state variable" (MSV) solution (see (McCallum 1983)). To obtain it explicitly

[6]It should be noted that, like much of the literature on monetary policy, we do not explicitly introduce the budget constraint of the government to the analysis. This is justified by assuming that fiscal policy is set "passively" in the sense of (Leeper 1991) and ensures that the intertemporal budget constraint of the government is satisfied.

[7]See (Woodford 1999b), Section 3.1 for the derivation of the first order conditions using Lagrange multipliers in a very similar setup.

[8](Clarida, Gali, and Gertler 1999), p.1681 and (Woodford 1999a), appendix also derive this optimality condition.

we use the method of undetermined coefficients, expressing the REE in the form[9]

$$
\begin{aligned}
x_t &= b_x x_{t-1} + c_x u_t, \qquad (9)\\
\pi_t &= b_\pi x_{t-1} + c_\pi u_t. \qquad (10)
\end{aligned}
$$

Under RE we have $E_t\pi_{t+1} = b_\pi(b_x x_{t-1} + c_x u_t) + c_\pi \rho u_t$, so that substituting into (2) and (8) yields

$$
\begin{aligned}
\lambda\pi_t &= -\alpha[(b_x - 1)x_{t-1} + c_x u_t],\\
\pi_t &= \lambda(b_x x_{t-1} + c_x u_t) + \beta[b_\pi(b_x x_{t-1} + c_x u_t) + c_\pi \rho u_t] + u_t.
\end{aligned}
$$

This implies the system of equations

$$
\begin{aligned}
b_\pi &= -\frac{\alpha}{\lambda}(b_x - 1),\\
c_\pi &= -\frac{\alpha}{\lambda}c_x,\\
b_\pi &= \lambda b_x + \beta b_\pi b_x,\\
c_\pi &= \lambda c_x + \beta(b_\pi c_x + c_\pi \rho) + 1
\end{aligned} \qquad (11)
$$

that determine the coefficients in (9) and (10).

The first and third equations in (11) lead to a quadratic equation in b_x

$$\beta b_x^2 - \gamma b_x + 1 = 0,$$

where $\gamma = 1 + \beta + \lambda^2/\alpha$. Given a solution for b_x, the solution for b_π is obtainable from $b_\pi = (\alpha/\lambda)(1 - b_x)$. Finally, the solutions for c_π and c_x are obtained from the second and fourth equation in (11), which are linear, given the solution for b_π. The quadratic in b_x has two solutions, but the solution of interest is[10]

$$\bar{b}_x = (2\beta)^{-1}[\gamma - (\gamma^2 - 4\beta)^{1/2}]. \qquad (12)$$

This delivers a stationary REE for all values of structural parameters, since $0 < \bar{b}_x < 1$, and corresponds to the policy optimum. We will therefore refer to the REE, given by (9)-(10), (11) and (12), as the *optimal REE*.

3 Optimal Interest-Rate Setting

Thus far we have formulated the concept of optimal monetary policy under RE and reviewed the derivation of the corresponding REE using the existing literature. This derivation did not rely on the aggregate demand curve (1), which depends on the interest rate and which can be used to determine the interest rate that implements the desired optimal equilibrium. Computation of the appropriate interest rate will lead to a

[9]The derivation follows closely (McCallum and Nelson 2000).
[10]The other root for b_x is always larger than one and therefore generates explosive time paths.

functional relationship that will be called a *reaction function*, since it aims to set interest rates so that the optimality condition (8) will be exactly met. Interest-rate rules that respond to endogenous and exogenous variables, but do not respect (8), are instead called *instrument rules* and we will analyze some instrument rules below in Section 4. We note that the terminology used by different authors appears to vary in the literature.[11]

As has become apparent from the earlier literature (see the references below), interest-rate setting in the form of a reaction function can be implemented in different ways depending on what is assumed to be known in the policy optimization. We now consider three possibilities, following and extending the analysis in (Evans and Honkapohja 2003) for discretionary policy. For each form of the reaction function we will test its performance in two ways.

First, we will determine if the resulting REE is determinate. This means that it is the unique stationary REE under the reaction function. If a solution is indeterminate there exist other stationary RE solutions nearby and, as is well known, these can include a dependence on extraneous variables or "sunspots." Second, we determine whether the REE corresponding to the reaction function implementing optimal policy is stable under adaptive learning by private agents. Here we formally analyze whether the RE solution is E-stable, since E-stability is known to determine whether the solution is locally stable if private agents update their forecasts using least squares or closely related learning schemes. We remark that these are independent criteria: we will find both cases of indeterminate but E-stable REE and cases of determinate but E-unstable REE, as well as cases where these criteria agree. Our aim is to look for reaction functions for the interest rate that induce both determinacy and stability under learning.

3.1 The Fundamentals-Based Reaction Function

In the REE constructed above, the inflation and output gap forecasts satisfy (9) and (10) with the parameter values $\bar{b}_\pi$, $\bar{c}_\pi$, $\bar{b}_x$ and $\bar{c}_x$, where $\bar{b}_x$ is given in (12) and

$$\begin{aligned}
\bar{b}_\pi &= (\alpha/\lambda)(1-\bar{b}_x), \\
\bar{c}_x &= -[\lambda + \beta\bar{b}_\pi + (1-\beta\rho)(\alpha/\lambda)]^{-1}, \\
\bar{c}_\pi &= -(\alpha/\lambda)\bar{c}_x.
\end{aligned}$$

RE are thus given by

$$\begin{aligned}
E_t\pi_{t+1} &= \bar{b}_\pi\bar{b}_x x_{t-1} + (\bar{b}_\pi\bar{c}_x + \bar{c}_\pi\rho)u_t, \\
E_t x_{t+1} &= \bar{b}_x^2 x_{t-1} + (\bar{b}_x + \rho)\bar{c}_x u_t.
\end{aligned}$$

Inserting these expectations and (9) into (1), and solving for the interest rate, yields

$$i_t = \psi_x x_{t-1} + \psi_g g_t + \psi_u u_t, \tag{13}$$

[11] Our terminology largely agrees with that of (Svensson and Woodford 2003) and (Svensson 2003). They call the optimality condition (8) a "specific targeting rule" and the setting of the interest rate instrument, with (8) satisfied, a "reaction function" of the policy maker.

with

$$\psi_x = \bar{b}_x[\varphi^{-1}(\bar{b}_x - 1) + \bar{b}_\pi], \tag{14}$$

$$\psi_g = \varphi^{-1}, \tag{15}$$

$$\psi_u = [\bar{b}_\pi + \varphi^{-1}(\bar{b}_x + \rho - 1)]\bar{c}_x + \bar{c}_\pi \rho. \tag{16}$$

We refer to (13) as the *fundamentals-based reaction function*, since its derivation is based solely on the model (1) and (2), the optimality condition (8) and the assumption that the economy is in a stationary REE.

We emphasize that the derivation of this interest-rate rule presupposes RE on the part of both the private agents and the policymaker. In the REE specified it indeed implements the optimal policy, as is evident from the way (13) was derived. (13) states that interest rates are set so as to respond to the lagged output gap and observable exogenous shocks. The dependence on lagged output gap reflects the commitment aspect of the optimal policy.[12] We note that interest-rate setting according to (13) is quite similar to the "reaction functions" in equation (2.30) in (Svensson and Woodford 2003) and (3.5) in (Svensson 2003). Their models differ from the model in this paper, but the setting of interest rates according to lagged output and observable exogenous variables is the key common feature for their setups and (13).[13]

We are now ready to analyze the model with interest-rate setting according to (13) for determinacy and stability under learning. For this purpose, combining (1), (2) and (13), we write the reduced form of the model in terms of general (possibly non-rational) expectations as

$$\begin{aligned}\begin{pmatrix} x_t \\ \pi_t \end{pmatrix} &= \begin{pmatrix} 1 & \varphi \\ \lambda & \beta + \lambda\varphi \end{pmatrix} \begin{pmatrix} E_t^* x_{t+1} \\ E_t^* \pi_{t+1} \end{pmatrix} + \\ &\quad \begin{pmatrix} -\varphi\psi_x & 0 \\ -\lambda\varphi\psi_x & 0 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ \pi_{t-1} \end{pmatrix} + \begin{pmatrix} -\varphi\psi_u \\ 1 - \lambda\varphi\psi_u \end{pmatrix} u_t. \end{aligned} \tag{17}$$

3.1.1 Does the Fundamentals-Based Reaction Function Yield Determinacy?

To analyze determinacy, we apply well-known methodology, see e.g. the Appendix of Chapter 10 of (Evans and Honkapohja 2001). Key technical details are given in Appendix A.2. The basic steps are to rewrite the model in first-order form and to compare the number of non-predetermined variables with the number of eigenvalues of the forward-looking matrix that lie inside the unit circle. When these numbers are equal the model is determinate and has a unique nonexplosive solution. Intuitively, each root inside the unit circle provides a side condition that ties down one non-predetermined variable. If there are fewer eigenvalues inside the unit circle than non-predetermined variables then the

[12] The corresponding interest rate function under discretion does not depend on x_{t-1}, see (Evans and Honkapohja 2003). In fact, the coefficients ψ_g and ψ_u with $\bar{b}_x = \bar{b}_\pi = 0$, are identical to the discretionary case.

[13] Our model does not include the unobservable judgement variables that are introduced in (Svensson 2003) to capture further model uncertainties.

model is indeterminate and there exist multiple nonexplosive solutions. In particular, in the indeterminate case there exist multiple stationary solutions that depend on sunspot variables. In contrast to the optimal REE, these other REE will not satisfy (8), the necessary conditions for an optimum.[14]

The conditions for determinacy are given in Appendix A.3. Whether the determinacy condition holds depends on the structural parameters of the model, and we have

Proposition 1 *Under the fundamentals-based reaction function there are parameter regions in which the model is determinate and other parameter regions in which it is indeterminate.*

As an illustration we consider three different calibrations found in the literature.[15]

Calibration W: $\beta = 0.99$, $\varphi = (0.157)^{-1}$ and $\lambda = 0.024$.
Calibration CGG: $\beta = 0.99$, $\varphi = 4$ and $\lambda = 0.075$.
Calibration MN: $\beta = 0.99$, $\varphi = 0.164$ and $\lambda = 0.3$.

These are taken, respectively, from (Woodford 1999b), (Clarida, Gali, and Gertler 2000) and (McCallum and Nelson 1999). Straightforward numerical calculations show that for small values of α the steady state is indeterminate, while for larger values of α the model is determinate. (With the calibrated parameter values the borderlines are approximately $\alpha = 0.16$, 0.47 and 278, for the three calibrations.) Determinacy thus arises only for some values of α. The domain of values for α that gives determinacy depends sensitively on the calibration, but in general sufficient flexibility in inflation targeting is needed to ensure determinacy of equilibrium under the reaction function (13).[16]

We remark that we are here treating α as a free policy preference parameter as is often done in the applied literature. If instead (4) is obtained as the approximation to the welfare of the representative consumer, the situation is more complicated as α, φ and λ all depend on deep preference and price setting parameters. Because there are more than three deep structural parameters however, there are degrees of freedom for α given β, φ and λ.[17]

[14] Other stationary REE that satisfy (2) cannot satisfy (8) because, as previously noted, the system (2) and (8) has a unique stationary RE solution.

[15] Both the (Clarida, Gali, and Gertler 2000) and (Woodford 1999b) calibrations are for quarterly data. However, (Woodford 1999b) uses quarterly interest rates and measures inflation as quarterly changes in the log price level, while (Clarida, Gali, and Gertler 2000) use annualized rates for both variables. We adopt the Woodford measurement convention, and therefore our CGG calibration divides by 4 both the σ and κ values reported by (Clarida, Gali, and Gertler 2000).

[16] Similar results are found if parameters other than α are varied. For example, for $\beta = 0.99, \lambda = 0.024$ and $\alpha = 0.2$, large values of φ lead to determinacy but small values of φ generate indeterminacy.

[17] In (Rotemberg and Woodford 1999) φ is determined by a parameter of the utility function for aggregate consumption. α and λ depend on this and two other preference parameters as well independent price setting parameters. A detailed analysis of the feasible range of $(\alpha, \varphi, \lambda)$ would require a separate study.

3.1.2 Instability Under Learning with the Fundamentals-Based Reaction Function

Derivation of the interest-rate reaction function (13) presupposed that economic agents in the model have RE. However, suppose now that private agents have possibly non-rational expectations, which they try to correct through adaptive learning. We assume that the policymaker does not explicitly take this private agent learning into account, and continues to set policy according to (13). We are thus analyzing whether, when interest rates are set according to (13), the optimal REE is robust to transient errors in forecasting by private agents. (The formulation will be analogous in later sections when some other reaction function or instrument rule for interest-rate setting is considered.)

In this analysis we employ the standard methodology of adaptive learning in macroeconomics, see (Evans and Honkapohja 2001) for an extensive treatise on the subject. We now briefly explain the formulation of the system under adaptive learning, more specifically under least squares learning, and provide a definition of the stability of an REE under learning. For convenience, Appendix A.1 provides the stability conditions for the settings needed in this paper.

The central idea is the assumption that at each period t private agents have a perceived law of motion (PLM) that they use to make forecasts. The PLM takes the form

$$y_t = a_t + b_t y_{t-1} + c_t v_t, \tag{18}$$

where we are using the vector notation

$$y_t = \begin{pmatrix} x_t \\ \pi_t \end{pmatrix}, v_t = \begin{pmatrix} g_t \\ u_t \end{pmatrix}.$$

The parameters (a_t, b_t, c_t) are updated over time using least squares. (This updating might for example be done by an econometric forecasting firm that supplies forecasts to the agents). Note that for the reduced form (17) the optimal REE can be written as

$$y_t = \bar{a} + \bar{b} y_{t-1} + \bar{c} v_t,$$

where $\bar{a} = 0$ and where the second column of $\bar{b}$ is zero. The PLM (18) has the same form as this REE, but in general the parameters (a_t, b_t, c_t) need not equal the REE values $(0, \bar{b}, \bar{c})$.

Given the PLM and the current value of v_t, the forecast functions of the private agents are $E_t^* y_{t+1} = a_t + b_t E_t^* y_t + c_t E_t^* v_{t+1}$ or

$$E_t^* y_{t+1} = a_t + b_t(a_t + b_t y_{t-1} + c_t v_t) + c_t F v_t, \tag{19}$$

where (a_t, b_t, c_t) are the parameter values of the forecasts functions that agents have estimated on the basis of past data up to and including period $t-1$. Note that we are assuming that current exogenous variables, and lagged but not current endogenous variables, are in the information set when forecasts are made. This is in line with much of the literature, and we will refer to this as the main information assumption, though at certain points in the text we will consider the implications of an alternative information assumption in which expectations are allowed to depend on current endogenous variables. Stability conditions for the different information assumptions are given in Appendix A.1.3.

These forecasts are used in decisions for period t, which yields the temporary equilibrium, also called the actual law of motion (ALM), for $y_t = (x_t, \pi_t)$ with the given PLM. The temporary equilibrium or ALM provides a new data point and agents are then assumed to re-estimate the parameters (a_t, b_t, c_t) with data through period t and use the updated forecast functions for period $t+1$ decisions. Together with v_{t+1} these in turn yield the temporary equilibrium for period $t+1$ and the learning dynamics continues with the same steps in subsequent periods. The REE $(0, \bar{b}, \bar{c})$ is said to be *stable under learning* if the sequence (a_t, b_t, c_t) converges to $(0, \bar{b}, \bar{c})$ over time.

Appendix A.1 gives the stability conditions for convergence to an REE under least squares learning. The central idea is to obtain a mapping T from the PLM parameters (a, b, c) to the implied ALM parameters, $T(a, b, c)$. The REE corresponds to a fixed point of this map and one can define a stability condition, known as E-stability, in terms of a differential equation describing partial adjustment of the PLM parameters towards the ALM parameters. E-stability turns out to provide the conditions for stability of an REE under least squares and closely related learning rules.

Earlier work by (Evans and Honkapohja 2003) showed that discretionary policy, using interest-rate setting based on fundamentals, leads to instability because learning by private agents fails to lead the economy to the REE corresponding to the optimal policy without commitment. It would seem possible that the full commitment policy implemented with (13) would perform better than discretion in this respect, because of the feedback of the output gap on interest rates. However, we have:

Proposition 2 *The fundamentals-based reaction function leads to instability under learning for all structural parameter values.*

The proof is given in the Appendix A.3.

The source of the instability lies in the interaction between the IS curve (1) and the price setting curve (2). The simplest intuition is obtained from considering a PLM (a, b, c) in which all of the parameters are held fixed at the optimal REE values, except for a_π, the intercept term in the PLM for inflation. In this case the mapping from PLM to ALM becomes one-dimensional and takes the form

$$T_{a_\pi}(a_\pi) = constant + (\beta + \lambda\varphi)a_\pi.$$

Since β is close to one and $\beta, \lambda, \varphi > 0$, for most parameter values we have $\beta + \lambda\varphi > 1$. A value of $a_\pi > 0$ will therefore tend to be adjusted upward, away from the equilibrium value. Intuitively, $a_\pi > 0$ corresponds to an exogenous positive shock to inflation expectations. This directly increases inflation by β times the shock. In addition via (1) the inflation expectations shock lowers the real interest rate, increasing output by φ times the shock, and through (2) this raises inflation indirectly by $\lambda\varphi$ times the shock.

If $\beta + \lambda\varphi > 1$ then revisions to expected inflation in the direction of actual inflation will lead to a cumulative movement away from equilibrium and we have instability.

Under least squares learning the dynamics are, of course, much more complicated and in particular all of the parameters (a, b, c) adjust to forecast errors. The proof of Proposition 2 shows that under the fundamentals-based interest-rate policy, the system is always locally unstable, even in the case $\beta + \lambda\varphi < 1$.[18]

In summary, under private agent learning, the policymaker's ability to commit to optimal policies is not sufficient to stabilize the economy, if the policy reaction function is based on observable exogenous shocks and the lagged output gap in the way suggested by the standard theory for optimal policy. We emphasize that under the fundamentals-based rule the problem of instability arises even if the optimal REE is determinate.

3.1.3 Alternative Information Assumption

Thus far we have treated expectations as determined before the current values of endogenous variables are realized, as is evident from (19). This would be natural if agents obtain these forecasts from an econometric forecasting firm prior to entering the market place. In this section we consider an alternative possibility that allows forecasts to be functions also of the current values of endogenous variables, so that

$$E_t^* y_{t+1} = a + b y_t + cF v_t.$$

This means that current decisions and forecasts of the agents are simultaneously determined. Private agents must now be regarded as entering the market place with the most recent estimate of the *forecast functions* (obtained from the forecasting firm), which are incorporated into the consumption and pricing plans. We remark that this stronger information assumption gives additional scope to monetary policy, since changes in interest rates will also have an immediate indirect effect on inflation and output expectations.[19]

Indeterminacy under the fundamentals-based reaction function is, of course, not affected since this is a property of the model under RE. Stability under learning can in general be affected by this alternative information assumption. For the model at hand we obtain:

Proposition 3 *Under the alternative information assumption and the fundamentals-based reaction function there are parameter regions in which the model is stable under learning and other parameter regions in which it is unstable under learning.*

[18] An interesting question is whether instrument rules of the form $i_t = \psi_x x_{t-1} + \psi_g g_t + \psi_u u_t$ always yield unstable REE under learning even when the coefficients are not chosen to deliver the optimal reaction function i.e. (14)-(16). It can be shown that stable (and determinate) cases do exist if the parameters satisfy $1 - \beta^2 - \lambda\varphi\beta > 0$.

[19] One could also consider reducing the amount of information available to agents when making pricing and/or consumption decisions. There are several specific possibilities and this would require a separate study.

We illustrate the result using the three calibrations in Section 3.1.1. Instability arises for sufficiently small values of α. For the W, CGG and MN calibrations the borderlines are approximately $\alpha = 0.004$, 0.301 and 1.830, respectively. Thus stability under learning, as well as determinacy, remains problematic since instability arises for many values of the structural and policy parameters.

3.1.4 Price-Level Formulation

The commitment optimality condition (8) can also be written in terms of the log of the price level p_t as

$$\lambda(p_t - p_{t-1}) = -\alpha(x_t - x_{t-1}).$$

This will be satisfied if

$$x_t = -\frac{\lambda}{\alpha}p_t + k,$$

for any constant k. This suggests that the optimal REE can be written in terms of x_t and p_t and it can be verified that the optimal REE satisfy

$$\begin{aligned} p_t &= \bar{b}_x p_{t-1} + \bar{c}_p u_t + \bar{a}_p \\ x_t &= \bar{b}_p p_{t-1} + \bar{c}_x u_t + \bar{a}_x, \end{aligned}$$

where $\bar{b}_x$ is as before in (12), $\bar{b}_p = -\frac{\lambda}{\alpha}\bar{b}_x$, and the other parameters depend on the model parameters and the value of k. This is a stationary process in (p_t, x_t). Calculating the expectations $E_t p_{t+1}$, $E_t x_{t+1}$, and inserting these and the REE p_t equation into the IS curve (1), we obtain an alternative fundamentals-based reaction function

$$i_t = \eta_p p_{t-1} + \varphi^{-1} g_t + \eta_u u_t + \eta_0, \tag{20}$$

where

$$\begin{aligned} \eta_p &= \bar{b}_x(1 - \bar{b}_x)(\frac{\lambda}{\alpha\varphi} - 1), \\ \eta_u &= (1 - \bar{b}_x)(\frac{\lambda}{\alpha\varphi} - 1)\bar{c}_p, \\ \eta_0 &= (\frac{\lambda}{\alpha\varphi} - 1)\bar{b}_x \bar{a}_p. \end{aligned}$$

We focus on the issue whether the optimal REE becomes stable under learning if the interest rate is set according to (20). We restrict attention to the main information assumption in which forecasts depend only on exogenous and lagged endogenous variables. The PLM takes the general form (18), where now $y_t' = (x_t, p_t)$. In Appendix A.3 we show that the optimal REE are not always stable under learning. In particular, instability occurs when $\varphi > \frac{\lambda}{\alpha}$.

Thus reformulating the fundamentals-based reaction function in terms of the exogenous shocks and the lagged price level does not ensure stability under learning. We now show how the instability problems associated with fundamentals-based reactions functions can be overcome if private expectations are observable and interest-rate policy conditions appropriately on their values.

3.2 An Expectations-Based Reaction Function

The computation deriving the fundamentals-based reaction function in Section 3.1 relied heavily on the assumption that the economy is in the optimal REE. We now obtain a different reaction function for interest-rate setting, under optimal monetary policy, which does not make direct use of the RE assumption. Instead, recognizing the possibility that private agents may have non-rational expectations during the learning transition, the policy rule is obtained by combining the optimality condition, the price-setting equation and the IS curve, for given private expectations.[20] This leads to a monetary policy in which interest rates depend on observed private expectations as well as on fundamentals. We call this rule the *expectations-based reaction function.*

Formally, combine the price-setting equation (2) and the optimality condition (8), treating private expectations as given. This leads to

$$x_t = \frac{\lambda}{\alpha+\lambda^2}\left[\frac{\alpha}{\lambda}x_{t-1} - \beta E_t^* \pi_{t+1} - u_t\right].$$

Next, substitute this expression into the IS curve (1) and solve for i_t. This yields the expectations-based reaction function for interest-rate setting:

$$i_t = \delta_L x_{t-1} + \delta_\pi E_t^* \pi_{t+1} + \delta_x E_t^* x_{t+1} + \delta_g g_t + \delta_u u_t, \tag{21}$$

where

$$\begin{aligned}
\delta_L &= \frac{-\alpha}{\varphi(\alpha+\lambda^2)}, \\
\delta_\pi &= 1 + \frac{\lambda\beta}{\varphi(\alpha+\lambda^2)}, \\
\delta_x &= \varphi^{-1}, \\
\delta_g &= \varphi^{-1}, \\
\delta_u &= \frac{\lambda}{\varphi(\alpha+\lambda^2)}.
\end{aligned}$$

Looking at the rule (21) it can be seen that its coefficients stipulate a relatively large response to expected inflation ($\delta_\pi > 1$) and that effects coming from the expected output gap and the aggregate demand shock are fully neutralized ($\delta_x = \delta_g = \varphi^{-1}$). The positive coefficients on private expectations are crucial for ensuring stability of the REE and the sizes of the coefficients are chosen so that the economy is led to the optimal REE.

The reduced form of the economy under (21) is

$$\begin{aligned}
\begin{pmatrix} x_t \\ \pi_t \end{pmatrix} &= \begin{pmatrix} 0 & -\frac{\lambda\beta}{\alpha+\lambda^2} \\ 0 & \frac{\alpha\beta}{\alpha+\lambda^2} \end{pmatrix} \begin{pmatrix} E_t^* x_{t+1} \\ E_t^* \pi_{t+1} \end{pmatrix} + \\
& \begin{pmatrix} \frac{\alpha}{\alpha+\lambda^2} & 0 \\ \frac{\alpha\lambda}{\alpha+\lambda^2} & 0 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ \pi_{t-1} \end{pmatrix} + \begin{pmatrix} -\frac{\lambda}{\alpha+\lambda^2} \\ \frac{\alpha}{\alpha+\lambda^2} \end{pmatrix} u_t.
\end{aligned} \tag{22}$$

[20] This general approach was suggested and studied in (Evans and Honkapohja 2003) in the context of discretionary policy. As before, we assume that the policy maker does not explicitly take into account private agents' learning rules in the policy optimization.

We now consider both determinacy and the stability under learning for the expectations-based reaction function (21).

The methodology of Appendix A.2 can be applied to the reduced form (22). In Appendix A.3 we obtain the following result.

Proposition 4 *Under the expectations-based reaction function (21) the REE is determinate for all structural parameter values.*

It is clearly a desirable property of our proposed monetary policy rule that it does not permit the existence of other suboptimal stationary REE. However, as we have seen in the case of the fundamentals-based reaction function, having a determinate REE does not always ensure that it is attainable under learning. To analyze stability under learning we can again use the general matrix framework in Appendix A.1. As in the preceding section we endow private agents with the PLM, compute the corresponding forecast function and substitute them into (22). This yields the temporary equilibrium or ALM and it is possible to study whether least squares learning converges to the REE under the expectations-based reaction function (21).

The next proposition shows that our interest-rate rule performs well (see Appendix A.3 for the formal proof).

Proposition 5 *Under the expectations-based reaction function (21), the optimal REE is stable under learning for all structural parameter values.*

Partial intuition for this result can be seen from the reduced form (22). An increase in inflation expectations now leads to an increase in actual inflation that is smaller than the change in expectations since $\alpha\beta/(\alpha+\lambda^2) < 1$. This dampened effect arises from the interest-rate reaction to changes in $E_t^*\pi_{t+1}$ and is a crucial element of the stability result. Of course, stability under learning requires convergence over time of all the PLM parameters, and Proposition 5 thus provides a remarkably strong result: Under the interest-rate setting rule (21), learning is stable and the economy is guided specifically to the optimal REE, and this result holds for all possible values of the structural parameters.

As one check on the robustness of the result we consider also the alternative information assumption that allows forecasts to be functions also of current endogenous variables. We remark that in this case interest rates, inflation, output and expectations are all simultaneously determined.

We continue to have stability under the expectations-based reaction function:

Proposition 6 *Under the alternative information assumption and the expectations-based reaction function (21), the optimal REE is stable under learning for all structural parameter values.*

Our analysis has shown that the reaction function (21) is a robust method for implementing optimal monetary policy with commitment, passing both of the performance tests we discussed earlier. Because the optimal REE is determinate under the expectations-based reaction function, there are no nearby sunspot equilibria that are consistent with the policy. Because it is stable under learning, the reaction function is robust to expectational errors by private agents. These positive results complement the analysis of (Evans and Honkapohja 2003) for the corresponding implementation of optimal discretionary policy and show that the policy optimum is obtainable with a well-designed interest-rate rule.

We remark that we have chosen our recommended rule carefully to ensure both determinacy and stability under learning for all parameter values. In the literature alternative interest-rate setting rules have appeared, which can be interpreted as expectations-based reaction functions but which do not meet our tests. For example, the interest-rate reaction function

$$i_t = (1 - \frac{\lambda}{\alpha\varphi})E_t\pi_{t+1} + \varphi^{-1}g_t. \tag{23}$$

is suggested in (Clarida, Gali, and Gertler 1999), Section 4.2.2. Replacing $E_t\pi_{t+1}$ with $E_t^*\pi_{t+1}$ leads to a policy reaction function based in part on observed expectations. This policy rule is consistent with the optimal policy under commitment under the RE assumption. However, as Clarida, Gali and Gertler note, this reaction function can lead to indeterminacy. Furthermore, it can be shown that if $\beta + \lambda^2/\alpha > 1$ the optimal REE is not stable under learning.

3.3 Discussion

We illustrate our results by two simulations using the MN calibration of Section 3.1.1 and $\alpha = 0.5$. Figure 1 plots results for the fundamentals-based reaction function. The figure shows an explosive path for the inflation rate over the first 120 periods. The PLM coefficients, not shown, also follow explosive paths. Other simulations for the fundamentals-based rule show a variety of unstable paths. Of course, faced with such a path, the policymaker would alter the policy rule and private agents would also be motivated to alter their learning rule. However, Figure 1 illustrates the stability problems inherent with the fundamentals-based rules: under this policy rule the economy will be subject to expectational instability.

Figure 2 shows the time paths for PLM parameters a and b from an illustrative simulation under least squares learning when the policymaker employs the expectations-based rule. There is now near convergence of the PLM parameters over 400 periods. (The PLM parameters c, not shown, also exhibit convergence). The numerical results show that the economy asymptotically reaches the optimal RE solution. This is further illustrated by Figure 3, which shows the deviation of x_t and π_t from the REE path that would be generated from the same sequence of stochastic shocks.[21]

[21]For discussion of recursive least squares algorithms, see (Evans and Honkapohja 2001). The simulations require specification of the "gain" sequences, which measures responsiveness to forecast errors in least squares type learning. In the simulation for Figure 1 the gain has been set at a small constant value. In the simulation for Figures 2 and 3 the gain is set at a small constant value for the first 100 periods and then declines at rate t^{-1}.

FIGURES 1, 2 AND 3 HERE

The key to our stability results is that monetary authorities raise interest rates, *ceteris paribus*, in response to increases in inflation and output forecasts by private agents, and lower interest rates in response to decreases in private expectations. Given the fundamentals u_t, g_t and x_{t-1}, overly optimistic or pessimistic forecasts by private agents have the potential to destabilize the economy under least squares learning. Our expectations-based policy is designed to offset this tendency and to guide the economy to the optimal REE.

Several points should be made concerning our results. First, although we have demonstrated our results in the context of least squares learning, the stability results will obtain under various generalizations of least squares.[22] In fact, the stability results for the expectations-based reaction function hold even for some forecast rules that do not converge to RE. For example, suppose private agents forecast both output and prices using the simple adaptive expectations rules:

$$\begin{aligned} x_{t+1}^e &= \gamma x_{t-1} + (1-\gamma)x_t^e, \\ \pi_{t+1}^e &= \gamma \pi_{t-1} + (1-\gamma)\pi_t^e, \end{aligned}$$

where $0 < \gamma < 1$. This forecast rule has a venerable history, but it is not rational in the current model. Nonetheless it can be shown that under our expectations-based rule the economy is stable for all $0 < \gamma < 1$, though it will not, of course, converge to the optimal REE.

Second, since our recommended policy reaction function depends explicitly on private expectations, it is desirable to have high quality observations or estimates of private forecasts. However, our stability results extend to the case in which the reaction function depends on private expectations observed with a white noise measurement error. In this case there is convergence to an REE that deviates from optimality by an amount depending on the measurement error variance.

Even if contemporaneous observations of expectations are not available, it may nonetheless be possible to either fully implement or approximate our policy, provided suitable auxiliary assumptions are made about the expectation formation process. Most obviously, if it is known that agents make forecasts based on a PLM of the form (18), with coefficients estimated using least squares, then policymakers can construct accurate proxies for private expectations. To do so, policymakers would proxy $E_t^*\pi_{t+1}$ and $E_t^*x_{t+1}$ by linear functions of $y_{t-1}' = (x_{t-1}, \pi_{t-1})$ and $v_t' = (g_t, u_t)$ with estimated coefficients, following the same procedure used by private agents. This procedure can be shown to lead to the optimal REE even if policymakers have different initial coefficient estimates or have different starting points for their data sets than private agents.[23]

[22] See e.g. the weighting schemes in (Marcet and Sargent 1989) and inertial behavior in (Evans, Honkapohja, and Marimon 2001).

[23] This was shown for the discretionary case in (Honkapohja and Mitra 2003), and the argument can be extended to the current context.

Third, our discussion has implicitly assumed that the coefficients of the structural model (1) and (2) are known to the policymaker. This, however, is stronger than necessary. In our analysis of discretionary policy, in (Evans and Honkapohja 2003), we showed that an expectations-based policy could be implemented using estimated structural parameters and that the REE was stable under simultaneous learning by private agents and policymakers. An analogous argument is applicable here in the case of optimal policy with commitment.

Finally, we remark that (Jensen and McCallum 2002) have recently shown that modifying the optimality condition (8) to $\lambda\pi_t = -\alpha(x_t - \beta x_{t-1})$ appears to improve the policy performance, because it partially compensates for the timeless perspective neglect of the first period optimality condition. Fundamentals and expectations-based reaction functions can be derived corresponding to this modified optimality condition. It can be shown that our stability and instability results remain unchanged.

4 Approximating Optimal Policy

(McCallum and Nelson 2000) have recently suggested that, in place of interest-rate setting by a reaction function satisfying the optimality condition (8), there are well performing instrument rules that can approximate this condition. These instrument rules specify that the interest rate is moved towards a specified target value in response to deviations from the commitment optimality condition (8).[24] In this section we analyze the performance of such rules for determinacy and stability under learning.

4.1 The Approximate Targeting Instrument Rule

To begin, we consider instrument rules of the form[25]

$$i_t = \pi_t + \theta[\pi_t + (\alpha/\lambda)(x_t - x_{t-1})]. \tag{24}$$

From now on we will call this rule the *approximate targeting instrument rule*, or simply the *approximate targeting rule.* This terminology describes better the underpinnings of (24) than the general term "instrument rule", which is used in (McCallum and Nelson

[24]See (McCallum 1999) for a general discussion of this approach.

[25](McCallum and Nelson 2000) include a constant real interest target but this does not affect our results. They also suggest adding a lagged nominal interest rate to the rule, but this term is dropped in their numerical results. For simplicity, we also ignore such an inertial term. (Bullard and Mitra 2001) analyze policies with interest rate inertia using the learning approach.

2000).[26] Substituting (24) into the model (1)-(2) leads to the reduced form

$$\begin{aligned} &\begin{pmatrix} 1+\alpha\varphi\theta\lambda^{-1} & \varphi(1+\theta) \\ -\lambda & 1 \end{pmatrix} \begin{pmatrix} x_t \\ \pi_t \end{pmatrix} \\ = &\begin{pmatrix} 1 & \varphi \\ 0 & \beta \end{pmatrix} \begin{pmatrix} E_t^* x_{t+1} \\ E_t^* \pi_{t+1} \end{pmatrix} + \\ &\begin{pmatrix} \alpha\varphi\theta\lambda^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ \pi_{t-1} \end{pmatrix} + \begin{pmatrix} g_t \\ u_t \end{pmatrix}. \end{aligned} \tag{25}$$

To assess determinacy we rewrite (25) in first-order form and compute the roots of the forward-looking matrix. Details are given in Appendix A.3. The eigenvalues do not lend themselves to clear theoretical results and thus we have studied them numerically. For the calibrated examples in Section 3.1.1 we find numerically that the steady state seems to be determinate for all values of α and θ.

Next, we consider learning stability of the REE under the instrument rule (24). We apply the general methodology of Appendix A.1, see the explanations in Appendix A.3, and restrict attention to the main information assumption. Again, theoretical results cannot be obtained, so that numerical analysis must be used. Using the calibrations given in Section 3.1.1 and setting $\rho = \mu = 0.35$ we have found stability under learning for all values of α and θ.

We conclude that the approximate targeting rule (24) appears to lead to both determinacy and stability under learning. Moreover it has the attractive feature that for large θ it leads to REE that are close to the optimal policy under the timeless perspective.

4.2 Variants of the Approximate Targeting Rule

As pointed out by (McCallum and Nelson 2000), a difficulty with the approximate targeting rule (24) is that it presupposes that the policymaker can observe current output gap and inflation when setting the interest rate. (McCallum and Nelson 2000) propose some alternative formulations that do not require observations on contemporaneous x_t and π_t. We explore some of these in this section.

Several possibilities appear natural. One possibility is to replace actual values of x_t and π_t by their forecasts, i.e. set the interest rate according to

$$i_t = \tilde{E}_t\pi_t + \theta[\tilde{E}_t\pi_t + (\alpha/\lambda)(\tilde{E}_t x_t - x_{t-1})].$$

Here $\tilde{E}_t(.)$ denotes the expectations of the policymaker. However, (McCallum and Nelson 2000) find that this rule performs very poorly under REE. (McCallum and Nelson 2000) find that making the approximate targeting rules forward looking leads to better performance under RE. In this case the policymaker adjusts the current interest rate in response to the discrepancy from the optimality condition (8) anticipated for

[26] (24) is a particular type of instrument rule in the terminology of (Svensson and Woodford 2003) and (Svensson 2003). The widely studied Taylor rules constitute another example of instrument rules.

the next period. This suggests interest-rate setting according to

$$i_t = \tilde{E}_t\pi_{t+1} + \theta[\tilde{E}_t\pi_{t+1} + (\alpha/\lambda)(\tilde{E}_t x_{t+1} - \tilde{E}_t x_t)]. \tag{26}$$

The expectations of the central bank could be modeled in several ways. One possibility would come from working out the implied REE, i.e. setting $\tilde{E}_t\pi_{t+1} = E_t\pi_{t+1}$ etc. and substituting these expressions into (26). This would lead to an instrument rule of the form $i_t = \zeta_0 + \zeta_g g_t + \zeta_u u_t$. However, this is known to lead to both indeterminacy and instability under learning, see (Evans and Honkapohja 2003). Another interpretation is to assume that the expectations of the policymaker are formed like those of private agents, i.e. $\tilde{E}_t y_{t+1} = E_t^* y_{t+1}$, by using vector autoregressions that are updated by recursive least squares.[27] This leads to an expectations-based rule, which, however, is different from the well-performing reaction function (21) analyzed above.

We next analyze determinacy and learnability of the economy with the instrument rule (26) using the latter interpretation for expectations (and our main information assumption). Substituting (26), with $\tilde{E}_t(.) = E_t^*(.)$, into (1) yields the reduced form

$$\begin{aligned}\begin{pmatrix} x_t \\ \pi_t \end{pmatrix} &= \begin{pmatrix} 1-\alpha\varphi\theta\lambda^{-1} & -\varphi\theta \\ \lambda-\alpha\varphi\theta & \beta-\varphi\theta\lambda \end{pmatrix}\begin{pmatrix} E_t^* x_{t+1} \\ E_t^*\pi_{t+1} \end{pmatrix} + \\ &\quad \begin{pmatrix} \alpha\varphi\theta\lambda^{-1} & 0 \\ \alpha\varphi\theta & 0 \end{pmatrix}\begin{pmatrix} E_t^* x_t \\ E_t^*\pi_t \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} g_t \\ u_t \end{pmatrix}.\end{aligned} \tag{27}$$

Details of the conditions for determinacy and learning stability are given in Appendix A.3.

Determinacy for the forward looking approximate targeting rule depends on the values of the parameters. We consider the alternative calibrated values in Section 3.1.1 and examine the issue numerically. Determinacy obtains for sufficiently small values of the reaction parameter θ, but larger values of θ lead to indeterminacy. The boundary between determinacy and indeterminacy depends on the model parameters and, in particular, on the degree of flexibility α in inflation targeting. Indeterminacy obtains for smaller values of θ, with more flexible inflation targeting, as illustrated in the following table for which the calibrations in Section 3.1.1 were used. (In the table $\theta > \hat{\theta}$ leads to indeterminacy.)

Table 1. Approximate critical value $\hat{\theta}$ for indeterminacy

α	0.01	0.05	0.1	0.5	1.0	2.0
W	0.371	0.075	0.038	0.008	0.004	0.002
CGG	1.643	0.365	0.185	0.037	0.019	0.009
MN	56.09	25.19	14.92	3.50	1.79	0.904

The critical value of θ depends sensitively on the values of structural parameters λ, φ and β, as well as the policy parameter α. Here W, CGG and MN refer to the

[27] A further possibility would be to assume that the policy maker uses its own forecasts in (26). See (Honkapohja and Mitra 2003) for such an analysis for optimal discretionary policies and Taylor rules.

three calibrations introduced in Section 3.1.1. Restricting θ to be relatively small is problematic since, under RE, rules with a small value of θ imply that deviations from optimality lead to only small corrections towards meeting the optimality condition. We remark that (McCallum and Nelson 2000) often consider large values of the reaction parameter in their quantitative analyses.

Numerical analysis also indicates that learning stability obtains for sufficiently small values of the reaction parameter θ, while larger values of θ can destabilize the economy under forward looking approximate targeting rules. To illustrate this we again revert to the calibrations of Section 3.1.1 and we also set $\rho = \mu = 0.35$. The following table illustrates the dependence of the critical value for θ on the degree of flexibility α in inflation targeting. (In the table $\theta > \bar{\theta}$ leads to instability.)

Table 2. Approximate critical value $\bar{\theta}$ for learning instability

α	0.01	0.05	0.1	0.5	1.0	2.0
W	0.396	0.076	0.038	0.008	0.004	0.002
CGG	3.495	0.413	0.197	0.038	0.019	0.009
MN	∞	2384	70.78	4.30	1.98	0.950

The conclusion regarding learning stability of forward looking approximate targeting rules (26) broadly resembles our findings for the determinacy of these rules. Again, the results depend sensitively on the model parameters, but large values of θ in most cases imply instability. Moreover, less strict inflation targeting leads more easily to instability. In contrast, sufficiently strict inflation targeting can in some cases yield E-stability of the MSV solution for all values of θ, though this result depends on the calibration.

Interestingly, determinacy is here a more stringent requirement than stability under learning. This is in contrast to the case of the fundamentals-based reaction function in which, under the main information assumption, stability under learning is a stricter requirement. The performance of approximate targeting rule (26) can be described in terms of welfare losses relative to those at the commitment optimum with the rule (21). Table 3 describes the ratio of welfare losses from following the rule (26) with the constraint that the MSV REE is determinate.[28]

Table 3. Ratio of the welfare loss from (26) rule to that from (21) rule.

α	0.01	0.05	0.1	0.5	1	2
W	1.32	3.19	5.58	24.36	47.54	93.35
CGG	1.10	1.47	2.04	6.69	12.31	23.31
MN	1.63	1.52	1.72	4.07	7.68	15.95

The results in Table 3 are not surprising, given the numbers in Tables 1 and 2. The performance of the approximate target rule depends sensitively on the value of θ and the constraint dictated by determinacy is quite stringent. If only E-stability is sought,

[28]In this table we set $\sigma_g = 1$ and $\sigma_u = 0.5$.

then the ratio of welfare losses can be much smaller and indeed with sufficiently strict inflation targeting ($\alpha = 0.01$ or $= .05$) even the optimum can be achieved for the CGG or MN calibrations. However, the W calibration leads to a different conclusions as the ratio is 1.31 even for $\alpha = 0.01$.

5 Concluding Remarks

This paper has analyzed determinacy and stability under learning for alternative interest-rate reaction functions or instrument rules that aim to implement optimal monetary policy under commitment. Determinacy is desirable because it implies that there do not exist other (nonoptimal) REE near the solution of interest. Stability under learning is desirable because it indicates that if private agents follow least squares learning they will converge over time to the optimal REE.

Our analysis leads to the conclusion that the two desiderata are met by a policy that sets interest rates according to our expectations-based reaction function. In this monetary policy reaction function, interest rates respond to private expectations as well as to fundamentals, i.e., exogenous shocks and the lagged output gap. This interest-rate reaction function performs well as it unambiguously delivers both determinacy and stability under learning for the economy, with the economy converging over time to the optimal REE.

Attempting to implement optimal policy through the fundamentals-based formulation does not perform well. Under this policy, problems with both indeterminacy and instability under learning can occur, depending on values of model parameters and the information available to agents under learning. The dependence on the lagged output gap implied by commitment is not sufficient to guarantee convergence under learning when interest-rate setting is carried out using the reaction function based solely on fundamentals.

We also considered a class of approximate targeting instrument rules in which interest rates respond to the deviation from the optimality condition. Such rules may or may not be satisfactory, depending on the information available to policymakers at the time interest rates are set. If contemporaneous observations of inflation and output gap are available to the policymaker, our numerical results indicate that these instrument rules do deliver both determinacy and stability under learning. However, it is arguably more realistic to assume that this contemporaneous data is not available, and for this reason a forward looking approximation of the target has been suggested as the basis for interest-rate setting. Both indeterminacy and instability problems can arise for such formulations unless the reaction parameter is set at a sufficiently low value. However, for low reaction parameters the resulting REE can deviate substantially from the optimal policy.

More generally, we reiterate that in monetary policy design expectations must be treated as potentially subject to deviations from rational expectations. This is necessary even if private agents are following a natural econometric forecasting procedure consistent with the REE. Optimal policy should be designed so that under private agent learning the economy is guided to the REE.

A Appendices

A.1 Stability Under Learning, General Methodology

A.1.1 Temporary Equilibrium

The starting point for models of adaptive learning is that agents have less information than is presumed under RE. Instead, private agents optimize using subjective (possibly non-rational) probability distributions over future variables. Given these subjective distributions, the standard Euler equations provide necessary conditions for optimal decisions, and we assume that the Euler equations for the current period specify the behavioral rule that gives current decisions as functions of the expected state next period. These Euler equations are then supplemented by rules for forecasting next period's values of the state variables. Thus, given their forecasts, agents make decisions for the current period according to the Euler equations. This kind of behavior is boundedly rational but, in our view, reasonable, since agents are attempting to make optimal decisions based on a perceived law of motion for the state variables.[29]

For the model at hand we give a detailed discussion making use of the general equilibrium framework presented in (Woodford 1996). Although we maintain the representative agent assumption, so that agents have identical expectations and make the same decisions, it will be useful to let i index individual firms or households. Consider first the Phillips curve (2). Let $\mathcal{P}_t^i$ be the price being set by those firms that can do so, P_t the average price index and $\hat{\mathcal{P}}_t^i$ the deviation of the relative price $\mathcal{P}_t^i/P_t$ from its stationary value. Woodford shows that $\hat{\mathcal{P}}_t^i$ can be expressed as a linear function of current output and discounted sums of expected future outputs and inflations. This derivation can be viewed as using subjective expectations that need not be rational. Assuming that the law of iterated expectations holds at the level of the individual agent, quasi-differencing allows us to express $\hat{\mathcal{P}}_t^i$ as a linear function of $E_t^{i*}\hat{\mathcal{P}}_{t+1}^i$, x_t and $E_t^{i*}\pi_{t+1}$, where of course E_t^{i*} denotes the expectation of firm i. However, $\hat{\mathcal{P}}_{t+1}^i = \hat{\mathcal{P}}_{t+1}$ and there is a proportional relationship between inflation and $\hat{\mathcal{P}}_t$. Firms will thus observe from the data that $\hat{\mathcal{P}}_t^i$ is exactly proportional to π_t, and thus $\hat{\mathcal{P}}_t^i$ can be rewritten as a linear function of expected inflation $E_t^{i*}\pi_{t+1}$ and current output x_t.[30] This defines the optimal price-setting schedule for $\mathcal{P}_t^i$, as a function of P_t, x_t and $E_t^{i*}\pi_{t+1}$, which firms take to the market place. In addition, we allow for an exogenous shock u_t to the price-setting schedule. In the temporary equilibrium, with identical firms and homogeneous forecasts and using again the relationship between $\hat{\mathcal{P}}_t$ and inflation, we obtain (2).

Consider next the IS curve (1). The linearized Euler equation, which is standard, is

[29] Recently, (Preston 2002) has studied the performance of standard instrument rules for monetary policy when agents have a different behavioral rule in which long-horizon forecasts can matter. The E-stability conditions turn out to be unchanged. For the relationship between these approaches see (Honkapohja, Mitra, and Evans 2002).

[30] We are making the simplifying assumption that potential output is constant so that output can be identified with x_t. This assumption is easily relaxed.

given by $c_t^i = E_t^{i*} c_{t+1}^i - \varphi(i_t - E_t^{i*}\pi_{t+1})$.[31] (This assumes that government purchases enter the utility function in an additively separable way.) Although c_{t+1}^i will be determined by the household itself, a forecast is required to determine its optimal current consumption. Let ξ_t denote the proportion of government purchases in GDP, and let $\hat{\xi}_t = -\ln(1-\xi_t)$. Then from market clearing $c_t^i = c_t = x_t - \hat{\xi}_t$. We assume that households observe from past data that $c_t^i = x_t - \hat{\xi}_t$ and make use of this relationship for forecasting their future consumption. For convenience, we further assume that $\hat{\xi}_t$ follows a known $AR(1)$ process $\hat{\xi}_t = \mu\hat{\xi}_{t-1} + \tilde{\xi}_t$. Then $E_t^{i*} c_{t+1} = E_t^{i*} x_{t+1} - E_t^{i*}\xi_{t+1}$, which leads to the consumption schedule

$$c_t^i = E_t^{i*} x_{t+1} - \varphi(i_t - E_t^{i*}\pi_{t+1}) - \mu\hat{\xi}_t$$

submitted to the market place. Finally, it is assumed that the government also comes to the market place with its plan to purchase the proportion ξ_t of output. In the temporary equilibrium, with identical households and homogeneous forecasts, we obtain (1), where $g_t = (1-\mu)\hat{\xi}_t$.

Given private expectations, these schedules together with the monetary policy rule determine a temporary equilibrium according to (1)-(2). Thus the values of π_t, x_t and i_t are simultaneously determined through market clearing, in the usual way, by the pricing and consumption schedules. To complete the description of the temporary equilibrium, we need to be specific on the formation of expectations. The main case considered in the text assumes that expectations are functions only of lagged endogenous variables and observable current shocks, see (19), and are thus predetermined when the plans are brought to the market. This would be natural if forecasts were obtained from an econometric forecasting firm before going to the market place. In the alternative formulation considered in Section 3.1.3, the agents instead obtain forecasting functions from the firm and they plug in observations of current endogenous variables at the market place.[32] Thus in the alternative formulation π_t, x_t, i_t and the forecasts are all simultaneously determined.

Under either information assumption the temporary equilibrium for the current period provides a new data point for the agents. Given this new data, the forecast functions are updated at the start of the following period using standard adaptive learning rules such as least squares. The question of interest is whether this kind of (adaptive) learning behavior converges over time to REE of interest. More specifically, the question is whether the estimated parameters of the forecast functions converge over time to their REE values. (Note that the REE can be viewed as a fixed point in the adjustment of forecast functions.)

A.1.2 Stability Conditions

When agents adjust their forecast functions over time, the dynamics of the economy is mathematically specified by a stochastic recursive algorithm, which is a special type

[31] See e.g. (McCallum and Nelson 1999) or (Woodford 1996).

[32] We remark that, since the model is based on representative household-producers, it seems most natural to have the same information available in both pricing and consumption plans.

of nonlinear time varying stochastic system. The conditions for convergence of such dynamics are formally obtained from the local stability conditions of an associated ordinary differential equation.[33] The latter conditions are in turn governed by what are called *expectational or E-stability conditions.* (Evans and Honkapohja 2001) provides an extensive analysis of adaptive learning and its implications in macroeconomics (see also the other references in footnote 2). In this paper we will simply exploit this connection between convergence of learning dynamics and E-stability and present the E-stability conditions for a general matrix model

$$y_t = A + ME_t^* y_{t+1} + QE_t^* y_t + Ny_{t-1} + Pv_t. \tag{28}$$

Clearly, the model of monetary policy with different ways for setting the interest rate lead to particular cases of (28) by setting

$$y_t = \begin{pmatrix} x_t \\ \pi_t \end{pmatrix} \text{ or } y_t = \begin{pmatrix} x_t \\ p_t \end{pmatrix} \text{ and } v_t = \begin{pmatrix} g_t \\ u_t \end{pmatrix}.$$

Some reduced forms in the main text have $Q = 0$ or $N = 0$.

For (28) with $N \neq 0$ an MSV REE takes the form

$$y_t = \bar{a} + \bar{b} y_{t-1} + \bar{c} v_t.$$

To define E-stability we consider PLMs of the form

$$y_t = a + b y_{t-1} + c v_t. \tag{29}$$

Computing expectations and inserting into (28) yields the ALM

$$y_t = A + (Q + M(I + b))a + (Mb^2 + Qb + N)y_{t-1} + (Qc + M(bc + cF) + P)v_t. \tag{30}$$

This equation defines the crucial mapping from PLM to ALM

$$T(a, b, c) = (A + (Q + M(I + b))a, Mb^2 + Qb + N, Qc + M(bc + cF) + P).$$

An MSV REE $(\bar{a}, \bar{b}, \bar{c})$ is a fixed point of this map. E-stability conditions can be obtained using the methods of Chapter 10 of (Evans and Honkapohja 2001). The stability conditions can be stated in terms of the derivative matrices

$$\begin{aligned} DT_a &= Q + M(I + \bar{b}) && (31) \\ DT_b &= \bar{b}' \otimes M + I \otimes M\bar{b} + I \otimes Q && (32) \\ DT_c &= F' \otimes M + I \otimes M\bar{b} + I \otimes Q, && (33) \end{aligned}$$

where $\otimes$ denotes the Kronecker product and $\bar{b}$ denotes the REE value of b.

[33] This approach was exploited in a learning context by (Marcet and Sargent 1989). (Woodford 1990) used these techniques to study the stability of sunspot equilibria.

Remark 7 *The necessary and sufficient condition for E-stability is that all eigenvalues of $DT_a - I$, $DT_b - I$ and $DT_c - I$ have negative real parts.*[34]

When $N = 0$, the MSV solution takes the form

$$y_t = a + hv_t, \tag{34}$$

where in the REE the coefficients satisfy $a = (M + Q)a$ and $h = MhF + Qh + P$.

For E-stability we use the PLM (34) with general values for a and h. E-stability conditions now require that the eigenvalues of the matrices

$$\begin{aligned} DT_a - I &= M + Q - I && (35) \\ DT_h - I &= F' \otimes M + I \otimes Q - I && (36) \end{aligned}$$

have negative real parts.

A.1.3 Alternative Information Assumption

We now assume that the current values of the endogenous variables are in the information set of the agents, so that necessarily $Q = 0$ in the general model (28). The forecast function from the PLM is now simply

$$E_t^* y_{t+1} = a + by_t + cFv_t.$$

The corresponding E-stability condition is that all the eigenvalues of

$$\begin{aligned} DT_a - I &= (I - M\bar{b})^{-1} M - I, \\ DT_b - I &= [(I - M\bar{b})^{-1} N]' \otimes [(I - M\bar{b})^{-1} M] - I, \\ DT_c - I &= F' \otimes [(I - M\bar{b})^{-1} M] - I \end{aligned}$$

have negative real parts.

A.2 Determinacy

The general methodology for ascertaining determinacy is given in the Appendix to Chapter 10 of (Evans and Honkapohja 2001). For models with reduced form (28) we first focus on the special case in which $y_t' = (x_t, \pi_t)$, $Q = 0$ and the second column of N is zero. Writing $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ and $N = \begin{pmatrix} n_{11} & 0 \\ n_{21} & 0 \end{pmatrix}$, assuming rational expectations, introducing the new variable $x_t^L \equiv x_{t-1}$, and noting that for any random variable z_{t+1} we have $E_t z_{t+1} = z_{t+1} + \varepsilon_{t+1}^z$ where $E_t \varepsilon_{t+1}^z = 0$, we can rewrite (28) as

$$\begin{pmatrix} 1 & 0 & -n_{11} \\ 0 & 1 & -n_{12} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ \pi_t \\ x_t^L \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{t+1} \\ \pi_{t+1} \\ x_{t+1}^L \end{pmatrix} + other,$$

[34] We are excluding the exceptional cases where one or more eigenvalue has zero real part.

where "other" includes terms that are not relevant in assessing determinacy. Assuming $n_{11} \neq 0$ this can be rewritten as

$$\begin{pmatrix} x_t \\ \pi_t \\ x_t^L \end{pmatrix} = J \begin{pmatrix} x_{t+1} \\ \pi_{t+1} \\ x_{t+1}^L \end{pmatrix} + other \tag{37}$$

where

$$J = \begin{pmatrix} 1 & 0 & -n_{11} \\ 0 & 1 & -n_{12} \\ 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Because this model has one predetermined variable, i.e. x_t^L, the condition for determinacy is that exactly two eigenvalues of J lie inside the unit circle and one eigenvalue outside. If one or no roots lie inside the unit circle (with the other roots outside), then the model is indeterminate.

For models with $N = 0$ the reduced form can be rewritten as

$$(I - Q)y_t = My_{t+1} + other$$

where y_t is assumed not to include predetermined variables. In this case the determinacy condition is that $J = (I - Q)^{-1}M$ have both roots inside the unit circle. If one or both roots lie outside the unit circle then the model is indeterminate.

A.3 Derivations

Proof of Proposition 1. Applying the methodology of Section A.2 to the reduced form (17) we obtain

$$\begin{aligned} J &= \begin{pmatrix} 1 & 0 & \varphi\psi_x \\ 0 & 1 & \lambda\varphi\psi_x \\ 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \varphi & 0 \\ \lambda & \beta + \lambda\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & \beta & \lambda \\ (\varphi\psi_x)^{-1} & {\psi_x}^{-1} & -(\varphi\psi_x)^{-1} \end{pmatrix}. \end{aligned}$$

Since the model has two free variables, determinacy requires that exactly two eigenvalues of J are inside the unit circle. Straightforward numerical calculations for the calibrated example show that two eigenvalues of J lie outside the unit circle, and one lies inside, for small values of α, so that the steady state is indeterminate, while for larger values of α exactly one root lies outside the unit circle, and the model is determinate. We remark that continuity of eigenvalues implies that both regions contain open sets of parameters.

Proof of Proposition 2. We apply the general methodology outlined above in Appendix A.1, when the general model (28) takes the specific form (17). In this case

$Q = 0$ and

$$M = \begin{pmatrix} 1 & \varphi \\ \lambda & \beta + \lambda\varphi \end{pmatrix}, \; N = \begin{pmatrix} -\varphi\psi_x & 0 \\ -\lambda\varphi\psi_x & 0 \end{pmatrix} \text{ and } P = \begin{pmatrix} 0 & -\varphi\psi_u \\ 0 & 1 - \lambda\varphi\psi_u \end{pmatrix}.$$

In the E-stability conditions (31)-(33), the condition for b is independent of the other variables, while the conditions for a and c are dependent on b but not on each other. Because of this recursive structure, a necessary condition for stability is that $DT_a - I$, evaluated at the REE, has eigenvalues with negative real parts. This condition is equivalent to $\operatorname{tr}(DT_a - I) < 0$ and $\det(DT_a - I) > 0$.

Using the notation $b = (b_{ij})$, $j = 1, 2$ and evaluating variables at the REE, we have $b_{11} = \bar{b}_x$, $b_{21} = \bar{b}_\pi$ and $b_{12} = b_{22} = 0$. The coefficient matrix for a in (31) for the reduced form (17) has the explicit form

$$DT_a - I = \begin{pmatrix} \bar{b}_x + \varphi\bar{b}_\pi & \varphi \\ \lambda(\bar{b}_x + 1) + (\beta + \lambda\varphi)\bar{b}_\pi & (\beta + \lambda\varphi) - 1 \end{pmatrix}. \tag{38}$$

The determinant of the coefficient matrix (38) is

$$(\beta - 1)\bar{b}_x - \varphi\bar{b}_\pi - \lambda\varphi < 0$$

since the parameters λ, φ are positive, $0 < \beta < 1$ and the REE values $\bar{b}_x$ and $\bar{b}_\pi$ are positive. The result follows.

E-Stability Analysis in Section 3.1.3. The result can be verified numerically by computing the E-stability conditions in Appendix A.1.3.[35]

E-Stability Analysis in Section 3.1.4. The reduced form for this model has the general form (28) with $y_t = (x_t, p_t)'$,

$$\begin{aligned} A &= \begin{pmatrix} -\varphi\eta_0 \\ -\lambda\varphi\eta_0 \end{pmatrix}, \; M = \begin{pmatrix} 1 & \varphi \\ \lambda & \beta + \lambda\varphi \end{pmatrix}, \; Q = \begin{pmatrix} 0 & -\varphi \\ 0 & -\beta - \lambda\varphi \end{pmatrix}, \\ N &= \begin{pmatrix} 0 & -\varphi\eta_p \\ 0 & 1 - \lambda\varphi\eta_p \end{pmatrix} \text{ and } P = \begin{pmatrix} 0 & -\varphi\eta_u \\ 0 & 1 - \lambda\varphi\eta_u \end{pmatrix}. \end{aligned}$$

A sufficient condition for instability is that one of the eigenvalues of $DT_a - I = Q + M(I + \bar{b}) - I$ has a positive real part. Hence a sufficient condition for instability is that $\det(DT_a - I) < 0$. It is easily computed that

$$Q + M(I + \bar{b}) - I = \begin{pmatrix} 0 & \bar{b}_p + \varphi\bar{b}_x \\ \lambda & \lambda\bar{b}_p + (\lambda\varphi + \beta)\bar{b}_x - 1 \end{pmatrix}$$

and $\det(DT_a - I) = -\lambda(\bar{b}_p + \varphi\bar{b}_x)$. Using $\bar{b}_p = -\frac{\lambda}{\alpha}\bar{b}_x$ we get $\det(DT_a - I) = -\lambda(\varphi - \frac{\lambda}{\alpha})\bar{b}_x$ Since $0 < \bar{b}_x < 1$, this is negative when $\varphi > \frac{\lambda}{\alpha}$. Since the trace is given by $\operatorname{tr}(DT_a - I) =$

[35] The Mathematica routines are available on request.

$(\beta+\lambda(\varphi-\frac{\lambda}{\alpha}))\bar{b}_x-1$, it can be seen that the trace condition is satisfied if the determinant condition is satisfied. Hence E-stability holds if and only if $\varphi > \frac{\lambda}{\alpha}$.

Proof of Proposition 4. Applying the methodology of Section A.2 to the reduced form (22) we obtain

$$J = \begin{pmatrix} 1 & 0 & \frac{-\alpha}{\alpha+\lambda^2} \\ 0 & 1 & \frac{-\alpha\lambda}{\alpha+\lambda^2} \\ 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -\frac{\lambda\beta}{\alpha+\lambda^2} & 0 \\ 0 & \frac{\alpha\beta}{\alpha+\lambda^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \beta & \lambda \\ 0 & \frac{\beta\lambda}{\alpha} & \frac{\alpha+\lambda^2}{\alpha} \end{pmatrix}.$$

The roots of J are 0 and $(2\alpha)^{-1}\left(\alpha+\alpha\beta+\lambda^2 \pm \sqrt{(\alpha+\alpha\beta+\lambda^2)^2-4\alpha^2\beta}\right)$. It can be verified that the nonzero roots are real and positive, with one root less than one and the other root larger than one. Since exactly two roots are inside the unit circle, the result follows.

Proof of Proposition 5. We note that in this case the matrices in the E-stability conditions can be written as

$$DT_b - I = \begin{pmatrix} \frac{-\beta\lambda\bar{b}_\pi}{\alpha+\lambda^2}-1 & \frac{-\beta\lambda\bar{b}_x}{\alpha+\lambda^2} & 0 & 0 \\ \frac{\alpha\beta\bar{b}_\pi}{\alpha+\lambda^2} & \frac{\alpha\beta\bar{b}_x}{\alpha+\lambda^2}-1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{39}$$

$$DT_c - I = \begin{pmatrix} \frac{-\lambda\beta\bar{b}_\pi}{\alpha+\lambda^2}-1 & -\frac{\lambda\beta\rho}{\alpha+\lambda^2} & 0 & 0 \\ \frac{\alpha\beta\bar{b}_\pi}{\alpha+\lambda^2} & \frac{\alpha\beta\rho}{\alpha+\lambda^2}-1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{40}$$

and

$$DT_a - I = \begin{pmatrix} \frac{-\beta\lambda\bar{b}_\pi}{\alpha+\lambda^2}-1 & \frac{-\beta\lambda}{\alpha+\lambda^2} \\ \frac{\alpha\beta\bar{b}_\pi}{\alpha+\lambda^2} & \frac{\alpha\beta}{\alpha+\lambda^2}-1 \end{pmatrix}. \tag{41}$$

Looking at the coefficient matrix (39), it has two eigenvalues equal to -1 while the remaining two eigenvalues are those of the 2×2 matrix in the top left corner of DT_b-I. The trace of this 2×2 matrix is given by

$$\frac{-\beta\lambda\bar{b}_\pi}{\alpha+\lambda^2}+\frac{\alpha\beta\bar{b}_x}{\alpha+\lambda^2}-2,$$

which is negative since the only positive term is less than one. Its determinant is equal to

$$\frac{\beta\lambda\bar{b}_\pi}{\alpha+\lambda^2}-\frac{\alpha\beta\bar{b}_x}{\alpha+\lambda^2}+1,$$

which is positive as the only negative term is less than one absolute value (since $\beta < 1$ and $0 < \bar{b}_x < 1$). Thus the matrix (39) is stable (i.e. all of its eigenvalues have negative real parts).

Next, consider the matrices (40) and (41). The matrix (40) has two eigenvalues equal to -1 and the remaining two are those of the 2×2 matrix in the top left corner. The trace of this 2×2 matrix is

$$\frac{-\lambda\beta\bar{b}_\pi}{\alpha+\lambda^2} + \frac{\alpha\beta\rho}{\alpha+\lambda^2} - 2.$$

The only positive term (if $\rho > 0$) is less than one and so the trace is always negative. (If $\rho < 0$, all terms are negative.) Its determinant is

$$\frac{\beta\lambda\bar{b}_\pi}{\alpha+\lambda^2} - \frac{\alpha\beta\rho}{\alpha+\lambda^2} + 1$$

and the only (possibly) negative term is less than one and so the determinant is positive. Thus $DT_c - I$ is a stable matrix. Finally, we note that the top left 2×2 matrix with $\rho = 1$ is identical to the matrix (41), so that the latter is also a stable matrix.

Proof of Proposition 6. The result is verified by computing the eigenvalues of $DT_j - I$ for $j = a, b, c$ of Appendix A.1.3 with M, N and P given by the reduced form (22). Let $\Delta = \alpha(1 + (1 - \bar{b}_x)\beta) + \lambda^2 > 0$. Apart from roots equal to -1 the eigenvalues are[36]

$$\begin{aligned}
&-\frac{\alpha(1-\bar{b}_x\beta)+\lambda^2}{\Delta},\\
&-\frac{\alpha^2(1+\beta-2\bar{b}_x\beta+(\bar{b}_x-1)^2\beta^2+2\alpha(1+\beta(1-\bar{b}_x))\lambda^2+\lambda^4}{\Delta},\\
&-\frac{\lambda^2+\alpha(1+\beta(1-\bar{b}_x-\mu))}{\Delta}, -\frac{\lambda^2+\alpha(1+\beta(1-\bar{b}_x-\rho))}{\Delta}.
\end{aligned}$$

The numerators of these expressions are all positive, so the result follows.

Details on Approximate Targeting Instrument Rules:

(1) Determinacy of the approximate targeting rule (24). Using a method analogous to Section A.2 we rewrite (25) as

$$\begin{aligned}
&\begin{pmatrix} 1+\alpha\varphi\theta\lambda^{-1} & \varphi(1+\theta) & -\alpha\varphi\theta\lambda^{-1} \\ -\lambda & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ \pi_t \\ x_t^L \end{pmatrix}\\
= &\begin{pmatrix} 1 & \varphi & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{t+1} \\ \pi_{t+1} \\ x_{t+1}^L \end{pmatrix} + other.
\end{aligned}$$

[36] The Mathematica routines for computing the eigenvalues are available on request.

This leads to an equation of the form (37) with

$$
\begin{aligned}
J &= \begin{pmatrix} 1+\alpha\varphi\theta\lambda^{-1} & \varphi(1+\theta) & -\alpha\varphi\theta\lambda^{-1} \\ -\lambda & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \varphi & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & \beta & \lambda \\ -\frac{\lambda}{\alpha\varphi\theta} & \frac{(-1+\beta+\beta\theta)\lambda}{\alpha\theta} & \frac{\lambda+\alpha\varphi\theta+(1+\theta)\lambda^2\varphi}{\alpha\varphi\theta} \end{pmatrix}.
\end{aligned}
$$

Determinacy requires exactly two roots of J inside the unit circle.

(2) E-stability of the approximate targeting rule (24). The matrices in (25) have the form $Q = 0$ and

$$
\begin{aligned}
M &= \begin{pmatrix} 1+\alpha\varphi\theta\lambda^{-1} & \varphi(1+\theta) \\ -\lambda & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \varphi \\ 0 & \beta \end{pmatrix}, \\
N &= \begin{pmatrix} 1+\alpha\varphi\theta\lambda^{-1} & \varphi(1+\theta) \\ -\lambda & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha\varphi\theta\lambda^{-1} & 0 \\ 0 & 0 \end{pmatrix} \text{ and} \\
P &= \begin{pmatrix} 1+\alpha\varphi\theta\lambda^{-1} & \varphi(1+\theta) \\ -\lambda & 1 \end{pmatrix}^{-1}.
\end{aligned}
$$

It is not possible to derive theoretical results from the E-stability conditions in Remark 7 and so we evaluated the conditions numerically.

(3) Determinacy of the forward looking approximate targeting rule (26). Write (27) in the form

$$
\begin{pmatrix} 1-\alpha\varphi\theta\lambda^{-1} & 0 \\ -\alpha\varphi\theta & 1 \end{pmatrix} \begin{pmatrix} x_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} 1-\alpha\varphi\theta\lambda^{-1} & -\varphi\theta \\ \lambda-\alpha\varphi\theta & \beta-\varphi\theta\lambda \end{pmatrix} \begin{pmatrix} x_{t+1} \\ \pi_{t+1} \end{pmatrix} + \mathit{other}.
$$

Determinacy requires that the eigenvalues of

$$
J = \begin{pmatrix} 1-\alpha\varphi\theta\lambda^{-1} & 0 \\ -\alpha\varphi\theta & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1-\alpha\varphi\theta\lambda^{-1} & -\varphi\theta \\ \lambda-\alpha\varphi\theta & \beta-\varphi\theta\lambda \end{pmatrix} = \begin{pmatrix} 1 & \frac{\varphi\theta\lambda}{\alpha\varphi\theta-\lambda} \\ \lambda & \frac{-\beta\lambda+\alpha\beta\theta\varphi+\theta\phi\lambda^2}{\alpha\varphi\theta-\lambda} \end{pmatrix}
$$

lie inside the unit circle. The condition for this is that $|\det(J)| < 1$ and $|\mathrm{tr}(J)| < 1+\det(J)$. The first condition is satisfied since $\det(J) = \beta$. The second condition can be shown to be satisfied if $\theta < \hat{\theta}$, where

$$
\hat{\theta} = \lambda\left(\alpha\varphi + \frac{\lambda^2\varphi}{2(1+\beta)}\right)^{-1}.
$$

For $\theta > \hat{\theta}$ the system is indeterminate. Numerical values for $\hat{\theta}$ are given in Table 1 of the main text.

(4) E-stability of the forward looking approximate targeting rule (26). The reduced form (27) is of the form (28) with $N = 0$. From Section A.1 the required conditions are that matrices (35) and (36) have negative real parts. First consider

$$M + Q = \begin{pmatrix} 1 & -\varphi\theta \\ \lambda & \beta - \varphi\theta\lambda \end{pmatrix}.$$

Since $\mathrm{tr}(M + Q - I) < 0$ and $\det(M + Q - I) > 0$, the first stability condition is met.

It can be shown that the 4×4 matrix $DT_h - I$ in (36) is block diagonal with the upper left hand block given by

$$DT_{UL} - I = \begin{pmatrix} (\alpha\theta\varphi\lambda^{-1} - 1)(1 - \mu) & -\mu\theta\varphi \\ \lambda\mu + \alpha\theta\varphi(1 - \mu) & \mu(\beta - \theta\lambda\varphi) - 1 \end{pmatrix}$$

and the lower right hand block formally the same except that ρ replaces μ. We thus only need to analyze $DT_{UL} - I$. It can be shown that (i) at $\theta = 0$ the trace is negative and the determinant is positive and (ii) the trace and the determinant are each monotonic in θ, but can be increasing or decreasing in θ depending on parameters. It follows that the approximate targeting rule leads to E-stability for $\theta > 0$ sufficiently small. However, depending on the parameter values there can exist $\bar{\theta} > 0$ such that the REE fails to be E-stable for all $\theta > \bar{\theta}$. The critical values $\bar{\theta}$ can be calculated numerically and are reported in Table 2 of the main text.

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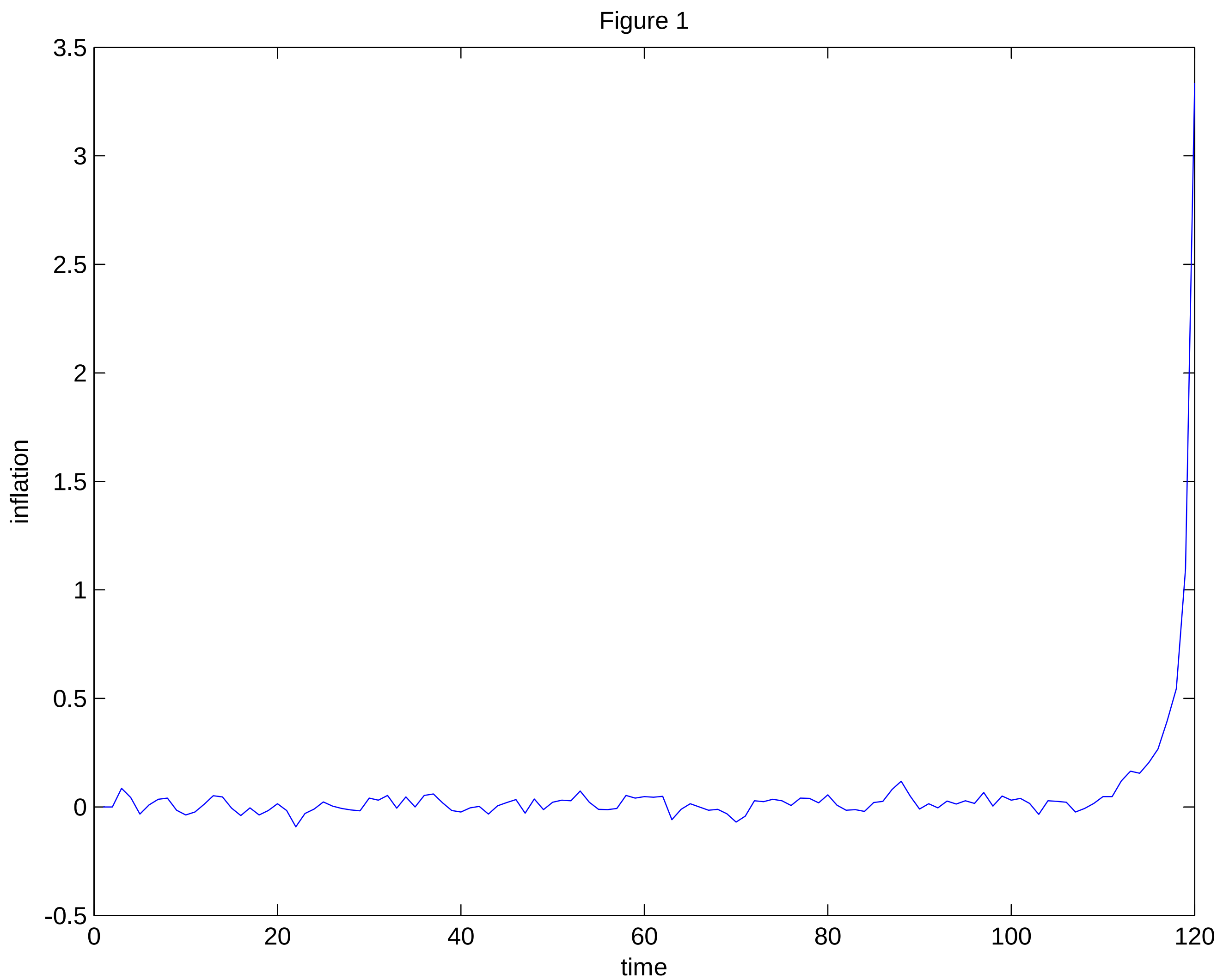


Figure 1
inflation
time
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
0
20
40
60
80
100
120

Figure 2

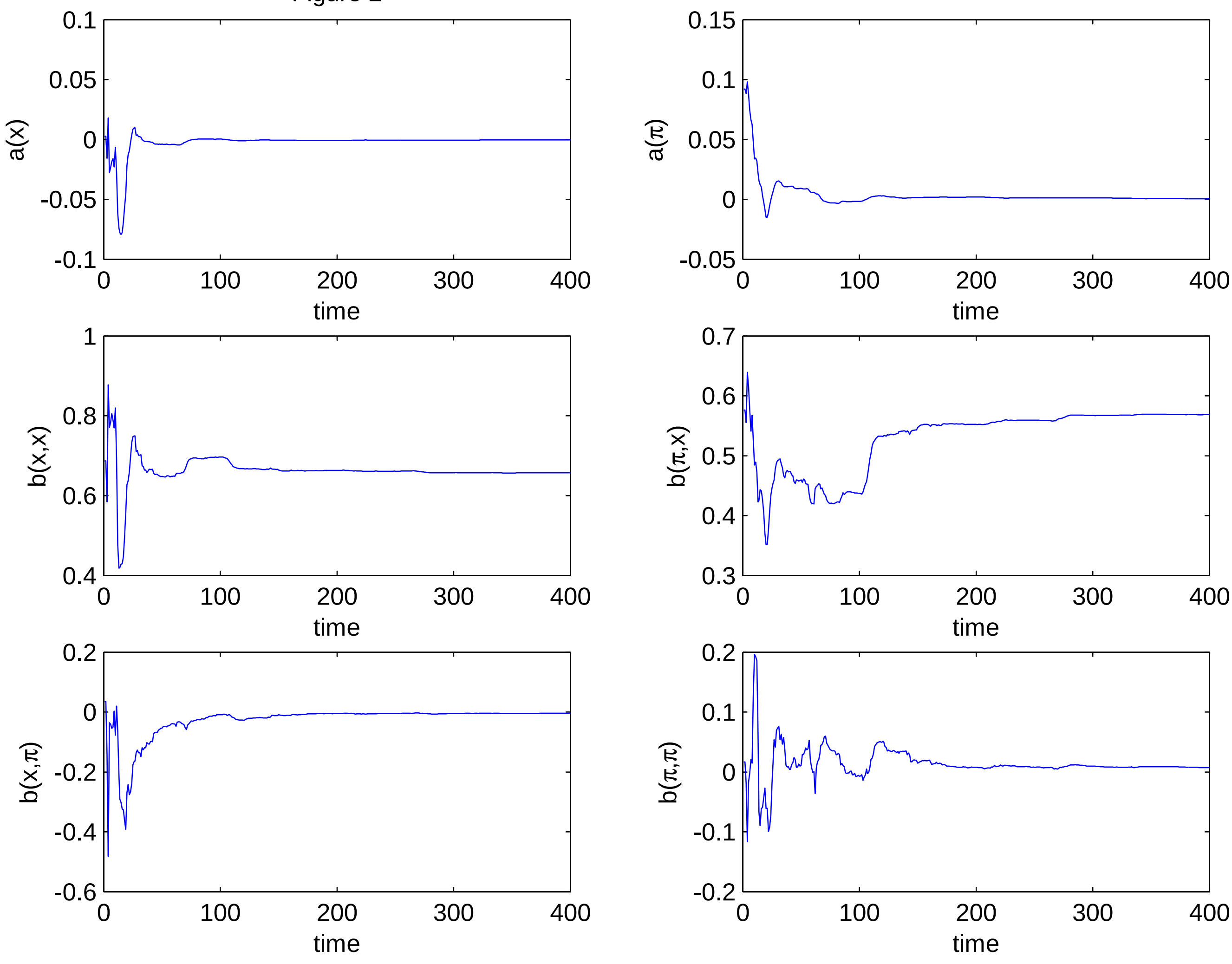
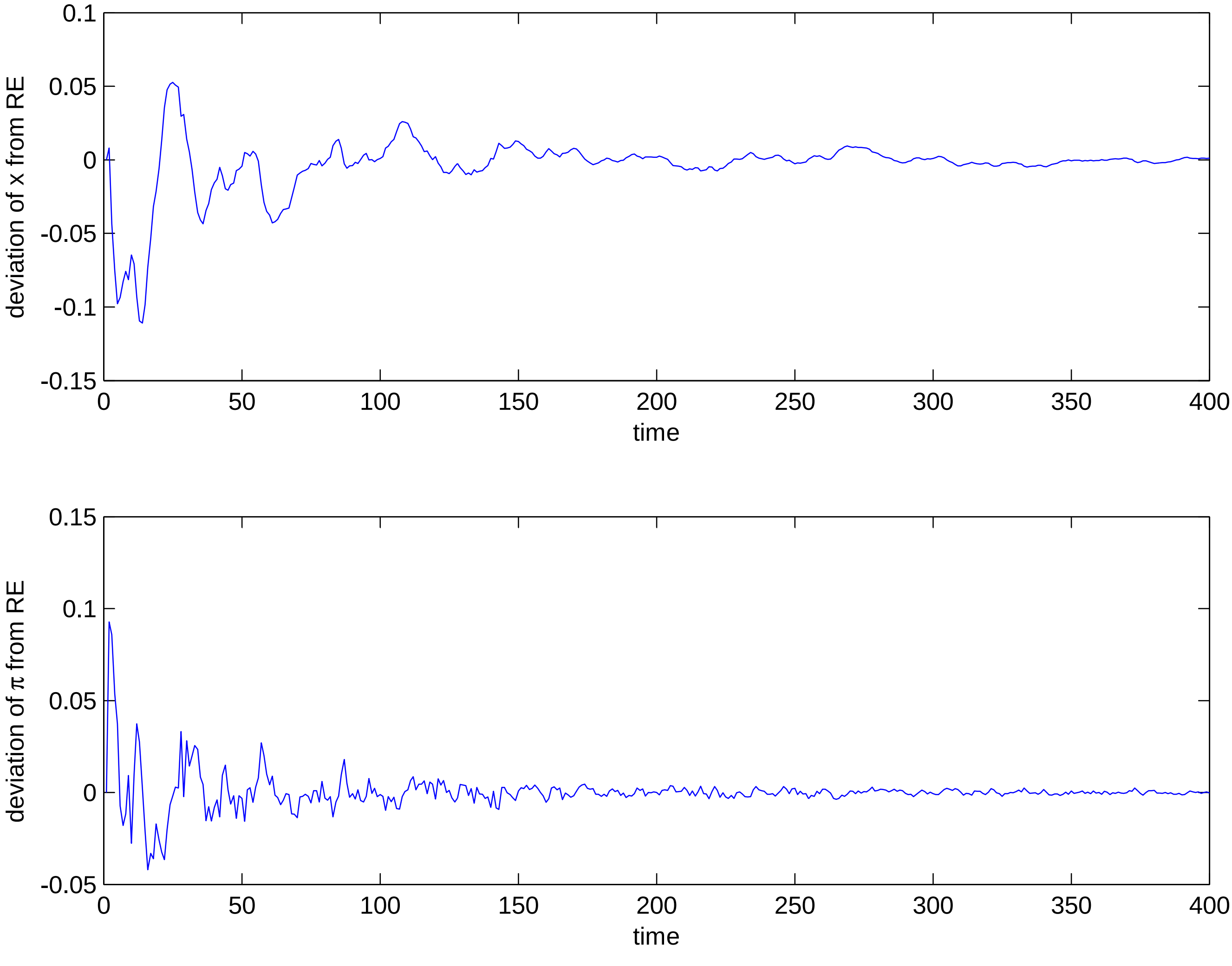


Figure 3


0.1
0.05
0
-0.05
-0.1
-0.15
deviation of x from RE
0
50
100
150
200
250
300
350
400
time
0.15
0.1
0.05
0
-0.05
deviation of π from RE
0
50
100
150
200
250
300
350
400
time